## Lecture 4

## Inverse Laplace Transform

**Formulas:**

| Laplace Transform | Inverse Laplace Transform |
|---|---|
| $L[1]=\frac{1}{s}$ | $L^{-1}[\frac{1}{s}]=1$ |
| $L[t]=\frac{1}{s^2}$ | $L^{-1}[\frac{1}{s^2}]=t$ |
| $L[t^2]=\frac{2!}{s^3}$ | $L^{-1}[\frac{2!}{s^3}]=t^2$ |
| $L[\sin at]=\frac{a}{s^2+a^2}$ | $L^{-1}[\frac{a}{s^2+a^2}]=\sin at$ |
| $L[\cos at]=\frac{s}{s^2+a^2}$ | $L^{-1}[\frac{s}{s^2+a^2}]=\cos at$ |
| $L[e^{at}]=\frac{1}{s-a}$ | $L^{-1}[\frac{1}{s-a}]=e^{at}$ |

**Problem 01:** Find $L^{-1}[F(s)]$, if $F(s)=\frac{2s-3}{s^2+16}+\frac{3s+10}{s^2-9}$

Solution:

$$F(s)=\frac{2s-3}{s^2+16}+\frac{3s+10}{s^2-9}$$

$$=\frac{2s}{s^2+16}-\frac{3}{s^2+16}+\frac{3s}{s^2-9}+\frac{10}{s^2-9}$$

$$=2\cdot\frac{s}{s^2+4^2}-\frac{3}{4}\cdot\frac{4}{s^2+4^2}+3\cdot\frac{s}{s^2-3^2}+\frac{10}{3}\cdot\frac{3}{s^2-3^2}$$

$$L^{-1}\left[F(s)\right]=2L^{-1}\left[\frac{s}{s^2+4^2}\right]-\frac{3}{4}\cdot L^{-1}\left[\frac{4}{s^2+4^2}\right]+3L^{-1}\left[\frac{s}{s^2-3^2}\right]+\frac{10}{3}\cdot L^{-1}\left[\frac{3}{s^2-3^2}\right]$$

$$\therefore L^{-1}\left[F(s)\right]=2\cos 4t-\frac{3}{4}\sin 4t+3\cosh 3t+\frac{10}{3}\sinh 3t.$$



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**Problem 02:** If $F(s)=\dfrac{5s-4}{s^2-4s+20}$ Find $L^{-1}[F(s)]$

Solution:

$$F(s)=\frac{5s-4}{s^2-4s+20}$$
$$=\frac{5s-4}{s^2-2\cdot s\cdot 2+4+16}$$
$$=\frac{5s-4}{(s-2)^2+4^2}$$
$$=\frac{5(s-2)+6}{(s-2)^2+4^2}$$

$$=5\cdot\frac{s-2}{(s-2)^2+4^2}+\frac{6}{(s-2)^2+4^2}$$
$$\therefore L^{-1}[F(s)]=5L^{-1}\left[\frac{s-2}{(s-2)^2+4^2}\right]+\frac{6}{4}\cdot L^{-1}\left[\frac{4}{(s-2)^2+4^2}\right]$$
$$\therefore L^{-1}[F(s)]=5e^{2t}\cos 4t+\frac{3}{2}e^{2t}\sin 4t$$

**Problem 03:** If $F(s)=\dfrac{8s+20}{s^2-12s+32}$ Find $L^{-1}[F(s)]$

Solution:

$$F(s)=\frac{8s+20}{s^2-12s+32}$$
$$=\frac{8s+20}{s^2-8s-2s+32}$$
$$=\frac{8s+20}{s(s-8)-4(s-8)}$$
$$=\frac{8s+20}{(s-8)(s-4)}$$
$$=\frac{8\cdot 8+20}{(s-8)(8-4)}+\frac{8\cdot 4+20}{(4-8)(s-4)}$$
$$=\frac{21}{s-8}-\frac{13}{s-4}$$
$$\therefore L^{-1}[F(s)]=21L^{-1}\left[\frac{1}{s-8}\right]-13L^{-1}\left[\frac{1}{s-4}\right]$$
$$=21e^{8t}-13e^{4t}$$



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**Problem 04:** *If* $F(s)=\dfrac{3s^2-5s+2}{s^3-6s^2+11s-6}$ **Find** $L^{-1}$ [F(s)]

