5.1. Convolution Theorem:

If $L^{-1}\{\bar{f}(s)\} = f(t)$ and $L^{-1}\{\bar{g}(s)\} = g(t)$, then $L^{-1}\{\bar{f}(s)\,\bar{g}(s)\} = \int_0^t f(u)\,g(t-u)\,du$

This theorem is also known as Faltung Theorem or Borel Theorem or Super position Theorem.

Example - 1: Find $L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}$

Since $L^{-1}\left\{\frac{1}{s^2}\right\} = t$ and $L^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t}$

$$\therefore L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = \int_0^t ue^{-u}(t-u)\,du$$

$$= \int_0^t (ut - u^2)e^{-u}\,du$$

$$= \left[-(ut-u^2)e^{-u} + \int(t-2u)e^{-u}du\right]_0^t$$

$$= \left[-(ut-u^2)e^{-u} - (t-2u)e^{-u} - 2\int e^{-u}du\right]_0^t$$

$$= \left[-(ut-u^2)e^{-u} - (t-2u)e^{-u} + 2e^{-u}\right]_0^t$$

$$= te^{-t} + 2e^{-t} + t - 2$$

Example: Apply convolution theorem to evaluate

(a) $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\}$

Solution: Since $\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$ and $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\} = \int_0^t \sin u \, e^{-(t-u)} \, du$$

$$= e^{-t}\int_0^t e^u \sin u \, du$$

Now let, $I = \int e^u \sin u \, du$

$$= -e^u \cos u + \int e^u \cos u \, du$$

$$= -e^u \cos u + e^u \sin u - \int e^u \sin u \, du$$

$$2I = e^u \sin u - e^u \cos u$$

$$I = \frac{1}{2}(e^u \sin u - e^u \cos u)$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\} = \frac{1}{2}\left[e^u \sin u - e^u \cos u\right]_0^t$$

$$= \frac{1}{2}(e^t \sin t - e^t \cos t + 1)$$

Ans.

Example-3: Find $L^{-1}\left\{\frac{1}{(s+2)^2(s-2)}\right\} = L^{-1}\left\{\frac{1}{(s+2)^2}\cdot\frac{1}{s-2}\right\}$

Solution: Since $L^{-1}\left\{\frac{1}{(s+2)^2}\right\} = e^{-2t}t$ and $L^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}$

$$\therefore L^{-1}\left\{\frac{1}{(s+2)^2(s-2)}\right\} = \int_0^t u\,e^{-2u}\,e^{2(t-u)}\,du$$

$$= e^{2t}\int_0^t u\,e^{-4u}\,du$$

$$I = \int u\,e^{-4u}\,du$$

$$= u\frac{e^{-4u}}{-4} + \frac{1}{4}\int e^{-4u}\,du$$

$$= -\frac{u\,e^{-4u}}{4} - \frac{1}{16}e^{-4u}$$

$$\therefore L^{-1}\left\{\frac{1}{(s+2)^2(s-2)}\right\} = e^{2t}\left[-\frac{u\,e^{-4u}}{4} - \frac{1}{16}e^{-4u}\right]_0^t$$

$$= -e^{2t}\left\{\frac{t\,e^{-4t}}{4} + \frac{1}{16}e^{-4t} - \frac{1}{16}\right\}$$

$$= -\frac{t\,e^{-2t}}{4} - \frac{e^{2t}}{16}(e^{-4t}-1)$$

Ans.

Heaviside Expansion Theorem (or Formula) Method:

Consider $\frac{P(s)}{Q(s)}$, where $P(s)$ and $Q(s)$ are polynomial functions and degree of $P(s)$ is less than the degree of $Q(s)$. The fraction can be written as

$$\frac{P(s)}{Q(s)} = \frac{A_1}{(s-\alpha_1)} + \frac{A_2}{(s-\alpha_2)} + \cdots + \frac{A_k}{(s-\alpha_k)} + \cdots + \frac{A_n}{(s-\alpha_n)}$$

Where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the n zeros of $Q(s)$

$$\therefore A_k = \lim_{s \to \alpha_k} \frac{P(s)}{Q(s)} (s - \alpha_k)$$

$$= \lim_{s \to \alpha_k} P(s) \cdot \lim_{s \to \alpha_k} \frac{s - \alpha_k}{Q(s)}$$

$$= P(\alpha_k) \cdot \lim_{s \to \alpha_k} \frac{1}{Q'(s)} \quad \text{[according to L'Hospital's rule]}$$

$$= \frac{P(\alpha_k)}{Q'(\alpha_k)}$$

$$\therefore \mathcal{L}^{-1}\left[\frac{P(s)}{Q(s)}\right] = \frac{P(\alpha_1)}{Q'(\alpha_1)} e^{\alpha_1 t} + \frac{P(\alpha_2)}{Q'(\alpha_2)} e^{\alpha_2 t} + \frac{P(\alpha_3)}{Q'(\alpha_3)} e^{\alpha_3 t} + \cdots$$

