DIMENSIONAL AND MODEL ANALYSIS

Model Analysis

7.6 MODEL TESTING

Since the beginning of the twentieth century, the engineers have started a new and scientific method to predict the performance of their structures and machines before they are made and installed. This is done by preparing models and testing them in the laboratory, so as to form some opinions about the working and behavior of the proposed structures, after their completion or actual installation. The structure, of which the model is prepared, is known as *prototype* and the model is known as the *physical model* or *scale model* or simply *model*.

The principal advantages of model testing are:

1. The behavior and working details of a hydraulic structure or machine can be easily predicted by performing experiments on its model.

2. If the hydraulic structure or machine is made directly, then in case of failure, it is very difficult to change its design. Moreover, it is very costly to build the same. Thus, making a model and testing it in the laboratory results in the saving of human labor and material.

3. With the help of model testing, a number of alternative designs can be studied. Finally the most economical, accurate and safe design may be selected.

4. In case when an existing structure is not functioning properly, its defects can be detected and rectified by model testing.

5. Sometimes, it is very difficult to design a particular portion of a complex structure or machine. In such cases the model testing becomes absolutely necessary to ascertain the safety and reliability of that particular portion of the prototype.

Following are the fields where the application of the model testing is of great use:

(a) Water power engineering
  (i) Turbines and pumps
  (ii) Hydraulic structures like dams, weirs, spillways, canal falls, etc.

(b) Irrigation engineering
  (i) Flood control
  (ii) River training
  (iii) Investigation of silting and scouring in rivers, irrigation canals, etc.

(c) Design of ships, harbors, etc.

(d) Design of aeroplanes for compressible flow.

- Model: It is a small scale replica of the actual structure
- Prototype: It is the actual structure

Hydraulic Similarity or Similitude:

In order to have a complete idea about the construction and working of a hydraulic structure or machine, it is very essential that the model should represent its prototype completely and fully in all respects. The similarity between the prototype and its model is known as hydraulic similarity or hydraulic similitude.

There are three kinds of similarities which a model should possess in order that it may reproduce the behaviour of its prototype:

- Geometric similarity
- Kinematic similarity
- Dynamic similarity

Geometric Similarity

The geometric similarity is said to exist between the model and the prototype, if both of them are identical in shape; but differ only in size. Or in other words, the geometric similarity is said to exist between the model and the prototype, if the ratios of all the corresponding linear dimensions are equal.

Let L = Length of the prototype,

B = width of the prototype,

D = Depth of the prototype, and

and l, b and d = Corresponding values for the model.

Now, if the geometric similarity exists, between the prototype and the model, then the linear ratio of the prototype and the model (also called scale ratio).

$$L_r = \frac{L}{l} = \frac{B}{b} = \frac{D}{d}$$

→ Prototype

→ Model

Similarly, area ratio of the prototype and the model,

$$A_r = \left(\frac{L}{l}\right)^2 = \left(\frac{B}{b}\right)^2 = \left(\frac{D}{d}\right)^2 = (L_r)^2$$

→ prototype

→ Model

and volume ratio of the prototype and the model,

$$V_r = \left(\frac{L}{l}\right)^3 = \left(\frac{B}{b}\right)^3 = \left(\frac{D}{d}\right)^3 = (L_r)^3$$

→ prototype

→ Model

L_r = scale ratio

Kinematic Similarity

The kinematic similarity is said to exist, between the model and the prototype, if both of them have identical motions or velocities. Or in other words, the kinematic similarity is said to exist between the model and the prototype, if the ratio of the corresponding velocities at corresponding points are equal.

Let V_1 = Velocity of liquid in prototype at point 1,

V_2 = Velocity of liquid in prototype at point 2,

v_1, v_2 = Corresponding values for the model.

Now, if the kinematic similarity exists, between the prototype and the model, then the velocity ratio of the prototype to the model,

$$v_r = \frac{V_1}{v_1} = \frac{V_2}{v_2} = \frac{V_3}{v_3} = \ldots$$

prototype

model

Dynamic Similarity

The dynamic similarity is said to exist, between the model and the prototype, if both of them have identical forces. Or in other words, the dynamic similarity is said to exist between the model and the prototype, if the ratios of the corresponding forces acting at corresponding points are equal.

Let F_1 = Force acting in the prototype at point 1, and

F_2 = Force acting in the prototype at point 2,

f_1, f_2 = Corresponding values for the model.

Now, if the dynamic similarity exists between the prototype and the model, then the force ratio of the prototype and the model,

$$F_r = \frac{F_1}{f_1} = \frac{F_2}{f_2} = \frac{F_3}{f_3} = \ldots$$

prototype

model

7.9 TYPES OF MODELS

All the models may be broadly classified into the following two types:

1. Undistorted model
2. Distorted model

Undistorted Model: A model is said to be undistorted when the horizontal and vertical scale ratios (model distance: prototype distance) are same. An undistorted model is geometrically similar to its prototype. The prediction of an undistorted model is comparatively easy and the model results can be easily transferred to the prototype as the basic condition of geometric similarity is satisfied.

Distorted Model: A model is said to be distorted when the vertical and horizontal scale ratios are different. The distorted model does not have true or complete geometric similarity. For example, models of rivers, harbors, reservoirs, etc. have very large horizontal dimensions, as compared to vertical ones. If a model of such a prototype having a complete geometrical similarity is made, then the depth of water in such a model becomes so small that it cannot be accurately measured. To overcome this difficulty, the vertical scale of the model is increased relative to the horizontal scale.

The prediction of a distorted model is relatively difficult, and the results of the models being distorted cannot be easily transferred to the prototype, as the basic condition of geometric similarity is not satisfied.

Velocity of Water in the Prototype for the given velocity in Undistorted Model

Consider an undistorted model geometrically similar to a proposed prototype like a weir, dam, spillway etc.

Let
h = Head of water over the model,
v = Velocity of water at a point in the model,
H, V = Corresponding values for the prototype.
$\frac{1}{s}$ = *Scale ratio of the model to the prototype.

We know that the velocity of water in the model,

$$v = c_v \sqrt{2gh} \quad \ldots(i)$$

Similarly, velocity of water on the corresponding point in the prototype,

$$V = C_v \sqrt{2gH} \quad \ldots(ii)$$

Dividing the equation (i) by (ii),

$$\frac{v}{V} = \frac{c_v \sqrt{2gh}}{C_v \sqrt{2gH}} = \frac{\sqrt{h}}{\sqrt{H}} \quad \ldots(\text{Taking } c_v = C_v)$$

$$= \sqrt{\frac{h}{H}} = \sqrt{\frac{1}{s}} \quad \ldots\left(\because \frac{h}{H} = \frac{1}{s}\right)$$

$$\therefore \quad V = v\sqrt{s}$$

Discharge of Prototype for the Given Discharge of an Undistorted Model

Consider an undistorted model geometrically similar to a prototype, like a weir, notch or spillway etc.