Solution:

$$F(s)=\frac{3s^2-5s+2}{s^3-6s^2+11s-6}$$

$$=\frac{3s^2-5s+2}{(s-1)(s^2-5s+6)}$$

$$=\frac{3s(s-1)-2(s-1)}{(s-1)\left(s^2-3s-2s+6\right)}$$

$$=\frac{(s-1)(3s-2)}{(s-1)\{s(s-3)-2(s-3)\}}$$

$$=\frac{(s-1)(3s-2)}{(s-1)(s-3)(s-2)}$$

$$=\frac{(3s-2)}{(s-3)(s-2)}$$

$$=\frac{(3\cdot 2-2)}{(2-3)(s-2)}+\frac{(3\cdot 3-2)}{(s-3)(3-2)}$$

$$=-\frac{4}{(s-2)}+\frac{7}{(s-3)}$$

$$=\frac{7}{(s-3)}-\frac{4}{(s-2)}$$

$$\therefore\ L^{-1}[F(s)]=7\,L^{-1}\left[\frac{1}{s-3}\right]-4\,L^{-1}\left[\frac{1}{s-2}\right]$$

$$=7\,e^{3t}-4\,e^{2t}$$



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**Problem 05:** Find the inverse Laplace transform of $F(s)=\frac{2s+1}{(s+2)^2(s-1)^2}$

Solution:

*Let,*

$$\frac{2s+1}{(s+2)^2(s-1)^2}=\frac{A}{s+2}+\frac{B}{(s+2)^2}+\frac{C}{s-1}+\frac{D}{(s-1)^2}$$

$$\Rightarrow 2s+1=A(s+2)(s-1)^2+B(s-1)^2+C(s-1)(s+2)^2+D(s+2)^2$$

If $s=1$, then $D=\frac{1}{3}$

If $s=-2$, then $B=-\frac{1}{3}$,

$A=0, C=0$,

$$F(s)=-\frac{1}{3}\cdot\frac{1}{(s+2)^2}+\frac{1}{3}\cdot\frac{1}{(s-1)^2}$$

$$\therefore L^{-1}[F(s)]=-\frac{1}{3}L^{-1}[\frac{1}{(s+2)^2}]+\frac{1}{3}L^{-1}[\frac{1}{(s-1)^2}]$$

$$f(t)=-\frac{1}{3}te^{-2t}+\frac{1}{3}te^{t}$$

**Problem 06:** *If* $F(s)=\frac{8s}{(s^2-2s+5)(s^2+6s+5)}$ Find $L^{-1}$ [F(s)]

**Answer:** $\therefore L^{-1}[F(s)]=\frac{1}{4}[e^{t}\sin 2t-e^{-3t}\sinh 2t$



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**Problem 07: Show that** $L^{-1}[\frac{F(s)}{s}] = \int_0^t f(u)du$

Solution:

$$Let\ g(t) = \int_0^t f(u)du$$

$$\Rightarrow g(0) = 0$$

$$Also, g'(t) = f(t)$$

*Now taking Laplace Transform on both sides, we have*

$$\Rightarrow L[g'(t)] = L[f(t)]$$

$$\Rightarrow S\,G(s) - g(0) = F(s)$$

$$\Rightarrow G(s) = \frac{F(s)}{s}$$

$$\Rightarrow L^{-1}[G(s)] = L^{-1}\left[\frac{F(s)}{s}\right]$$

$$\Rightarrow g(t) = L^{-1}\left[\frac{F(s)}{s}\right]$$

$$\Rightarrow \int_0^t f(u)du = L^{-1}\left[\frac{F(s)}{s}\right]$$

$$\therefore L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(u)du$$

**Problem08: Find** $L^{-1}\left[\frac{16}{s^3(s^2+4)}\right]$

**Solution:**

*Let,*

$$F(s) = \frac{1}{s^2+4}$$

$$= \frac{1}{s^2+2^2}$$

$$= \frac{1}{2}\frac{2}{s^2+2^2}$$

$$L^{-1}[F(s)] = L^{-1}\left[\frac{1}{2}\cdot\frac{2}{s^2+2^2}\right]$$

$$f(t) = \frac{1}{2}\sin 2t$$



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We know, $L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(u)du$

$$\therefore L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s^2+4}\right] = \frac{1}{2}\int_0^t \sin 2u\,du$$

$$= \frac{1}{2}\left[-\frac{\cos 2u}{2}\right]_0^t$$

$$= \frac{1}{4}\left[-\cos 2t + \cos 0\right]$$

$$= \frac{1}{4}\left[1-\cos 2t\right]$$

$$\therefore L^{-1}\left[\frac{16}{s(s^2+4)}\right] = 16\cdot\frac{1}{4}\left[1-\cos 2t\right]$$

$$= 4\left(1-\cos 2t\right)$$

$$\therefore L^{-1}\left[\frac{1}{s}\cdot\frac{16}{s(s^2+4)}\right] = 4\int_0^t (1-\cos 2u)\,du$$

$$= 4\left[u-\frac{\sin 2u}{2}\right]_0^t$$

$$= 4\left[t-\frac{\sin 2t}{2}\right]$$

$$\therefore L^{-1}\left[\frac{1}{s^2(s^2+4)}\right] = 4t - 2\sin 2t$$

*Similarly we can find*

$$\therefore L^{-1}\left[\frac{16}{s^3(s^2+4)}\right] = \cos 2t + 2t^2 - 1$$



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