Ordinary Differential Eqn's. with variable co-efficients

Example 1: Solve $ty'' + y' + ty = 0$, $y(0) = 1, y'(0) = 0$

Solution Taking Laplace transform, we have

$$L\{ty''\} + L\{y'\} + L\{ty\} = 0$$

$$\Rightarrow -\frac{d}{ds}\{s^2 Y(s) - sy(0) - y'(0)\} + \{sY(s) - y(0)\} - \frac{d}{ds}Y(s) = 0$$

which on using initial conditions yields

$$\Rightarrow -\left[s^2\frac{dY}{ds} + 2sY(s)\right] + 1 + sY(s) - 1 - \frac{dY(s)}{ds} = 0$$

$$\Rightarrow (s^2+1)\frac{dY}{ds} + sY(s) = 0$$

$$\Rightarrow \frac{dY(s)}{Y(s)} + \frac{s\,ds}{s^2+1} = 0$$

Integrating, we have

$$\log Y(s) + \frac{1}{2}\log(s^2+1) = \log A$$

$$\Rightarrow \log\{Y(s)\sqrt{s^2+1}\} = \log A$$

$$\Rightarrow Y(s) = \frac{A}{\sqrt{s^2+1}}$$

Taking inverse Laplace transform, we get

$y(t) = A J_0(t)$, where $J_0(t)$ is Bessel function of order zero.

Putting $t = 0$ and using initial condition $y(0) = 1$, we have $1 = A J_0(0) = A$

Hence the required solution is

$$y(t) = J_0(t).$$

Example-2: Solve $y''+ty'-2y=4,\ y(0)=-1,\ y'(0)=0$

Solution: Taking Laplace transform yields

$$L\{y''(t)\}+L\{ty'(t)\}-2L\{y(t)\}=4L\{1\}$$

$$\Rightarrow s^2Y(s)-sy(0)-y'(0)-\frac{d}{ds}\{sY(s)-y(0)\}-2Y(s)=\frac{4}{s}$$

$$\Rightarrow s^2Y(s)+s-\{s\frac{dY}{ds}+Y(s)\}-2Y(s)=\frac{4}{s}$$

$$\Rightarrow s\frac{dY(s)}{ds}-(s^2-3)Y(s)=s-\frac{4}{s}$$

$$\Rightarrow \frac{dY(s)}{ds}+\left(\frac{3}{s}-s\right)Y(s)=1-\frac{4}{s^2}$$

The integrating factor is $e^{\int(\frac{3}{s}-s)ds}=s^3e^{-\frac{s^2}{2}}$

Therefore,

$$\frac{d}{ds}\left[Y(s)\cdot s^3e^{-s^2/2}\right]=s^3e^{-\frac{s^2}{2}}-4se^{-s^2/2}$$

and so integration yields

$$Y(s)\,s^3e^{-s^2/2}=\int s^3e^{-\frac{s^2}{2}}ds-4\int se^{-s^2/2}ds$$

Putting $u=-s^2/2$, we get

$$Y(s)s^3e^{-s^2/2}=2\int ue^u du+4\int e^u du$$

$$=2\left(-\frac{s^2}{2}e^{-s^2/2}-e^{-s^2/2}\right)+4e^{-s^2/2}+A$$

$$=2e^{-s^2/2}-s^2e^{-s^2/2}+c$$

$$\Rightarrow Y(s)=\frac{2}{s^3}-\frac{1}{s}+\frac{c}{s^3}e^{s^2/2}$$

Since $Y(s)\to 0$ as $s\to\infty$, we must have $c=0$ and so

$$Y(s)=\frac{2}{s^3}-\frac{1}{s}$$

Taking inverse Laplace transform, we get

$$y(t)=t^2-1\ .$$

Example3: Solve $ty''+2y'+ty=0$, $y(0)=1$, $y(\pi)=0$

Solution: Let $y'(0)=A$ (constant). Taking Laplace transform of both sides, we obtain