Let a = Area of discharge of the model,

v = Actual velocity of water in the model,

q = Discharge of the model,

A, V, Q = Corresponding values for the prototype.

$\frac{1}{s}$ = Scale ratio of model to prototype.

We know that the discharge of the model,

$$q = \text{Area} \times \text{Velocity} = a \times v \quad \ldots(i)$$

Similarly, discharge through the corresponding area of the prototype,

$$Q = A \times V \quad \ldots(ii)$$

Dividing the equation (i) by (ii),

$$\frac{q}{Q} = \frac{a \times v}{A \times V} = \frac{a}{A} \times \frac{v}{V}$$

$$= \frac{1}{s^2} \times \frac{1}{\sqrt{s}} = \frac{1}{s^{2.5}}$$

$$\therefore \quad Q = q \times s^{2.5}$$

Force of Resistance of the Prototype for the Given Force of Resistance of an Undistorted Model

Consider an undistorted model geometrically similar to a prototype, like a ship, aeroplane etc.

Let
a = Area of model subjected to resistance,
v = Velocity of the model,
A, V = Corresponding values for the prototype, and
$\frac{1}{s}$ = Scale ratio of model to prototype.

We know that the force on the model

$$f = \text{Mass} \times \text{Acceleration}$$

$$= (\text{Volume} \times \text{Mass density}) \times \frac{\text{Velocity}}{\text{Time}}$$

$$= \frac{\text{Volume}}{\text{Time}} \times \text{Mass density} \times \text{Velocity}$$

$$= (\text{Area} \times \text{Velocity}) \times \text{Mass density} \times \text{Velocity}$$

$$= \text{Area} \times \text{Mass density} \times \text{Velocity}^2$$

$$= a \times \rho \times v^2 \quad \ldots(i)$$

Similarly, force on the prototype,

$$F = A \times \rho \times V^2 \quad \ldots(ii)$$

Dividing equation (i) by (ii),

$$\frac{f}{F} = \frac{a}{A} \times \frac{\rho}{\rho} \times \frac{v^2}{V^2} = \frac{1}{s^2} \times \frac{v^2}{(v\sqrt{s})^2} = \frac{1}{s^3}$$

or $$F = f \times s^3$$

Note : If the model is tested in a different fluid (*i.e.* other than the fluid, in which the prototype is to move, the corresponding value for the force will become:

$$F = f \times \frac{\rho_p}{\rho_m} \times s^3$$

where ρ_p and ρ_m are the mass densities of the fluids, in which the prototype and the model is to move.

Speed of the Prototype for the Given Speed of an Undistorted Model

Consider a undistorted model geometrically similar to a prototype, like a centrifugal pump or turbine

Let d = Diameter of the model impeller,
v = Tangential velocity of the model impeller,
n = Speed of the model runner,
D, V, N = Corresponding values for the prototype runner, and
$\frac{1}{s}$ = Scale ratio of model to prototype,

We know that tangential velocity of the model impeller,

$$v = \frac{\pi d n}{60} \quad \text{or} \quad n = \frac{60 v}{\pi d} \qquad \ldots(i)$$

Similarly, speed of the prototype runner,

$$N = \frac{60\ V}{\pi D} \qquad \ldots(ii)$$

Divising the equation (i) by (ii),

$$\frac{n}{N} = \frac{\frac{60\ v}{\pi d}}{\frac{60\ V}{\pi D}} = \frac{v}{V} \times \frac{D}{d}$$

$$= \frac{1}{\sqrt{s}} \times s = \sqrt{s} \qquad \ldots\left(\because \frac{D}{d} = s\right)$$

or $$N = \frac{n}{\sqrt{s}}$$

Time of Emptying a Prototype for the Given Time of Emptying of an Undistorted Model

Consider an undistorted model, geometrically similar to a prototype, like a tank or a reservoir.

Let

v = Total volume of water in the model,

q = Rate of discharge of water in the model,

V, Q = Corresponding values for the prototype, and

$\frac{1}{s}$ = Scale ratio of the model to the prototype.

We know that time of emptying the model,

$$t = \frac{\text{Total volume}}{\text{Rate of discharge}} = \frac{v}{q} \qquad \ldots(i)$$

and time of emptying the prototype,

$$T = \frac{V}{Q} \qquad \ldots(ii)$$

Dividing equation (i) by (ii),

$$\frac{t}{T} = \frac{\frac{v}{q}}{\frac{V}{Q}} = \frac{v}{V} \times \frac{Q}{q} = \frac{1}{s^3} \times s^{2.5} = \frac{1}{\sqrt{s}}$$

$$\therefore \quad T = t \times \sqrt{s}$$

Velocity of Water in the Prototype for the Given Velocity of Water of a Distorted Model

Consider a distorted model of prototype like a weir, dam or spillway etc.

Let h = Head of water over the model,

v = Velocity of water at some point in the model,

H, V = Corresponding values for the prototype.

$\frac{1}{s_H}$ = Horizontal scale ratio of the model to the prototype,

$\frac{1}{s_V}$ = Vertical scale ratio of the model and the prototype:

We know that the velocity of water in the model,

$$v = c_v \sqrt{2gh} \qquad \ldots(i)$$

Similarly, the velocity of water in the prototype,

$$V = C_v \sqrt{2gH} \qquad \ldots(ii)$$

Dividing equation (i) by (ii),

$$\frac{v}{V} = \frac{c_v \sqrt{2gh}}{C_v \sqrt{2gH}} = \frac{\sqrt{h}}{\sqrt{H}} \qquad \ldots(\text{Taking } c_v = C_v)$$

$$= \sqrt{\frac{h}{H}} = \sqrt{\frac{1}{s_V}} = \frac{1}{\sqrt{s_V}}$$

$$\therefore \quad V = v\sqrt{s_V}$$

Discharge of Prototype for the Given Discharge of a Distorted Model

Consider a distorted model of a prototype like a weir,notch, spillway etc

Let l = Length of the model,

h = Height of the model, through which the discharge is taking place,

v = Actual velocity of water in the model,

L, H, V = Corresponding values for the prototype,

$\frac{l}{s_H}$ = Horizontal scale ratio of the model to the prototype, and

$\frac{l}{s_V}$ = Vertical scale ratio of the model to the prototype.

