SIMULTANEOUS DIFFERENTIAL EQUATIONS:

Example1 : Solve the simultaneous differential equations, $3x' + y' + 2x = 1$, $x' + 4y' + 3y = 0$ subject to the conditions $x(0) = 0$, $y(0) = 0$.

Solution: Taking Laplace transform, we get

$$3\{sX(x) - x(0)\} + \{sY(s) - y(0)\} + 2X(x) = \frac{1}{s}$$

and $$sX(s) - x(0) + 4\{sY(s) - y(0)\} + 3Y(s) = 0$$

Using the initial conditions, these equations reduce to,

$$(3s+2)X(s) + sY(s) = \frac{1}{s} \quad \ldots \quad \ldots(1)$$

and $$sX(s) + (4s+3)Y(s) = 0 \quad \ldots \quad \ldots(2)$$

Multiplying (1) and (2) by s and $(3s+2)$ respectively and then subtracting, we get

$$Y(s) = -\frac{1}{11s^2 + 17s + 6} = \frac{-1}{(11s+6)(s+1)}$$

and then using (4), we have

$$X(s) = \frac{4s+3}{s(11s+6)(s+1)}$$

We deal with $X(s)$ first. Using partial fraction, we have

$$X(s) = \frac{1}{2s} - \frac{3}{10}\left(\frac{1}{s + (6/11)}\right) - \frac{1}{5(s+1)}$$

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Taking inverse transform, we have

$$x(t) = \frac{1}{2} - \frac{3}{10} e^{-\frac{6}{11}t} - \frac{1}{5} e^{-t}$$

$$= \frac{1}{10}(5 - 3e^{-6/11\,t} - 2e^{-t})$$

Further, poles of $Y(s)$ are $-\frac{6}{11}$ and -1. Hence

$$y(t) = \lim_{s \to -\frac{6}{11}} \left(s + \frac{6}{11}\right) Y(s) e^{-6/11\,t}$$

$$+ \lim_{s \to -1} (s+1) Y(s) e^{-t} = \frac{1}{5}(e^{-t} - e^{-\frac{6}{11}t}).$$

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Example 2 : Solve the simultaneous differential equations $\frac{dx}{dt} = 2x - 3y$, $\frac{dy}{dt} = y - 2x$ subject to the conditions $x(0) = 8$, $y(0) = 3$.

Solution: Taking Laplace transform and using the given conditions, we have

$$sX(s) = 2X(s) - 3Y(s) + 8$$

and $$sY(s) = Y(s) - 2X(s) + 3$$

Thus $$(s-2)X(s) + 3Y(s) = 8$$

and $$2X(s) + (s-1)Y(s) = 3$$

Solving these algebraic equations, we get

$$X(s) = \frac{8s - 17}{s^2 - 3s - 4} = \frac{8s - 17}{(s+1)(s-4)}$$

and $$Y(s) = \frac{3s - 22}{s^2 - 3s - 4} = \frac{3s - 22}{(s+1)(s-4)}$$

Using partial fractions, these yields,

$$X(s) = \frac{5}{s+1} + \frac{3}{s-4}, \quad Y(s) = \frac{5}{s+1} - \frac{2}{s-4}$$

Hence taking inverse Laplace transform, we get

$$x(t) = 5e^{-t} + 3e^{4t}, \quad y(t) = 5e^{-t} - 2e^{4t}.$$

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Example 3 : Solve $\frac{dx}{dt} - y = e^t$, $\frac{dy}{dt} + x = \sin t$ subject to the conditions $x(0) = 1$, $y(0) = 0$.

Solution: Taking Laplace transform and using the given conditions, we have

$$s\,X(s) - Y(s) = \frac{1}{s-1} + 1 = \frac{s}{s-1}$$

$$s\,Y(s) + X(s) = \frac{1}{s^2+1}$$

Solving these equations, we get

$$X(s) = \frac{s^4 + s^2 + s - 1}{(s-1)(s^2+1)^2}$$

$$Y(s) = \frac{-s^4 + s^3 - 2s^2}{s(s-1)(s^2+1)^2} = \frac{-s^3 + s^2 - 2s}{(s-1)(s^2+1)^2}$$

Now $$X(s) = \frac{s^4 + s^2 + s - 1}{(s-1)(s^2+1)^2}$$

$$= \frac{A}{s-1} + \frac{Bs + C}{s^2+1} + \frac{Ds + E}{(s^2+1)^2}$$

Comparison of coefficients yields

$$A = B = C = \frac{1}{2} \quad \text{and} \quad E = 1$$

Thus $$X(s) = \frac{1}{2(s-1)} + \frac{s}{2(s^2+1)} + \frac{1}{2(s^2+1)} + \frac{1}{(s^2+1)^2}$$

