3.1. Laplace Transform of special functions.

(a) Bessel Function $J_n(t)$ is defined as:

$$J_n(t) = \frac{t^n}{2^n \lfloor n+1} \left[1 - \frac{t^2}{2(2n+3)} + \frac{t^4}{2.4.(2n+2)(2n+4)} - \cdots\right]$$

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2.4^2} - \cdots\cdots$$

$$\mathcal{L}\{J_0(t)\} = \frac{1}{s} - \frac{1}{2^2}\frac{2!}{s^3} + \frac{1}{2^2.4^2}\cdot\frac{4!}{s^5} - \cdots$$

Taylor series, refer to this: https://qr.ae/p2idcS

$$= \frac{1}{s}\left\{1 - \frac{1}{2}\cdot\frac{1}{s^2} + \frac{1}{2}\cdot\frac{3}{4}\left(\frac{1}{s^2}\right)^2 - \cdots\right\}$$

$$\therefore \mathcal{L}\{J_0(t)\} = \frac{1}{s}\left\{\left(1+\frac{1}{s^2}\right)^{-1/2}\right\} = \frac{1}{\sqrt{s^2+1}}$$

$$\Rightarrow \mathcal{L}\{J_0(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right) = \frac{1}{a}\frac{1}{\sqrt{(s/a)^2+1}} = \frac{1}{\sqrt{s^2+a^2}}$$

Again $J_0'(t) = -J_1(t)$

$$\therefore \mathcal{L}\{J_1(t)\} = -\mathcal{L}\{J_0'(t)\}$$

$$= -[s\mathcal{L}\{J_0(t) - J_0(0)]$$

$$\Rightarrow \mathcal{L}\{J_1(t)\} = 1 - \frac{s}{\sqrt{s^2+1}} = \frac{\sqrt{s^2+1} - s}{\sqrt{s^2+1}}$$

(b) Error function

Defined as $erf(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-u^2}du$

Complementary error function is defined as

$$erfc(t) = 1 - erf(t) = 1 - \frac{2}{\sqrt{\pi}}\int_0^t e^{-u^2}du$$

$$= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-u^2}du - \frac{2}{\sqrt{\pi}}\int_0^t e^{-u^2}du$$

$$= \frac{2}{\sqrt{\pi}}\int_t^\infty e^{-u^2}du$$

(i) Now $\mathcal{L}\{\text{erf}f(t)\} = \mathcal{L}\left\{\frac{2}{\sqrt{\pi}}\int_0^t e^{-u^2}du\right\}$

$$= \frac{2}{\sqrt{\pi}}\mathcal{L}\left\{\int_0^t e^{-u^2}du\right\} = \frac{2}{\sqrt{\pi}}\frac{\bar{f}(s)}{s}$$

where $\bar{f}(s) = \mathcal{L}\{e^{-t^2}\}$

$$= \int_0^\infty e^{-t^2}.e^{-st}\,dt$$

$$= e^{s^2/4}\int_0^\infty e^{-(t^2+st+\frac{s^2}{4})}\,dt$$

$$= e^{s^2/4}\int_0^\infty e^{-(t+s/2)^2}\,dt$$

$$= e^{s^2/4}\int_{s/2}^\infty e^{-u^2}du \qquad \left| t+\frac{s}{2}=u\right.$$

$$= e^{s^2/4}\,\text{erfc}(s/2)\cdot\frac{\sqrt{\pi}}{2}$$

$$\therefore \mathcal{L}\{\text{erf}f(t)\} = \frac{2}{\sqrt{\pi}}\cdot\frac{1}{s}\cdot e^{s^2/4}\,\text{erfc}\left(\frac{s}{2}\right)\cdot\frac{\sqrt{\pi}}{2}$$

$$= \frac{e^{s^2/4}}{s}\,\text{erfc}\left(\frac{s}{2}\right)$$

(ii) $\mathcal{L}\{\text{erf}\sqrt{t}\} = \mathcal{L}\left\{\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}du\right\}$

$$= \mathcal{L}\left\{\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}}\left(1-u^2+\frac{u^4}{2!}-\frac{u^6}{3!}+\cdots\right)du\right\}$$

$$= \frac{2}{\sqrt{\pi}}\mathcal{L}\left\{t^{1/2}-\frac{t^{3/2}}{3}+\frac{t^{5/2}}{5.2!}-\frac{t^{7/2}}{7.3!}+\cdots\right\}$$

