2. Laplace Transform of a Periodic Function–

Let f be a periodic function with period T, so that $f(t) = f(t+T)$, then

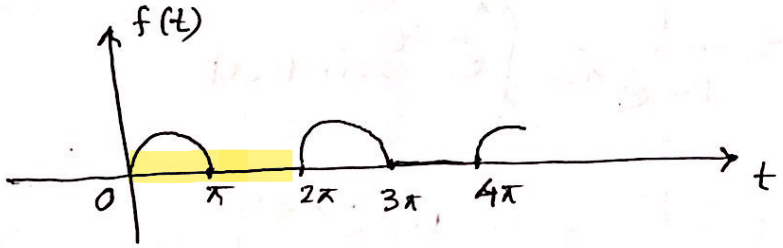
$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

Example 1: Find Laplace transform of the half-wave rectified sinusoidal

$$f(t) = \begin{cases} \sin t\,; & \text{for } 2n\pi < t < (2n+1)\pi \\ 0\,; & \text{for } (2n+1)\pi < t < (2n+2)\pi \end{cases}$$

for $n = 0, 1, 2, \ldots$

Soln: The graph of the half-wave rectified sinusoidal function f is shown below:



The function is of period 2π, so,

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s}} \int_0^{\pi} e^{-st} \sin t\,dt$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-st}(-s\sin t - \cos t)}{s^2+1} \right]_0^{\pi}$$

$$= \frac{1}{1-e^{-2\pi s}} \left(\frac{1+e^{-\pi s}}{s^2+1} \right) = \frac{1}{(1-e^{-\pi s})(s^2+1)}$$

$\int e^{ax} \sin bx\,dx = \frac{e^{at}}{a^2+b^2}(a\sin bx - b\cos bx)$

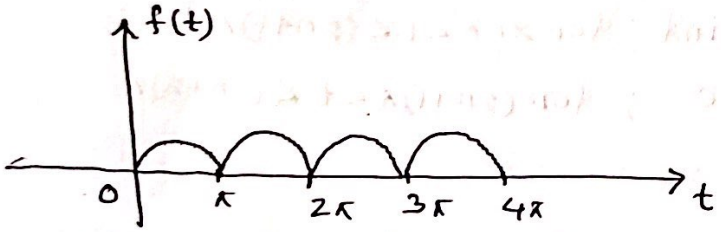
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Example-2: Find the Laplace transform of full rectified sine wave defined by the expression

$$f(t) = \begin{cases} \sin t; & \text{for } 0 < t < \pi \\ -\sin t; & \text{for } \pi < t < 2\pi \end{cases}$$

$$f(t) = f(t+\pi)$$

Solution: The graph of the rectified sine wave is shown in the figure



The function f has period π, and therefore

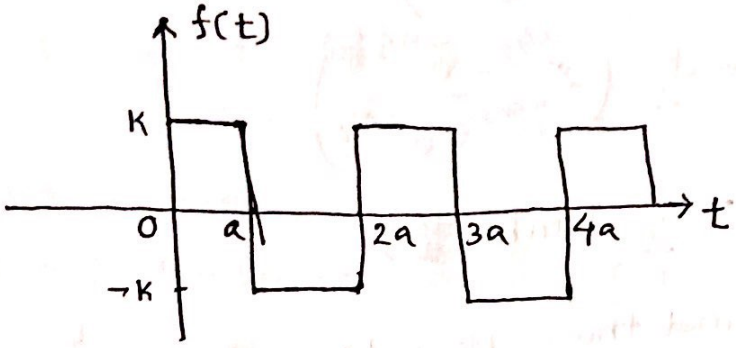
$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-\pi s}} \int_0^{\pi} e^{-st} \sin t \, dt$$

$$= \frac{1}{1-e^{-\pi s}} \left[\frac{1+e^{-\pi s}}{s^2+1} \right]$$

$$= \frac{1+e^{-\pi s}}{(1-e^{-\pi s})(s^2+1)}$$

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Example-3: Find the Laplace transform of square wave, whose output is given below:



Solution: It is a periodic function with period 2a.

Here $f(t) = \begin{cases} K & ; 0 < t < a \\ -K & ; a < t < 2a \end{cases}$

$$\therefore \mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st} \cdot K\,dt + \int_a^{2a} e^{-st}(-K)\,dt \right]$$

$$= \frac{K}{1-e^{-2as}} \left[\int_0^a e^{-st}\,dt - \int_a^{2a} e^{-st}\,dt \right]$$

$$= \frac{K}{1-e^{-2as}} \left[\frac{e^{-st}}{-s}\Big|_0^a - \frac{e^{-st}}{-s}\Big|_a^{2a} \right]$$

$$= \frac{K}{1-e^{-2as}} \left[-\frac{e^{-as}}{s} + \frac{1}{s} + \frac{1}{s}e^{-2as} - \frac{1}{s}e^{-as} \right]$$