Hence $$x(t) = \frac{1}{2}e^t + \frac{1}{2}\cos t + \frac{1}{2}\sin t + \frac{1}{2}(\sin t - t\cos t)$$

$$= \frac{1}{2}\left[e^t + \cos t + 2\sin t - t\cos t\right]$$

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Now consider $Y(s)$. We have

$$Y(s) = \frac{-s^3 + s^2 - 2s}{(s-1)(s^2+1)^2}$$

$$= \frac{A}{s-1} + \frac{Bs+C}{(s^2+1)} + \frac{Ds+E}{(s^2+1)^2}$$

Comparing coefficients, we get

$$A = -\frac{1}{2}, \quad B = \frac{1}{2}, \quad C = -\frac{1}{2}, \quad D = 2, \quad E = 0$$

and so $Y(s) = \frac{-1}{2(s-1)} + \frac{s}{2(s^2+1)} - \frac{1}{2(s^2+1)} + \frac{2s}{(s^2+1)^2}$

Hence, $y(t) = -\frac{1}{2}e^t + \frac{1}{2}\cos t - \frac{1}{2}\sin t + t\sin t$

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Example 4 : The co-ordinates (x, y) of a particle moving along a plane curve at any time t are given by

$$\frac{dy}{dt} + 2x = \sin 2t, \quad \frac{dx}{dt} - 2y = \cos 2t, \quad t > 0$$

If at $t = 0$, $x = 1$ and $y = 0$, show by using transforms, that the particle moves along the curve $4x^2 + 4xy + 5y^2 = 4$.

Solution : Using Laplace transform, we get,

$$s\,Y(s) - y(0) + 2X(s) = \frac{2}{s^2+4}$$

and $$s\,X(s) - x(0) - 2\,Y(s) = \frac{s}{s^2+4}$$

Using the given conditions, we have

$$s\,Y(s) + 2X(s) = \frac{2}{s^2+4}$$

and $$s\,X(s) - 2Y(s) = 1 + \frac{s}{s^2+4} = \frac{s^2+s+4}{s^2+4}$$

Solving for $X(s)$ and $Y(s)$ and using partial fractions, we have

$$X(s) = \frac{s^3+s^2+4s+4}{(s^2+4)^2} = \frac{s}{s^2+4} + \frac{1}{s^2+4}$$

$$= \frac{s}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4}$$

$$Y(s) = \frac{-2s^2-8}{(s^2+4)^2} = -\frac{2}{s^2+4}$$

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Hence taking inverse transform, we get,

$$x(t) = \cos 2t + \frac{1}{2}\sin 2t$$

$$y(t) = -\sin 2t$$

We observe that, $4x^2 + 4xy + 5y^2 = 4(\cos^2 2t + \sin^2 2t)$

$$= 4$$

and hence the particle moves along the curve $4x^2 + 4xy + 5y^2 = 4$.

Example5 : Solve the following system of equations:

$$x(t) - y''(t) + y(t) = e^{-t} - 1$$

$$x'(t) + y'(t) - y(t) = -3e^{-t} + t$$

Subject to $x(0) = 0, \quad y(0) = 1, \quad y'(0) = -2$

Solution: Taking Laplace transform yields

$$X(s) - \{s^2 Y(s) - sy(0) - y'(0)\} + Y(s) = \frac{1}{s+1} - \frac{1}{s}$$

and $$sX(s) - x(0) + sY(s) - y(0) - Y(s) = \frac{-3}{s+1} + \frac{1}{s^2}$$

Using the given conditions, we have

$$X(s) - s^2 Y(s) + s - 2 + Y(s) = \frac{-1}{s(s+1)}$$

and, $$sX(s) + sY(s) - 1 - Y(s) = \frac{-3s^2 + s + 1}{(s+1)s^2}$$

or, $$X(s) - (s^2 - 1) Y(s) = 2 - s - \frac{1}{s(s+1)}$$

$$= \frac{-s^3 + s^2 + 2s - 1}{s(s+1)}$$

and $$sX(s) + (s-1) Y(s) = 1 - \frac{3s^2 - s - 1}{(s+1)s^2}$$

$$= \frac{s^3 - 2s^2 + s + 1}{(s+1)s^2}$$

Solving for $X(s)$ and $Y(s)$, we have

$$X(s) = \frac{1}{s^2(s+1)} = \frac{1}{s^2} + \frac{1}{s+1} - \frac{1}{s}$$

$$Y(s) = \frac{s^2 - s - 1}{s^2(s+1)} = \frac{1}{s+1} - \frac{1}{s^2}$$

Hence, taking inverse Laplace transform, we get

$$x(t) = t + e^{-t} - 1, \quad y(t) = e^{-t} - t.$$

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