Example - 1: $\mathcal{L}^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\}$

Soln: Here $P(s) = 3s+1$

$$Q(s) = (s-1)(s^2+1) = s^3 - s^2 + s - 1$$

$$Q'(s) = 3s^2 - 2s + 1$$

$\therefore \alpha_1 = 1, \quad \alpha_2 = i, \quad \alpha_3 = -i$

$$\mathcal{L}^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\} = \frac{P(1)}{Q'(1)}e^t + \frac{P(i)}{Q'(i)}e^{it} + \frac{P(-i)}{Q'(-i)}e^{-it}$$

$$= \frac{4}{2}e^t + \frac{3i+1}{-2-2i}e^{it} + \frac{-3i+1}{-2+2i}e^{-it}$$

$$= 2e^t + \left(-1 - \tfrac{1}{2}i\right)e^{it} + \left(-1 + \tfrac{1}{2}i\right)e^{-it}$$

$$= 2e^t - \left(e^{it} + e^{-it}\right) - \frac{1}{2i}\left(e^{it} - e^{-it}\right)$$

$$= 2e^t - 2\cos t - \sin t$$

Example 2: Use Heaviside expansion thm.

(a) $\mathcal{L}^{-1}\left\{\frac{s+5}{(s+1)(s^2+1)}\right\}$

Solution: $P(s) = s+5$

$Q(s) = (s+1)(s^2+1) = s^3+s^2+s+1$

$Q'(s) = 3s^2+2s+1$

$\alpha_1 = -1, \quad \alpha_2 = i, \quad \alpha_3 = -i$

$$\mathcal{L}^{-1}\left\{\frac{s+5}{(s+1)(s^2+1)}\right\} = \frac{P(-1)}{Q'(-1)}e^{-t} + \frac{P(i)}{Q'(i)}e^{it} + \frac{P(-i)}{Q'(-i)}e^{-it}$$

$$= \frac{4}{2}e^{-t} + \frac{5+i}{2i-2}e^{it} + \frac{5-i}{-2-2i}e^{-it}$$

$$= 2e^{-t} + \frac{(5+i)(i+1)}{2\cdot(-2)}e^{it} + \frac{(5-i)(1-i)}{-2\cdot 2}e^{-it}$$

$$= 2e^{-t} - \frac{1}{4}(5i-1+5+i)e^{it} - \frac{1}{4}(5-i-5i-1)e^{-it}$$

$$= 2e^{-t} - \frac{6i}{4}e^{it} - e^{it} - e^{-it} + \frac{3}{2}ie^{-it}$$

$$= 2e^{-t} - \frac{3}{2}i(e^{it}-e^{-it}) - (e^{it}+e^{-it})$$

$$= 2e^{-t} + \frac{3}{2i}(e^{it}-e^{-it}) - (e^{it}+e^{-it})$$

$$= 2e^{-t} + 3\sin t - 2\cos t.$$

(b) $\mathcal{L}^{-1}\left\{\frac{s-1}{(s+3)(s^2+2s+2)}\right\}$

Solution:

$$P(s) = s-1$$

$$Q(s) = (s+3)(s^2+2s+2)$$

$$= s^3 + 3s^2 + 2s^2 + 6s + 2s + 6$$

$$= s^3 + 5s^2 + 8s + 6$$

$$Q'(s) = 3s^2 + 10s + 8$$

$$\alpha_1 = -3, \quad \alpha_2 = -1+i, \quad \alpha_3 = -1-i$$

$$\mathcal{L}^{-1}\left\{\frac{s-1}{(s+3)(s^2+2s+2)}\right\} = \frac{P(-3)}{Q'(-3)}e^{-3t} +$$

$$\frac{P(-1+i)}{Q'(-1+i)}e^{(-1+i)t} + \frac{P(-1-i)}{Q'(-1-i)}e^{-(1+i)t}$$

$$= -\frac{4}{5}e^{-3t} + \frac{1}{5}e^{-t}(4\cos t - 3\sin t)$$

Ans.

(c) $\mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$

Solution: $P(s) = 2s^2 - 4$

$Q(s) = (s+1)(s-2)(s-3) = s^3 - 4s^2 + s + 6$

$Q'(s) = 3s^2 - 8s + 1$

$\alpha_1 = -1, \quad \alpha_2 = 2, \quad \alpha_3 = 3$

$$\mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\} = \frac{P(-1)}{Q'(-1)}e^{-t} + \frac{P(2)}{Q'(2)}e^{2t} + \frac{P(3)}{Q'(3)}e^{3t}$$

$$= -\frac{2}{12}e^{-t} + \frac{4}{-3}e^{2t} + \frac{14}{4}e^{3t}$$

$$= -\frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}.$$

Ans.