We know that the discharge of the model,

$$q = \text{Area} \times \text{Velocity} = (\text{Length} \times \text{Height}) \times \text{Velocity}$$
$$= l \times h \times v \quad \ldots(i)$$

Similarly, discharge of the prototype,

$$Q = L \times H \times V \quad \ldots(ii)$$

Dividing equation (i) by (ii),

$$\frac{q}{Q} = \frac{l \times h \times v}{L \times H \times V} = \frac{l}{L} \times \frac{h}{H} \times \frac{v}{V}$$
$$= \frac{1}{s_H} \times \frac{1}{s_V} \times \frac{1}{\sqrt{s_V}}$$
$$Q = q \times s_H \times s_V^{1.5}$$

Sample Problems on Undistorted And Distorted Model

Math: The discharge over a model, which is reduced to 1:100 in all its dimensions is 1.5 litre/s. What is the corresponding discharge in the prototype?

$$Q = q \times (s)^{2.5} = 1.5 \times (100)^{2.5} = 150000 \text{ litre/sec} = 150 \text{ m}^3 / \text{s}$$

Math: The resistance on a ship is to be found out from the model tests on 1:60 model in wind tunnel. If the drag on the model was found t 0.005 N, what will be the drag on the prototype? Take the density of the air as 1.2 kg/m^3

$$F = f \times (\rho_p / \rho_m) \times s^3 = 0.005 \times (1030/1.2) \times (60)^3$$

$$= 927000 \text{ N}$$

The mass density of sea water is 1030 kg/m^3

Math : A model of spillway is constructed to a scale of 1:30 in a flume. The length of the spillway is 30m. If the discharge over the spillway at the head of 6m (depth of flow over spillway) is 443·6 cumecs, calculate the corresponding head and discharge of the model required for this model study.

$\Rightarrow$ Given, $\frac{1}{S} = \frac{1}{30}$ or $S = 30$; $L_p = 30\,m$; $H = 6\,m$, $Q = 443{\cdot}6\ m^3/s$

Now, $S = \frac{H}{h} \Rightarrow h = \frac{H}{S} = \frac{6}{30} = 0{\cdot}2m$

Again, $Q = q \times (S)^{2{\cdot}5}$

$\Rightarrow 443{\cdot}6 = q \times (S)^{2{\cdot}5}$

$\Rightarrow 443{\cdot}6 = q \times (30)^{2{\cdot}5}$

$\therefore q = 0{\cdot}09\ m^3/s$

[q = Discharge of the model
Q = Discharge over the spillway]

Math: A pump in a sewerage disposal works is required to work at a speed of 250 r.p.m. Find the speed of the model, which has a scale ratio of 1:25.

$\Rightarrow$ $N = 250$ r.p.m.

$\frac{1}{5} = \frac{1}{25}$; $s = 25$

Now, $N = \frac{n}{\sqrt{5}} = \frac{n}{\sqrt{25}}$

$\Rightarrow 250 = \frac{n}{5}$

$\therefore n = 1250$ r.p.m

Math: A model of weir is made to a horizontal scale of $\frac{1}{40}$ and vertical scale $\frac{1}{9}$. Find the discharge of the prototype, if the model is discharging 1 litre/s.

$\Rightarrow$ Given, $\frac{1}{S_H} = \frac{1}{40}$; $S_H = 40$; $\frac{1}{S_V} = \frac{1}{9}$ $\therefore S_V = 9$

We know, $Q = q \times S_H \times S_V^{1.5}$

$= (1 \times 40 \times 9^{1.5})$

$= 1080$ litre/sec

EWCE 261

FLUID MECHANICS

3 Credit Hour

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DIMENSIONAL AND MODEL ANALYSIS

Introduction to Dimensional Analysis

In the recent past, there has been a rapid development in the field of Fluid Mechanics. As a result of this development, several new problems are met with by the engineers. These problems generally pertain to the design, construction as well as efficient working of various types of hydraulic structures and machines. Some of these problems may be easily solved by the mathematical analysis, whereas others may be difficult to solve due to their complex nature. The solution of such problems is considerably simplified by the use of dimensional analysis.

Strictly speaking, the dimensional analysis is a mathematical technique, which deals with the dimensions of the physical quantities involved in the phenomenon. It is based on the assumption that the phenomenon can be expressed by a dimensionally homogeneous equation, with certain variables. These days, the dimensional analysis is widely used in research work for developing design criteria and also for conducting model tests.

Fundamental Dimensions:

All physical quantities are measured by comparison. This comparison is always made with respect to some arbitrarily fixed value for each independent quantity, called dimension (*e.g.*, length, mass, time, temperature etc.). Since there is no direct relationship between these dimensions, therefore they are called fundamental dimensions or fundamental quantities. Some other quantities such as area, volume, velocity, force etc., cannot be expressed in terms of fundamental dimensions and thus may be called derived dimensions, derived quantities or secondary quantities.

There are two systems for fundamental dimensions namely *MLT* (*i.e.*, mass, length and time) and *FLT* (*i.e.*, force, length and time). The dimensional form of any quantity is independent of any system of units. Some engineers prefer to use force instead of mass as fundamental quantity, as the former is easy to measure. But in this book, we shall use the *MLT* system.

The following table gives the dimensions and units for the various physical quantities, which are important from the subject point of view :

Table 7.1 Dimensions and units of some physical quantities

Sl. No.	Quantity & symbol	SI unit	Dimension in MLT system	Dimension in FLT system
1.	Length (l)	m	L	L
2.	Area (A)	m^2	L^2	L^2
3.	Volume (V)	m^3	L^3	L^3
4.	Time (t)	s	T	T
5.	Velocity (V)	m/s	LT^{-1}	LT^{-1}
6.	Acceleration (a)	m/s^2	LT^{-2}	LT^{-2}
7.	Gravitational acceleration(g)	m/s^2	LT^{-2}	LT^{-2}
8.	Frequency (N)	/s	T^{-1}	T^{-1}
9.	Discharge (Q)	m^3/s	L^3T^{-1}	L^3T^{-1}
10.	Force (F)/Weight (W)	N	MLT^{-2}	F
11.	Power (P)	W	ML^2T^{-3}	FLT^{-1}
12.	Work/Energy (E)	N-m or J	ML^2T^{-2}	FL
13.	Pressure (p)	N/m^2	$ML^{-1}T^{-2}$	FL^{-2}
14.	Mass (M)	kg	M	FT^2L^{-1}
15.	Mass density (ρ)	kg/m^3	ML^{-3}	FT^2L^{-4}
16.	Specific weight (γ)	N/m^3	$ML^{-2}T^{-2}$	FL^{-3}
17.	Dynamic viscosity (μ)	$N\text{-}s/m^2$	$ML^{-1}T^{-1}$	FTL^{-2}
18.	Kinematic viscosity (ν)	m^2/s	L^2T^{-1}	L^2T^{-1}
19.	Surface tension (σ)	N/m	MT^{-2}	FL^{-1}
20.	Shear stress (τ)	N/m^2	$ML^{-1}T^{-2}$	FL^{-2}
21.	Modulus of elasticity (K)	N/m^2	$ML^{-1}T^{-2}$	FL^{-2}
22.	Angular velocity (ω)	rad/s	T^{-1}	T^{-1}
23.	Torque (T)	N-m	ML^2T^{-2}	FL
24.	Bulk Modulus	N/m^2	$ML^{-1}T^{-2}$	FL^{-2}

Dimensional Analysis

The dimensional analysis is a mathematical technique which deals with the dimensions of the physical quantities involved in the phenomenon.

