# Solution of differential equations

(a) Solve : $(D^2 + 3D + 2)y = \exp(-t)$

where $y(0) = 4$, $y'(0) = 1$

$L\{D^2 y\} = s^2 Y - sy(0) - y'(0)$, where $L\{y\} = Y(s)$

$= s^2 Y - 4s - 1$

$L\{3DY\} = 3\{sY - Y(0)\} = 3sY - 12$

$\Rightarrow s^2 Y - 4s - 1 + 3sY - 12 + 2Y = L\{e^{-t}\} = \frac{1}{s+1}$

$\Rightarrow (s^2 + 3s + 2)Y = 4s + 13 + \frac{1}{s+1}$

$$Y = \frac{4s+13}{(s+1)(s+2)} + \frac{1}{(s+1)^2(s+2)}$$

$$= \frac{-4}{s+2} + \frac{8}{s+1} + \frac{1}{(s+1)^2}$$, By partial Fraction

$\therefore y = L^{-1}\{Y\} = -4e^{-2t} + 8e^{-t} + te^{-t}$

(b) Solve : $y'' + a^2 y = F(t)$, where $y(0) = 1$, $y'(0) = -2$

Applying Laplace transform

$s^2 y - sy(0) - y'(0) + a^2 y = \bar{f}(s)$

or $(s^2 + a^2)y = s - 2 + f(s)$

$$y = \frac{s-2}{s^2+a^2} + \frac{f(s)}{s^2+a^2}$$

$$y = \frac{s}{s^2+a^2} - \frac{2}{s^2+a^2} + \frac{f(s)}{s^2+a^2}$$

$Y = L^{-1}\{Y\} = \cos at - \frac{1}{a}2\sin at + F(t) * \frac{1}{a}\sin at$

$Y = \cos at - \frac{2}{a}\sin at + \frac{1}{a}\int_0^t F(u)\sin a(t-u)\,du$

(c) Solve the initial value problem

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 8y = 0, \quad y(0) = 3, \; y'(0) = 6$$

Solution: The given equation is

$$y''(t) - 2y'(t) - 8y = 0, \quad y(0) = 3, \; y'(0) = 6$$

Laplace transform leads to

$$L\{y''(t)\} - 2L\{y'(t)\} - 8L\{y\} = 0$$

that is

$$s^2Y(s) - s\,y(0) - y'(0) - 2\{sY(s) - y(0)\} - 8Y(s) = 0$$

Using the initial conditions, we have

$$(s^2 - 2s - 8)Y(s) - 3s = 0$$

$$Y(s) = \frac{3s}{s^2 - 2s - 8} = 3\left[\frac{s-1+1}{(s-1)^2 - 9}\right]$$

$$Y(s) = 3\left[\frac{s-1}{(s-1)^2 - 9} + \frac{1}{(s-1)^2 - 9}\right]$$

Taking inverse Laplace transform, we get

$$y(t) = 3L^{-1}\left\{\frac{s-1}{(s-1)^2 - 9}\right\} + 3L^{-1}\left\{\frac{1}{(s-1)^2 - 9}\right\}$$

$$y(t) = 3e^t \cosh 3t + e^t \sinh 3t.$$

(d) Solve the initial value problem

$$y''' + y'' = e^t + t + 1, \quad y(0) = y'(0) = y''(0) = 0$$

Solution: Taking Laplace transform of both sides of the given equation, we have

$$L\{y'''(t)\} + L\{y''(t)\} = L\{e^t\} + L\{t\} + L\{1\}$$

$$\Rightarrow s^3Y(s) - s^2y(0) - s\,y'(0) - y''(0) + s^2Y(s) - s\,y(0) - y'(0) = \frac{1}{s-1} + \frac{1}{s^2} + \frac{1}{s}$$

Using the initial conditions, we have

$$(s^3 + s^2)Y(s) = \frac{1}{s-1} + \frac{1}{s^2} + \frac{1}{s}$$

$$\Rightarrow Y(s) = \frac{2s^2 - 1}{s^4(s-1)(s+1)}$$

Using partial fraction decomposition, we have

$$Y(s) = \frac{-1}{s^2} + \frac{1}{s^4} - \frac{1}{2(s+1)} + \frac{1}{2(s-1)}$$

Taking inverse transform yields

$$y(t) = L^{-1}\left\{-\frac{1}{s^2} + \frac{1}{s^4} - \frac{1}{2(s+1)} + \frac{1}{2(s-1)}\right\}$$

$$y(t) = -t + \frac{1}{6}t^3 - \frac{1}{2}e^{-t} + \frac{1}{2}e^{t}$$

Verification: we have

$$y'(t) = -1 + \frac{1}{2}t^2 + \frac{1}{2}e^{-t} + \frac{1}{2}e^{t}$$

$$y''(t) = t - \frac{1}{2}e^{-t} + \frac{1}{2}e^{t}$$

$$y'''(t) = 1 + \frac{1}{2}e^{-t} + \frac{1}{2}e^{t}$$

Adding $y''$ and $y'''$ we get

$$y'' + y''' = t + e^{t} + 1$$

(e) Solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = \sin t$, $y(0) = y'(0) = 0$