$$-\frac{d}{ds}\{s^2Y(s)-sy(0)-y'(0)\}+2\{sY(s)-y(0)\}-\frac{d}{ds}Y(s)=0$$

$$\Rightarrow -s^2Y'(s)-2sY(s)+y(0)+2sY(s)-2y(0)-Y'(s)=0$$

Using boundary conditions we get

$$-(s^2+1)Y'(s)-1=0$$

$$\Rightarrow Y'(s)=-\frac{1}{s^2+1}$$

Integrating

$$Y(s)=-\tan^{-1}s+B$$

Since $Y(s)\to 0$ as $s\to\infty$, we must have $B=\frac{\pi}{2}$

$$Y(s)=\frac{\pi}{2}-\tan^{-1}s=\tan^{-1}\left(\frac{1}{s}\right)$$

Taking inverse Laplace transform, we have

$$y(t)=L^{-1}\left\{\tan^{-1}\left(\frac{1}{s}\right)\right\}=\frac{\sin t}{t}$$

This solution clearly satisfies $y(\pi)=0$

Example 4 : Solve $ty'' - y' = -1$, $y(0) = 0$

Solution: Taking Laplace transform of both sides of the given equation,

$$-\frac{d}{ds}\{s^2 Y'(s) - sy(0) - y'(0)\} - \{sY(s) - y(0)\} = -\frac{1}{s}$$

$$\Rightarrow -s^2 Y'(s) - 2sY(s) + y(0) - sY(s) + y(0) = -\frac{1}{s}$$

$$\Rightarrow -s^2 Y'(s) - 3sY(s) = -\frac{1}{s}$$

$$\Rightarrow Y'(s) + \frac{3}{s} Y(s) = \frac{1}{s^3}$$

The integrating factor is

$$e^{\int \frac{3}{s} ds} = e^{3 \log s} = s^3$$

Therefore, $\frac{d}{ds}(Y(s)s^3) = \frac{1}{s^3} s^3 = 1$

Integrating, $Y(s)s^3 = s + A$ (constant)

and so, $Y(s) = \frac{1}{s^2} + \frac{A}{s^3}$

Taking inverse Laplace transform, we get

$$y(t) = t + Bt^2$$

where B is constant. Obviously, the solution satisfies $y(0) = 0$.

Example 5 : Solve $ty'' + (t+1)y' + 2y = e^{-t}$, $y(0) = 0$.

Solution: Taking Laplace transform of both sides gives

$$-\frac{d}{ds}\{s^2Y(s) - s\,y(0) - y'(0)\} - \frac{d}{ds}\{s\,Y(s) - y(0)\} + \{s\,Y(s) - y(0)\} + 2\,Y(s) = \frac{1}{s+1}$$

$$\Rightarrow -s^2Y'(s) - 2sY(s) - \{s\,Y'(s) + Y(s)\} + s\,Y(s) - y(0) + 2Y(s) = \frac{1}{s+1}$$

$$\Rightarrow -s^2Y'(s) - sY'(s) - 2sY(s) + sY(s) + Y(s) = \frac{1}{s+1}$$

$$\Rightarrow -(s^2+s)Y'(s) - (s-1)Y(s) = \frac{1}{s+1}$$

$$\Rightarrow Y'(s) + \frac{s-1}{s^2+s} = \frac{-1}{s(s+1)^2}$$

The integration factor is

$$e^{\int \frac{s-1}{s^2+s}\,ds} = e^{\int(-\frac{1}{s} + \frac{2}{s+1})\,ds} = \frac{(s+1)^2}{s}$$

Therefore, $\frac{d}{ds}\left(Y(s)\,\frac{(s+1)^2}{s}\right) = \frac{-1}{s(s+1)^2}\cdot\frac{(s+1)^2}{s} = -\frac{1}{s^2}$

Integrating, we get, $Y(s)\frac{(s+1)^2}{s} = \int -\frac{1}{s^2}\,ds = \frac{1}{s} + C$

and so, $Y(s) = \frac{1}{(s+1)^2} + \frac{Cs}{(s+1)^2}$

By initial value theorem $y(0) = \lim_{s\to 0} s\,Y(s) = 0$ and so $C = 0$. Hence,

$$Y(s) = \frac{1}{(s+1)^2}$$

Taking inverse Laplace transform, we get

$$y(t) = te^{-t}.$$