Application:

1. To know if any equation is dimensionally homogeneous or not.(Example in next slide)
2. To find out dimension. Suppose, Area = Length x Width = L x L = L^2
3. In unit conversion.
4. To solve complex engineering problem.
5. In model and prototype related topics.

Dimensional Homogeneity

An equation is called dimensionally homogeneous if the dimensions have identical powers on both sides. Such an equation is also known as a rational equation. A dimensionally homogeneous equation would essentially be independent of the system of units (i.e. SI, Metric or English). Let us consider the most common equation of hydraulics

$$Q = AV$$

Dimensionally,

Example 1

$$L^3T^{-1} = L^2 \times LT^{-1}$$
$$= L^3T^{-1}$$

Example 2. .

$$P = \frac{F}{A}; \quad F = ma; \quad a = \frac{v}{t} = \frac{\text{distance}}{t \times t} = \frac{L}{T \times T} = LT^{-2}$$

$$P = \frac{ma}{A} = \frac{MLT^{-2}}{L^2} = MLT^{-2} \times L^{-2} = ML^{-1}T^{-2}$$

$$\gamma = \frac{W}{V} = \frac{mg}{V} = \frac{M \cdot LT^{-2}}{L^3} = ML^{-2}T^{-2}$$

$$\therefore \gamma h = ML^{-2}T^{-2} \times L = ML^{-1}T^{-2}$$

$$\therefore P = \gamma h = ML^{-1}T^{-2}$$

So, both sides of the equation have the same dimensions. Therefore, the equation is dimensionally homogeneous or rational.

7.2 DIMENSIONAL ANALYSIS

Dimensional analysis is a mathematical technique which deals with the dimensions of physical quantities involved in the phenomenon. Each physical phenomenon can be expressed by an equation giving relationship between different quantities. In general, any variable present in a physical phenomenon is a dimensional quantity. Dimensional analysis helps us to determine a convenient arrangement of quantities or variables in a physical relationship. This is accomplished by forming a number of non-dimensional groups out of a given number of dimensional quantities so that variables can be reduced. These days the dimensional analysis is widely used in research work for developing design criteria and also for conducting model test.

The use of dimensional analysis can be summarized as follows:

1. It helps to find whether an equation of any flow phenomenon is rational or not. The equation is called *rational* if it is dimensionally homogeneous. This is to say that dimensional analysis can be applied only when a phenomenon can be expressed by a dimensionally homogeneous equation.
2. The relationship between various physical quantities in an equation, governing a particular flow phenomenon, can be known.
3. It helps the reduction of a number of variables involved in a flow phenomenon with which the performance of experiments becomes easy.
4. Rational formula for the flow phenomenon can be derived.
5. It helps in making suitable smaller-sized models in which experiments can be performed to predict the performance of prototypes.
6. Dimensional analysis, on the whole, facilitates planning of reliable scientific experiments.

Methods of Dimensional Analysis:

Dimensional analysis enables the mathematical formulation of a physical phenomenon when the variables which are involved in the phenomenon are known. The methods of dimensional analysis enable us to combine the variables involved into compact non-dimensional groups and obtain a functional relationship between them. There are two methods of dimensional analysis:

1. Rayleigh's method
2. Buckingham's method, generally known as Buckingham's π-theorem.

Rayleigh's Method

This method was originally proposed by Lord Rayleigh in 1899. In this method, the dependent variable is expressed as a function of the exponents (or powers) of the independent variables. The dependent variable is one for which information is required, whereas the independent variables are those which govern the variation of the dependent variable. If y is the dependent variable and x_1, x_2, x_3, ... are the independent variables involved in a physical process, then the functional relation may be written as

$$y = f(x_1^a, x_2^b, x_3^c, ...)$$

where a, b, c, ...are the exponents or powers of x_1, x_2, x_3, ..., respectively The values of a, b, c, ... are obtained using the principle of dimensional homogeneity.

MATH EXAMPLE ON RAYLEIGH'S METHOD

Math Example of Rayleigh's Method:

Question: If the capillary rise (h) depends upon the specific weight (γ), surface tension of the liquid (σ) and the radius of the tube (r), show that

$$h = r\, f\left(\frac{\sigma}{\gamma r^2}\right)$$

Solution Let the functional relationship be

$$h \propto \gamma^a . \sigma^b . r^c \qquad \text{(i)}$$

Substituting the respective dimensions in (i), we get

$$L \propto (ML^{-2}T^{-2})^a . (MT^{-2})^b . (L)^c \qquad \text{(ii)}$$

Now, for dimensional homogeneity, equating the powers of M, L and T on both sides of (ii), we get

for M,	$0 = a + b$	(iii)
for L,	$1 = -2a + c$	(iv)
for T,	$0 = -2a - 2b$	(v)

From (iii), we get $a = -b$

From (iv), we get $c = 1 + 2a = 1 - 2b$

Now, substituting the values of a and c in (i), we get

$$h \propto \gamma^{-b} . \sigma^b . r^{1-2b}$$

or, $h \propto r\left(\frac{\sigma}{\gamma r^2}\right)^b$ or, $\frac{h}{r} \propto \left(\frac{\sigma}{\gamma r^2}\right)^b$

$\therefore$ The functional relationship may be expressed as

$$\frac{h}{r} = f\left(\frac{\sigma}{\gamma r^2}\right) \quad \text{or,} \quad h = r f\left(\frac{\sigma}{\gamma r^2}\right)$$

Example 7.2

Derive a rational formula for pipe flow having given the following quantities affecting the flow phenomenon: F is the boundary friction force, μ is the viscosity, ρ is the mass density, V is the velocity of flow, D is the pipe diameter and k is the surface roughness.