$$= \frac{2}{\sqrt{\pi}}\left[\frac{\lfloor 3/2}{s^{3/2}}-\frac{\lfloor 5/2}{3\,s^{5/2}}+\frac{\lfloor 7/2}{5.2!.s^{7/2}}-\cdots\right]$$

$$= \frac{1}{s^{3/2}}-\frac{1}{2}\cdot\frac{1}{s^{5/2}}+\frac{1.3}{2.4}\cdot\frac{1}{s^{7/2}}-\cdots$$

$$= \frac{1}{s^{3/2}}\left[1-\frac{1}{2}\cdot\frac{1}{s}+\frac{1.3}{2.4}\frac{1}{s^2}-\frac{1.3.5}{2.4.6}\frac{1}{s^3}+\cdots\right]$$

$$\Rightarrow \mathcal{L}\{\text{erf}\sqrt{t}\} = \frac{1}{s^{3/2}}\left(1+\frac{1}{s}\right)^{-1/2} = \frac{1}{s\sqrt{s+1}}$$

(c) Sine Integral

Defined as $s_i(t) = \int_0^t \frac{\sin u}{u}\, du$

Put $u = tv$

$\therefore \int_0^t \frac{\sin u}{u}\, du = \int_0^1 \frac{\sin tv}{v}\, dv$

$\therefore L\left\{\int_0^t \frac{\sin u}{u}\, du\right\} = L\left\{\int_0^1 \frac{\sin tv}{v} dv\right\}$

$= \int_0^\infty e^{-st}\left\{\int_0^1 \frac{\sin tv}{v}\, dv\right\} dt$

$= \int_0^1 \frac{1}{v}\left\{\int_0^\infty e^{-st} \sin tv\, dt\right\} dv$

$= \int_0^1 \frac{1}{v} \cdot \frac{v}{s^2+v^2}\, dv$

$= \int_0^1 \frac{dv}{s^2+v^2}$

$= \frac{1}{s} \tan^{-1}\frac{v}{s}\Big|_0^1$

$= \frac{1}{s} \tan^{-1}\frac{1}{s}$.

Example- Show that $\mathcal{L}\{t J_0(2t)\} = \frac{s}{(s^2+4)^{3/2}}$

Solution: We know

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 \cdot 4^2} - \cdots$$

$$\mathcal{L}\{J_0(t)\} = \frac{1}{s} - \frac{1}{2^2}\cdot\frac{2!}{s^3} + \frac{1}{2^2\cdot 4^2}\frac{4!}{s^5} - \cdots$$

$$= \frac{1}{s}\left\{1 - \frac{1}{2}\frac{1}{s^2} + \frac{1}{2}\cdot\frac{3}{4}\left(\frac{1}{s^2}\right)^2 - \cdots\right\}$$

$$= \frac{1}{s}\left\{\left(1+\frac{1}{s^2}\right)^{-\frac{1}{2}}\right\} = \frac{1}{s\sqrt{\frac{s^2+1}{s^2}}}$$

$$= \frac{1}{(s^2+1)^{1/2}} = \bar{f}(s)$$

$$\therefore \mathcal{L}\{J_0(2t)\} = \frac{1}{2}\bar{f}\left(\frac{s}{2}\right)$$

$$= \frac{1}{2}\frac{1}{\left(\frac{s^2}{4}+1\right)^{1/2}}$$

$$= \frac{1}{(s^2+4)^{1/2}}$$

$$\therefore \mathcal{L}\{t J_0(2t)\} = -\frac{d}{ds}\cdot\frac{1}{(s^2+4)^{1/2}}$$

$$= \frac{2s}{2(s^2+4)^{3/2}}$$

$$= \frac{s}{(s^2+4)^{3/2}}$$

Example- Show that $\int_0^\infty e^{-t} J_0(t)\,dt = \frac{1}{\sqrt{2}}$

Solution: we know $\mathcal{L}\{J_0(t)\} = \frac{1}{(s^2+1)^{1/2}}$

$$\Rightarrow \int_0^\infty e^{-st} J_0(t)\,dt = \frac{1}{(s^2+1)^{1/2}}$$

Taking $s = 1$

$$\int_0^\infty e^{-t} J_0(t)\,dt = \frac{1}{\sqrt{2}}$$

Example: Show that $\mathcal{L}\{t\,J_1(t)\} = \frac{1}{(s^2+1)^{3/2}}$

Solution: we know $J_0'(t) = -J_1(t)$

$$\Rightarrow \mathcal{L}\{J_1(t)\} = -\mathcal{L}\{J_0'(t)\}$$

$$= -\left[s\,\mathcal{L}\{J_0(t)\} - J_0(0)\right]$$

$$= 1 - \frac{s}{\sqrt{s^2+1}}$$

$$\mathcal{L}\{t\,J_1(t)\} = -\frac{d}{ds}\left\{1 - \frac{s}{\sqrt{s^2+1}}\right\}$$