Solution: Taking Laplace transform of both sides of the given equation, we have

$$L\{y''(t)\} + 2L\{y'(t)\} - 3L\{y(t)\} = L\{\sin t\}$$

$$\Rightarrow s^2Y(s) - sy(0) - y'(0) + 2\{sY(s) - y(0)\} - 3Y(s) = \frac{1}{s^2+1}$$

Using the given initial conditions, we have

$$s^2Y(s) + 2sY(s) - 3Y(s) = \frac{1}{s^2+1}$$

$$Y(s) = \frac{1}{(s^2+1)(s^2+2s-3)}$$

$$= \frac{s-1}{2(s^2+1)} - \frac{s+1}{2(s^2+2s-3)}$$

$$\Rightarrow Y(s) = \frac{s}{2(s^2+1)} - \frac{1}{2}\frac{1}{s^2+1} - \frac{1}{2}\left[\frac{s+1}{(s+1)^2-4}\right]$$

Taking inverse Laplace transform, we have

$$y(t) = \frac{1}{2}\cos t - \frac{1}{2}\sin t - \frac{1}{2}e^{-t}\sinh 2t$$

(f) Solve $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = t^2e^{3t}$, $y(0)=2$, $y'(0)=6$

Solution: Taking Laplace transform, we get

$$s^2Y(s) - sy(0) - y'(0) - 6(sY(s) - y(0)) + 9Y(s) = \frac{2}{(s-3)^3}$$

using initial conditions, we have

$$s^2Y(s) - 2s - 6 - 6sY(s) + 12 + 9Y(s) = \frac{2}{(s-3)^3}$$

$$\Rightarrow (s^2 - 6s + 9)Y(s) = 2(s-3) + \frac{2}{(s-3)^3}$$

$$\Rightarrow Y(s) = \frac{2}{s-3} + \frac{2}{(s-3)^5}$$

Taking inverse Laplace transform

$$y(t) = 2e^{3t} + \frac{1}{12}t^4e^{-3t}$$

(g) Solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t}\sin t$, $y(0)=0$, $y'(0)=1$

Solution: Taking Laplace transform, we get

$$s^2Y(s) - sy(0) - y'(0) + 2\{sY(s) - y(0)\} + 5Y(s) = \frac{1}{(s+1)^2+1}$$

using initial conditions, we get

$$s^2Y(s) - 1 + 2sY(s) + 5Y(s) = \frac{1}{(s+1)^2+1}$$

$$\Rightarrow (s^2+2s+5)Y(s) = 1 + \frac{1}{(s+1)^2+1}$$

$$\Rightarrow Y(s) = \frac{1}{s^2+2s+5} + \frac{1}{(s^2+2s+5)(s^2+2s+2)}$$

$$= \frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$$

Using partial fractions, we have

$$Y(s) = \frac{2}{3(s^2+2s+5)} + \frac{1}{3(s^2+2s+2)}$$

$$Y(s) = \frac{2}{3\{(s+1)^2+4\}} + \frac{1}{3\{(s+1)^2+1\}}$$

Taking inverse transform, we get

$$y(t) = \frac{1}{3}e^{-t}\sin 2t + \frac{1}{3}e^{-t}\sin t$$

(b) Solve $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = \sin t \quad (t \geq 0)$ subject to the conditions $x(0) = x'(0) = 0$

Solution: Taking Laplace transform on both sides of the given equation

$$s^2X(s) - sx(0) - x'(0) + 6\{sX(s) - x(0)\} + 9X(s) = \frac{1}{s^2+1}$$

Using the initial conditions, we have

$$(s^2+6s+9)X(s) = \frac{1}{s^2+1}$$

$$X(s) = \frac{1}{(s^2+1)(s+3)^2}$$

$$= \frac{A}{s+3} + \frac{B}{(s+3)^2} + \frac{Cs+D}{s^2+1}$$

Comparing coefficients of the powers of s, we get

$$A = \frac{3}{50}, \quad B = \frac{1}{10}, \quad C = -\frac{3}{50}, \quad D = \frac{2}{25}$$

$$\therefore X(s) = \frac{3}{50(s+3)} + \frac{1}{10(s+3)^2} - \frac{3s}{50(s^2+1)} + \frac{2}{25(s^2+1)}$$

Taking inverse Laplace transform

$$x(t) = \frac{3}{50}e^{-3t} + \frac{1}{10}te^{-3t} - \frac{3}{50}\cos t + \frac{2}{25}\sin t$$