Solution Assuming that $F = f(\mu, \rho, V, D, k)$ (i)

$$F \propto \mu^a . \rho^b . V^c . D^d . k^e \quad \text{(ii)}$$

or, dimensionally

$$MLT^{-2} \propto (ML^{-1}T^{-1})^a . (ML^{-3})^b . (LT^{-1})^c . (L)^d . (L)^e \quad \text{(iii)}$$

Equating indices of M, L and T on both sides, we get

For M, $1 = a + b$ (iv)
For L, $1 = -a - 3b + c + d + e$ (v)
For T, $-2 = -a - c$ (vi)

From (iv), we have $b = 1 - a$ (vii)
From (vi), we have $c = 2 - a$ (viii)
From (v), we have $d = 1 + a + 3b - c - e = 1 + a + 3 - 3a - 2 + a - e$
$= 2 - a - e$ (substituting for b and c) (ix)

Eliminating b, c and d from (ii), we have,

$$F \propto (\mu)^a \cdot (\rho)^{1-a} \cdot (v)^{2-a} \cdot (D)^{2-a-e} \cdot (K)^e$$

$$\Rightarrow F \propto \rho v^2 D^2 \cdot (\mu \cdot \rho^{-1} \cdot v^{-1} \cdot D^{-1})^a \cdot (K D^{-1})^e$$

$$\Rightarrow F \propto \rho v^2 D^2 \left(\frac{\mu}{\rho v D}\right)^a \cdot \left(\frac{K}{D}\right)^e$$

$$\therefore \frac{F}{\rho v^2 D^2} = f\left(\frac{\mu}{\rho v D}, \frac{K}{D}\right)$$

Buckingham's Method or Buckingham's pi Theorem

Buckingham's Method or Buckingham's π-Theorem

This method was originally proposed by Edgar Buckingham in 1915. The Rayleigh's method of dimensional analysis becomes cumbersome when a large number of variables are involved. In order to overcome this difficulty, the Buckingham's method may be conveniently used. It states "*If there are n variables in a dimensionally homogeneous equation, and if these variables contain m fundamental dimensions, they may be grouped into n-m non-dimensional independent parameters.*"

Mathematically, if a variable x_1 depends upon a number of independent variables x_2, x_3, x_4, ..., x_n, the functional equation may be written as

$$x_1 = \phi(x_2, x_2, x_3, ..., x_n)$$

where ϕ is a function. This equation may be written in its general form as

$$f(x_1, x_2, x_2, x_3, ..., x_n) = 0$$

where f is another function.

If $\pi_1, \pi_2, \pi_3, ...$ represent dimensionless groupings of the variables x_1, x_2, x_3, ..., x_n with m dimensions involved, then according to Buckingham's π - theorem, an equation of the form

$$F(\pi_1, \pi_2, \pi_3, ..., \pi_{n-m}) = 0 \qquad (7.1)$$

exists, where F is still another function.

Steps of Buckingham's Methods are given below:

1. Write the functional relationship with the given data.
2. Then write the equation in its general form.
3. Now choose m *repeating variables* and write separate expression for each π-term. Each π-term will contain the repeating variables and one of the remaining variables. The repeating variables are written in exponential form.
4. Use the principle of dimensional homogeneity to find out the values of the exponents or powers by obtaining simultaneous equations.
5. Now substitute the values of the exponents or powers in the π-terms.
6. After the π-terms are determined, write the functional relation in the required form.

Following Points must be considered while selecting repeating variables:

1. The number of repeating variables should be equal to m, the number of fundamental dimensions.
2. The repeating variables should not be dimensionless and the repeating variables among themselves should not form a dimensionless number.
3. Independent variables should, as far as possible, be selected as repeating variables.

Generally, the repeating variables selected are ρ, V and l, i.e. the first repeating variable will represent the fluid property, the second will represent the flow characteristics and the third will represent the geometrical characteristics of the body.

Geometric Property (L, D, H)

Flow Property (v, a, ω, α)

Fluid Property (μ, ρ)

Buckingham's Pi Theorem Math Example



Math: Prove that the discharge over a spillway is given by the relation-

$$Q = VD^2 f\left[\frac{\sqrt{gD}}{V}, \frac{H}{D}\right]$$

where V = Velocity of flow; D = Depth at the throat; H = Head of water and g = Acceleration due to gravity.

Solution. Given : Discharge = Q; Velocity of flow = V; Depth at the throat = D; Head of water = H and acceleration due to gravity = g.

Let the functional relationship be,

$$Q = f[V, D, H, g]$$

The above equation in its general form may be written as,

$$f(Q, V, D, H, g) = 0$$

Here, n = 5
m = 2
So we need (n-m)=(5-2)=3 pi terms

A little consideration will show, that in the above equation, the primary dimensions are only two. Thus $m = 2$. Taking V and D as repeating variables, we get

$$\pi_1 = (V)^{a1} \cdot (D)^{b1} \cdot Q \qquad \ldots(i)$$

*. The most suitable variables are those, which contain flow characteristics (such as velocity, mass, density etc.).

Substituting the dimensions in both sides of the above equation,

or $$M^0L^0T^0 = [LT^{-1}]^{a1} \times [L]^{b1} \times [L^3T^{-1}]$$

Now for dimensional homogeneity, equating the powers of M, L and T on both sides of the equation,

For T, $$0 = -a_1 - 1 \text{ or } a_1 = -1$$

For L, $$0 = a_1 + b_1 + 3$$

or $$b_1 = -3 - a_1 = -3 - (-1) = -2$$

Substituting the values, of a_1 and b_1 in equation (i),

$$\pi_1 = V^{-1} \cdot D^{-2} \cdot Q$$

$$= \frac{Q}{VD^2} \quad \text{...}(ii)$$

Similarly, $$\pi_2 = (V)^{a2} (D)^{b2} H$$

Substituting the dimensions in both sides of the above equation,

or $$M^0L^0T^0 = [LT^{-1}]^{a2} \times [L]^{b2} \times [L]$$

Now for dimensional homogeneity, equating the powers of M, L and T on both sides of the equation,

For T, $$0 = -a_2 \quad \text{or} \quad a_2 = 0$$

For L, $$0 = a_2 + b_2 + 1 \quad \text{or} \quad b_2 = -a_2 - 1 = 0 - 1 = -1$$

Thus $$\pi_2 = V^0 \cdot D^{-1} \cdot H$$

$$= \frac{H}{D} \quad \text{...}(iii)$$

and $$\pi_3 = (V)^{a3}(D)^{b3}\, g$$

Substituting the dimensions in both sides of the above equation,

or $$M^0L^0T^0 = [LT^{-1}]^{a3} \times [L]^{b3} \times [LT^{-2}]$$

Now for dimensional homogeneity, equating the powers of M, L and T on both sides of the equation,