$$= \frac{\sqrt{s^2+1} - s\left(\frac{1}{2}\sqrt{s^2+1}\right)\cdot 2s}{\left(\sqrt{s^2+1}\right)^2}$$

$$= \frac{s^2+1-s^2}{(s^2+1)^{3/2}}$$

$$= \frac{1}{(s^2+1)^{3/2}}$$

3.2 Unit step function or Heaviside unit function:

$$u(t-a) = \begin{cases} 0 & ; t < a \\ 1 & ; t > a \end{cases}$$

$$\therefore L\{u(t-a)\} = \int_0^\infty e^{-st}\, u(t-a)\, dt$$

$$= \int_0^a e^{-st}\, u(t-a)\, dt + \int_a^\infty e^{-st} u(t-a)\, dt$$

$$= 0 + \int_0^\infty e^{-st}\, dt$$

$$= -\frac{e^{-st}}{s}\Big|_0^\infty = \frac{e^{-as}}{s}$$

Unit impulse function or Dirac delta function

consider function

$$F_\epsilon(t) = \begin{cases} 1/\epsilon & ; 0 \le t \le \epsilon \\ 0 & ; t > \epsilon \end{cases}$$

This is defined in such always

$$\lim_{\epsilon \to 0} \int_0^\infty F_\epsilon(t)\, dt = 1$$

Function $\lim_{\epsilon \to 0} f_\epsilon(t)$ is defined as unit impulse function or Dirac delta function and denoted by $\delta(t)$

$$L\{F_\epsilon(t)\} = \int_0^\infty e^{-st} F_\epsilon(t)\, dt$$

$$= \int_0^\epsilon e^{-st} \frac{1}{\epsilon}\, dt + \int_\epsilon^\infty e^{-st} \cdot 0 \cdot dt$$

$$= \frac{1}{\epsilon} \int_0^\epsilon e^{-st}\, dt = \frac{1 - e^{-st}}{\epsilon s}$$

$$\therefore L\{\delta(t)\} = L\{\lim_{\epsilon \to 0} F_\epsilon(t)\} = \lim_{\epsilon \to 0} L\{F_\epsilon(t)\}$$

$$= \lim_{\epsilon \to 0} \frac{1 - e^{-st}}{\epsilon s} = 1$$

3.3 Special Techniques of finding Laplace Transform

(a) Laplace Transform of $J_0(t)$.

$J_0(t)$ satisfy Bessel's equation of zero order

$$t J_0''(t) + J_0'(t) + t J_0(t) = 0$$

Suppose $L\{J_0(t)\} = Y(s)$

Now $L\{t J_0''(t)\} + L\{J_0'(t)\} + L\{t J_0(t)\} = 0$

$$\Rightarrow -\frac{d}{ds}[s^2 Y - s J_0(0) - J_0'(0)] + [sY - J_0(0)] - \frac{d}{ds}Y = 0$$

$$\Rightarrow -\frac{d}{ds}[s^2 Y - s.1 - 0] + [sY - 1] - \frac{dY}{ds} = 0$$

$$\Rightarrow (s^2+1)\frac{dY}{ds} = -sY$$

$$\Rightarrow \frac{dY}{ds} = -\frac{sY}{s^2+1}$$

$$\Rightarrow \frac{dY}{Y} = -\frac{s\,ds}{s^2+1}$$

$$\Rightarrow \log Y = -\frac{1}{2}\log(s^2+1) + \log c$$

$$\therefore Y = \frac{c}{\sqrt{s^2+1}}$$

Now $\lim_{s\to\infty} s\,Y(s) = \lim_{s\to\infty} \frac{sc}{\sqrt{s^2+1}} = c$

and $\lim_{t\to 0} J_0(t) = 1$

From initial value theorem $c = 1$

$$\therefore Y(s) = \frac{1}{\sqrt{s^2+1}}$$

(b) $L\{\log t\}$

we know that for $k > -1$

$$\int_0^\infty e^{-st} t^k dt = \frac{\Gamma(k+1)}{s^{k+1}} = \Gamma(k+1) \cdot \frac{1}{s^{k+1}}$$