$$= \frac{-K}{s(1-e^{-2as})} \left[2e^{-as} - e^{-2as} - 1 \right]$$

$$= \frac{K}{s(1-e^{-as})(1+e^{-as})} \left(1-e^{-as}\right)^2$$

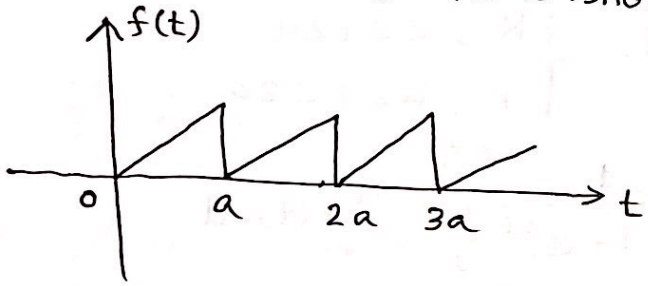
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$$\mathcal{L}\{f(t)\} = \frac{K}{s}\left(\frac{1-e^{-as}}{1+e^{-as}}\right)$$

$$= \frac{K}{s}\left(\frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}}\right)$$

$$= \frac{K}{s}\tanh\left(\frac{as}{2}\right)$$

Example-4: Find the Laplace transform of the sawtooth wave function, whose graph is shown in the figure.



Solution: It is a periodic function with period a.

$$f(t) = t \; ; \; 0 < t < a \quad \text{and} \quad f(t+a) = f(t)$$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-as}}\int_0^a e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-as}}\int_0^a t\,e^{-st}\,dt$$

$\int \underset{u}{t}\,\underset{v}{e^{-st}}\,du$

$$= \frac{1}{1-e^{-as}}\left\{\left[\frac{te^{-st}}{-s}\right]_0^a + \frac{1}{s}\int_0^a e^{-st}\,dt\right\}$$

$$= \frac{1}{1-e^{-as}}\left\{\frac{ae^{-as}}{-s} - \frac{1}{s^2}e^{-st}\Big|_0^a\right\}$$

$$= \frac{1}{1-e^{-as}}\left\{-\frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2}\right\}$$

$$= \frac{1}{1-e^{-as}}\left[-\frac{ae^{-as}}{s} + \frac{1-e^{-as}}{s^2}\right]$$

$$= \frac{1}{s^2} - \frac{ae^{-sa}}{s(1-e^{-as})}$$

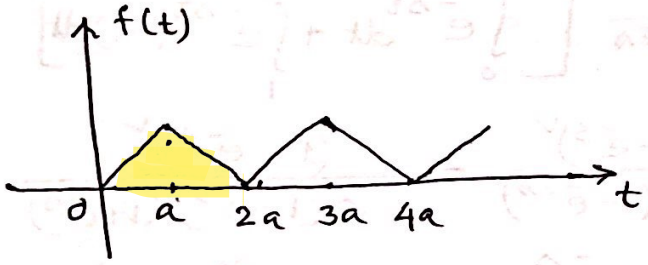
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Example-5: Find Laplace transform of the triangular wave defined by

$$f(t) = \begin{cases} t & \text{for } 0 \le t < a \\ 2a - t & \text{for } a < t < 2a \end{cases}$$

$$f(2a + t) = f(t)$$

Solution: The graph of triangular wave function is shown in the figure.



Using the formula for the Laplace transform of a periodic function, we have

$$\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-2as}} \int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1 - e^{-2as}} \left[\int_0^a t\, e^{-st}\,dt + \int_a^{2a} (2a - t)\, e^{-st}\,dt \right]$$

Now integration by parts gives

$$\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-2as}} \left(e^{-as} - 1\right)^2 \cdot \frac{1}{s^2}$$

$$= \frac{1}{s^2} \cdot \frac{1 - e^{-as}}{1 + e^{-as}} = \frac{1}{s^2} \frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}}$$

$$= \frac{1}{s^2} \tanh\left(\frac{as}{2}\right)$$

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Example-6: Find $\mathcal{L}\{f(t)\}$, where

$$f(t)=\begin{cases}1 ; & 0<t<1\\ -1 ; & 1<t<2\end{cases} ; \quad f(t+2)=f(t)$$

Solution: It is a periodic function of period 2.

$$\mathcal{L}\{f(t)\}=\frac{1}{1-e^{-2s}}\int_0^2 e^{-st} f(t)\,dt$$

$$=\frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}\,dt+\int_1^2 e^{-st}(-1)\,dt\right]$$

$$=\frac{(1-e^{-s})^2}{s(1-e^{-2s})}=\frac{(1-e^{-s})^2}{s(1-e^{-s})(1+e^{-s})}$$

$$=\frac{1-e^{-s}}{s(1+e^{-s})}=\frac{1}{s}\cdot\frac{e^{s/2}-e^{-s/2}}{e^{s/2}+e^{-s/2}}$$

$$=\frac{1}{s}\tanh(s/2)$$

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