For T, $0 = -a_3 - 2$ or $a_3 = -2$

For L, $0 = a_3 + b_3 + 1$ or $b_3 = -a_3 - 1 = -(-2) - 1 = +1$

Thus $$\pi_3 = V^{-2}.D^{-1}.g = \frac{gD}{V^2}$$

$$= \frac{\sqrt{gD}}{V} \qquad \ldots(iv)$$

Now the functional relationship be written as,

$$f\left[\frac{Q}{VD^2}, \frac{H}{D}, \frac{\sqrt{gD}}{V}\right] = 0$$

$$\frac{Q}{VD^2} = f\left[\frac{\sqrt{gD}}{V}, \frac{H}{D}\right]$$

$\therefore$ $$Q = VD^2 f\left[\frac{\sqrt{gD}}{V}, \frac{H}{D}\right]$$ **Ans.**

25.12. The revi... ...ilered as

EWCE 261

FLUID MECHANICS

3 Credit Hour

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DIMENSIONAL AND MODEL ANALYSIS

Math Example 2 (Buckingham's Pi theorem):

The resisting force (F) of a supersonic plane during flight can be considered as dependent upon the length of aircraft (l), Velocity (V), air viscosity (μ),air density (ρ),bulk modulus of air(k). Express the functional relationship between These variables and the resisting force.

solution

Given,
Resisting force =F
Length of the aircraft = l
Velocity of the aircraft = V
Air viscosity = μ
Air density = ρ
Bulk modulus of air=K

Here, total variables, n=6
Fundamental dimensions, m=3
So pi terms required=n-m=(6-3)=3

viscosity = μ, ...

Let the functional relationship be,

$$F = f[l, V, \mu, \rho, K]$$

The above equation, in its general form, may be written as,

$$f\,(F., l,\ V\ ,\ \mu\ ,\ \rho\ ,\ K) = 0$$

A little consideration will show, that in the above equation, the primary dimensions are three only. Thus $m = 3$. Taking l, V and ρ as repeating variables, we get

$$\pi_1 = (l)^{a1}, (V)^{b1}. (\rho)^{c1}. F$$

Substituting the dimension in both sides of the above equation,

$$\therefore \qquad M^0L^0T^0 = (L)^{a1}. (LT^{-1})^{b1}. (ML^{-3})^{c1}. (MLT^{-2})$$

Now for dimensional homogeneity, equating the powers of M, L and T on both sides of the equation,

For M, $\quad 0 = c_1 + 1 \quad$ or $\quad c_1 = -1$

For T, $\quad 0 = -b_1 - 2 \quad$ or $\quad b_1 = -2$

For L, $\quad 0 = a_1 + b_1 - 3c_1 + 1 \quad$ or $\quad a_1 = -2$

$$\therefore \qquad \pi_1 = (l)^{-2}. (V)^{-2} (\rho)^{-1}. F \qquad \ldots(i)$$

$$= \frac{F}{l^2V^2\rho}$$

Similarly $\qquad \pi_2 = (l)^{a2}. (V)^{b2}. (\rho)^{c2}. \mu$

Similarly $\pi_2 = (l)^{a2}.(V)^{b2}.(\rho)^{c2}.\mu$

Substituting the dimension in both sides of the above equation,

$\therefore$ $M^0L^0T^0 = (L)^{a2}.(LT^{-1})^{b2}.(ML^{-3})^{c2}.(ML^{-1}T^{-1})$

Now for dimensional homogeneity, equating the power of M, L and T on both sides of the equation,

For M, $0 = c_2 + 1$ or $c_2 = -1$

For T, $0 = -b_2 - 1$ or $b_2 = -1$

For L, $0 = a_2 + b_2 - 3c_2 - 1$ or $a_2 = -1$

$\therefore$ $\pi_2 = (l)^{-1}.(V)^{-1}.(\rho)^{-1}.\mu$

$= \frac{\mu}{lV\rho}$ ...(ii)

and $\pi_3 = (l)^{b3}.(V)^{b3}.(\rho)^{c3}.K$

Substituting the dimension in both sides of the above equation,

$\therefore$ $M^0L^0T^0 = (L)^{a3}.(LT^{-2})^{b3}.(ML^{-3})^{c3}.(ML^{-1}T^{-2})$

Now for dimensional homogeneity, equating the powers of M, L and T on both sides of the equation,

For M, $0 = c_3 + 1$ or $c_3 = -1$

For T, $0 = -b_3 - 2$ or $b_3 = -2$

For L, $0 = a_3 + b_3 - 3c_3 - 1$ or $a_2 = 0$

$\therefore$ $\pi_3 = (l)^0.(V)^{-2}.(\rho)^{-1}.K$

$= \frac{K}{V^2\rho}$ ...(iii)

Now the functional relationship may be written as :

$$f\left(\frac{F}{l^2V^2\rho}, \frac{\mu}{lV\rho}, \frac{K}{V^2\rho}\right) = 0$$

or

$$F = f(l^2V^2\rho)\left(\frac{\mu}{lV\rho}, \frac{K}{V^2\rho}\right)$$ **Ans.**

Types of Forces Present in a Moving Liquid:

When a liquid is in motion, some forces are always involved in the phenomenon of flow. But there are always one or two forces, which dominate the other forces, and they govern the flow of the liquid and keep it in motion. The following which forces are present in a moving liquid, are important from the subject point of view:

- Inertia Force
- Viscous Force
- Gravity Force
- Surface Tension Force
- Pressure Force
- Elastic Force

Inertia Force:

The inertia force is the product of mass and acceleration of flowing liquid, and is always existing in the phenomenon of the liquid flow.

Viscous Flow:

The viscous force is the product of shear stress, due to viscosity and cross-sectional area of flow.

Gravity Force:

The gravity force is the product of mass and acceleration, due to gravity of a flowing liquid.

Surface Tension Force:

The surface tension force is the product of surface tension per unit length and length of the surface of flowing liquid.

Pressure Force:

The pressure force is the product of intensity of pressure and the area of a flowing liquid.

Elastic Force:

The elastic force is the product of the elastic stress and the area of a flowing liquid.