Differentiating w.r.to k

$$\int_0^\infty e^{-st} t^k \log t \, dt = \frac{\Gamma'(k+1)}{s^{k+1}} - \frac{\Gamma(k+1)}{s^{k+1}} \log s$$

put $k = 0$

$$\int_0^\infty e^{-st} \log t \, dt = \frac{\Gamma'(1) - \log s}{s^{0+1}}$$

$$\Rightarrow L\{\log t\} = \frac{\Gamma'(1) - \log s}{s}$$

(c) $L\{\sin\sqrt{t}\}$

Let $Y(t) = \sin\sqrt{t}$. $Y(t)$ satisfy the equation

$$4t Y'' + 2Y' + Y = 0$$

consider $L\{Y(t)\} = y$

Applying Laplace Transform on the equation

$$-4\frac{d}{ds}[s^2 y - sY(0) - Y'(0)] + 2[sy - Y(0)] + y = 0$$

$\therefore Y(0) = 0$ and $\frac{d}{ds} Y'(0) = 0$

$$\Rightarrow -4s^2\frac{dy}{ds} - 8sy + 2sy + y = 0$$

$$\Rightarrow 4s^2\frac{dy}{ds} + (6s-1)y = 0$$

$$\Rightarrow \frac{dy}{y} = \frac{1-6s}{4s^2} = \left(\frac{1}{4s^2} - \frac{3}{2}\frac{1}{s}\right) ds$$

$$\log y = -\frac{1}{4s} - \frac{3}{2}\log s + \log c$$

$$\Rightarrow y s^{3/2} = c\, e^{-1/4s}$$

For large s, $y \sim \frac{c}{s^{3/2}}$ ——(A)

For small t, $\sin\sqrt{t} \sim \sqrt{t}$

Now $\mathcal{L}\{\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}}$ ——(B)

Comparing (A) and (B), $c = \frac{\sqrt{\pi}}{2}$

$$\therefore y = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}$$

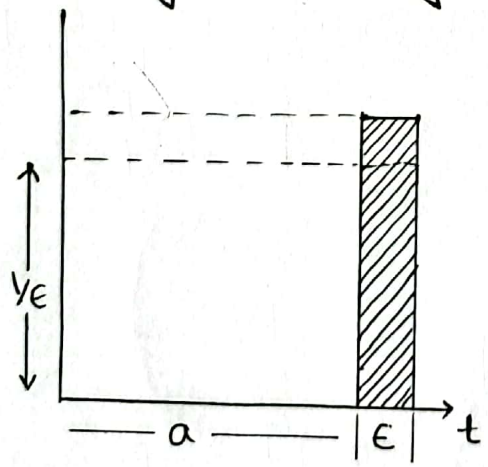
(d) $\mathcal{L}\{\delta(t-a)\}$

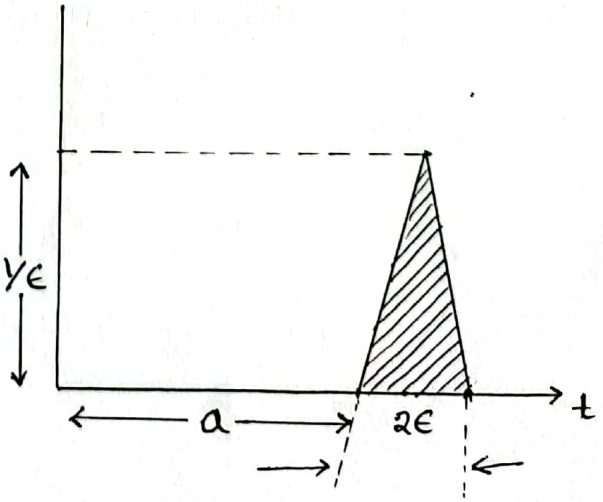
Property of Dirac delta function is

$$\delta(t-a) = 0 \quad , t \neq a$$

and $\int_0^\infty \delta(t-a)\,dt = 1$

Its diagram may be as follows





$\therefore f(t)\,\delta(t-a) = 0$ every where except at $t = a$

$$\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$$

$$\therefore \mathcal{L}\{\delta(t-a)\} = \int_0^\infty e^{-st}\,\delta(t-a)\,dt = e^{-sa}$$