Reynold's Number:

The Reynold's number is the ratio of inertia force to the viscous force. If viscous force is the predominating force in the model and the prototype, dynamic similarity is said to exist between the two, when the Reynold's number for the model and the prototype is the same. *e.g.*, Reynold's number.

$$R_N = \frac{\text{Inertia force}}{\text{Viscous force}}$$

$$= \frac{\text{Mass} \times \text{Acceleration}}{\text{Shear stress} \times \text{Cross-sectional area}}$$

$$= \frac{\text{Volume} \times \text{Mass Density} \times (\text{Velocity} / \text{Time})}{\text{Shear stress} \times \text{Cross-sectional area}}$$

$$= \frac{(\text{Volume} / \text{Time}) \times \text{Mass Density} \times \text{Velocity}}{\text{Shear stress} \times \text{Cross-sectional area}}$$

$$= \frac{\text{Cross-sectional area} \times \text{Velocity} \times \text{Mass density} \times \text{Velocity}}{\text{Shear stress} \times \text{Cross sectional area}}$$

$$= \frac{\text{Velocity} \times \text{Mass density} \times \text{Velocity}}{\text{Shear stress}}$$

$$= \frac{V \rho V}{\mu \left(\frac{dv}{dy}\right)} = \frac{V \rho V}{\mu \cdot \frac{V}{L}} = \frac{V\rho}{\frac{\mu}{L}} = \frac{V \cdot L}{\frac{\mu}{\rho}}$$

$$= \frac{V \cdot L}{\nu}$$

$\left(\because \frac{\mu}{\rho} = \nu\right)$

$Re = \frac{VL}{\nu}$ → ν → kinematic viscosity

μ → Dynamic viscosity

Froude's Number:

The Froude's number is the ratio of inertia force and gravity force. If gravity force is the predominating force in the model and prototype, dynamic similarity is said to exist between the two, when the Froude's number of the model and prototype is same.

∴ Ratio of these two forces

$$= \frac{\text{Inertia force}}{\text{Gravity force}}$$

$$= \frac{\text{Mass} \times \text{Acceleration}}{\text{Mass} \times \text{Acceleration due to gravity}}$$

$$= \frac{(\text{Velocity/Time})}{\text{Acceleration due to gravi}}$$

$$= \frac{\frac{V}{T}}{g} = \frac{V}{Tg} = \frac{V}{\frac{L}{V} \times g} = \frac{V^2}{Lg}$$

Now the Froude's number is the square root of this ratio. *i.e.*,

$$F_N = \sqrt{\frac{V^2}{Lg}} = \frac{V}{\sqrt{Lg}}$$

Weber's Number:

The Weber's number is the ratio of inertia force to the surface tension force. If the surface tension force is the predominating force in the model and prototype, dynamic similarity is said to exist between the two, when the Weber's number for the model and prototype is the same.

∴ Ratio of these two forces

$$= \frac{\text{Inertia force}}{\text{Surface tension force}}$$

$$= \frac{\text{Mass} \times \text{Acceleration}}{\text{Surface tension per unit length} \times \text{Length}}$$

$$= \frac{(\text{Volume} \times \text{Mass density})\ (\text{Velocity} / \text{Time})}{\text{Surface tension per unit length} \times \text{Length}}$$

$$= \frac{(\text{Volume} / \text{Time}) \times \text{Mass density} \times \text{Velocity}}{\text{Surface tension per unit length} \times \text{Length}}$$

$$= \frac{\text{Cross–sectional area} \times \text{Velocity} \times (\text{Mass density} \times \text{Velocity})}{\text{Surface tension per unit length} \times \text{Length}}$$

$$= \frac{L^2 V \rho V}{\sigma L} = \frac{\rho L V^2}{\sigma}$$

Now Weber's number is the square root of this ratio. *i.e.*,

$$W_N = \sqrt{\frac{\rho L V^2}{\sigma}} = \sqrt{\frac{V^2}{\frac{\sigma}{\rho L}}} = \frac{V}{\sqrt{\frac{\sigma}{\rho L}}}$$

Euler's Number:

The Euler's number is the ratio of inertia force and pressure force. If the pressure force is the predominating force in the model and the prototype, dynamic similarity is said to exist between the two, when the Euler's number for the model and prototype is same.

$$= \frac{\text{Inertia force}}{\text{Pressure Force}}$$

$$= \frac{\text{Mass} \times \text{Acceleration}}{\text{Intensity of pressure} \times \text{Area}}$$

$$= \frac{(\text{Volume} \times \text{Mass density}) \times (\text{Velocity} / \text{Time})}{\text{Intensity of pressure} \times \text{Area}}$$

$$= \frac{(\text{Volume} / \text{Time}) \times \text{Mass density} \times \text{Velocity}}{\text{Intensity of pressure} \times \text{Area}}$$

$$= \frac{(\text{Cross–sectional area}) \times \text{Velocity} \times \text{Mass density} \times \text{Velocity}}{\text{Intensity of pressure} \times \text{Area}}$$

$$= \frac{\text{Velocity} \times \text{Mass density} \times \text{Velocity}}{\text{Intensity of pressure}}$$

$$= \frac{V \rho V}{P} = \frac{V^2 \rho}{p}$$

Now Euler's number is the square root of this ratio. *i.e.*,

$$E_N = \frac{V^2 \rho}{p} = \sqrt{\frac{V^2}{\frac{p}{\rho}}} = \frac{V}{\sqrt{p/\rho}}$$

Note : The reciprocal of Eular number is called Newton number.

Mach Number or Cauchy Number:

The Mach's or Cauchy's number is the ratio of inertia force and the elastic force. If the elastic force is the predominating force in the model and the prototype, dynamic similarity is said to exist between the two when the Mach's number for the model and the prototype is the same.

∴ Ratio of these two forces

$$= \frac{\text{Inertia force}}{\text{Elastic force}}$$

$$= \frac{\text{Mass} \times \text{Acceleration}}{\text{Elastic stress} \times \text{Area}}$$

$$= \frac{(\text{Volume} \times \text{Mass density}) \times (\text{Velocity/Time})}{\text{Elastic stress} \times \text{Area}}$$

$$= \frac{(\text{Volume/Time}) \times \text{Mass density} \times \text{Velocity}}{\text{Elastic stress} \times \text{Area}}$$

$$= \frac{(\text{Cross–sectional area} \times \text{Velocity}) \times (\text{Mass density} \times \text{Velocity})}{\text{Elastic stress} \times \text{Area}}$$

$$= \frac{\text{Velocity} \times \text{Mass density} \times \text{Velocity}}{\text{Elastic stress}}$$

$$= \frac{V \rho V}{K} = \frac{V^2 \rho}{K}$$

Now Mach's number is the square root of this ratio. *i.e.*,

$$M_N = \frac{V^2 \rho}{K} = \sqrt{\frac{\frac{V^2}{K}}{\rho}} = \frac{V}{\sqrt{\frac{K}{\rho}}} = \frac{V}{\sqrt{k/\rho}}$$

K = elastic stress



MATH EXAMPLES

Example If the Froude Number of a model and its prototype are the same, find V_r, T_r, a_r, Q_r.

$\Rightarrow \quad Fr_p = Fr_m$

$\Rightarrow \quad \frac{V_p}{\sqrt{g_p L_p}} = \frac{V_m}{\sqrt{g_m L_m}}$

$\Rightarrow \quad \frac{V_p}{V_m} = \sqrt{\frac{L_p}{L_m}}$

$\Rightarrow V_r = \sqrt{L_r}$

$\therefore V_r = \sqrt{L_r}$

$$T_r = \frac{T_p}{T_m}$$

$$= \frac{\frac{L_p}{V_p}}{\frac{L_m}{V_m}}$$

$$= \left(\frac{L_p}{L_m} \times \frac{V_m}{V_p}\right)$$

$$= L_r \times \frac{1}{V_r}$$

$$= \frac{L_r}{\sqrt{L_r}} = \frac{L_r \cdot \sqrt{L_r}}{\sqrt{L_r} \cdot \sqrt{L_r}} = \frac{L_r \cdot \sqrt{L_r}}{(\sqrt{L_r})^2} = \sqrt{L_r}$$

g for model and prototype are same as both are in earth

$$a_r = \frac{V_r}{T_r}$$

$$= \frac{\sqrt{L_r}}{\sqrt{L_r}}$$

$$= 1$$

$$Q_r = \frac{Q_p}{Q_m}$$

$$= \frac{A_p \cdot V_p}{A_m \cdot V_m}$$

$$= \frac{A_r \cdot V_r}{1}$$

$$= \frac{L_r^2 \cdot \sqrt{L_r}}{1}$$

$$= L_r^{5/2}$$

p means prototype
m means model
V_r= velocity ratio
a_r = accelaration ratio
T_r = time ratio
Q_r=dicharge ratio

Force = mass × acceleration

$= \rho V \times$ acceleration

$= (\rho \times A \times L) \times \frac{V}{t}$ $\quad V$ = velocity

$= \rho A \times V \times \left(\frac{L}{t}\right)$

$= \rho A \times v \times v = \rho A v^2$

$\therefore F_r = \rho_r \cdot L_r^2 \cdot (\sqrt{L_r})^2$

$= \rho_r \cdot L_r^3$

If ρ is same for model & prototype, then

$F_r = L_r^3$

Example 7.4

The performance of a ship was predicted by making its model and testing it in a wind tunnel. The length of the prototype was 350 m and its model was 10 m long. The ν for air is 1.25 times that of water. The velocity of air around the model in the wind tunnel was measured as 35 m/s. Find the velocity of actual ship in water if the model has dynamic similarity with the prototype and the flow is governed by Reynolds law.

Solution Here, $L_p = 350$ m, $L_m = 10$ m, $\nu_m = 1.25\nu_p$, $V_m = 35$ m/s

Kinematic viscosity

We know, for dynamic similarity between the model and the prototype, the Reynolds number of model and prototype should be equal.

$$\therefore \frac{V_p L_p}{\nu_p} = \frac{V_m L_m}{\nu_m}$$

$$\therefore V_p = \frac{\nu_p}{\nu_m} \times \frac{L_m}{L_p} \times V_m = \frac{1}{1.25} \times \frac{10}{350} \times 35 = 0.8 \ m/s$$

Math: A channel model 250 mm deep is discharging water with a velocity of 1.5 m/s. Find the velocity of water in the channel 4 m deep, if the model has dynamic similarity with its prototype and the flow is governed by Froude law.

$V_m = 1.5$ m/s, $g_p = g_m$ (since the model and the prototype are in the same place)

We know, for dynamic similarity between the model and the prototype, the Froude number of model and prototype should be equal.

$$\therefore \frac{V_p}{\sqrt{g_p L_p}} = \frac{V_m}{\sqrt{g_m L_m}}$$

scale ratio, $L_r = (L_p/L_m) = (4/0.25) = 16/1$

$$\therefore V_p = \sqrt{\frac{g_p}{g_m}} \times \sqrt{\frac{L_p}{L_m}} \times V_m = \sqrt{1} \times \sqrt{16} \times 1.5 = 6 \text{ m/s}$$

Math: Water flows over a crest of 1:35 model spiilway and the velocity measured at a particular point is 0.4 m/s. What velocity does this represent in the prototype? The force exerted on a certain area of the model is measured to be 0.12 N. What would be the force on the corresponding area in the prototype? The model has dynamic similarity with the prototype and flow is governed by Froude number.

$\Rightarrow \frac{L_m}{L_p} = \frac{1}{35}$

$V_m = 0.4\ m/s$

$V_p = ?$

$F_p = F_m$

$\Rightarrow \left(\frac{V}{\sqrt{gL}}\right)_p = \left(\frac{V}{\sqrt{gL}}\right)_m$

g is same for model and prototype

$\Rightarrow V_m = V_p \left(\frac{g_m}{g_p}\right)^{\frac{1}{2}} \left(\frac{L_m}{L_p}\right)^{\frac{1}{2}}$

$= V_p \times 1 \times \left(\frac{1}{35}\right)^{\frac{1}{2}}$

Here, Density is same

$\Rightarrow V_p = V_m \times (35)^{\frac{1}{2}}$

$= 0.4 \times (35)^{\frac{1}{2}} = 2.37\ m/s$

Now, $F_r = L_r^3$

$\Rightarrow \left(\frac{F_m}{F_p}\right) = \left(\frac{L_m}{L_p}\right)^3$

$[F_m = 0.12\ N]$

$\Rightarrow \left(\frac{F_p}{F_m}\right) = \left(\frac{L_p}{L_m}\right)^3$

$= \left(\frac{L_p}{L_m}\right)^3 \times F_m = (35)^3 \times 0.12 = 5145\ N$

Example:

An airfoil moves at 650 km/hour through still air at 20°c. If the elastic stress and density of air at this temperature are 21×10^9 N/m^2 and 0.126 kg/m^3, find the Mach number.

=> Here, $V = 650$ km/hour

$$= \frac{650 \times 1000}{3600} \text{ m/s}$$

$$= 180.6 \text{ m/s}$$

$K = 21 \times 10^9$ N/m^2

$\rho = 0.126$ kg/m^3

Now, $$M_a = \frac{V}{\sqrt{\frac{K}{\rho}}} = \frac{180.6}{\sqrt{\frac{21 \times 10^9}{0.126}}}$$

$$= 0.14$$

See the math carefully, some corrections are made in the question