

MILITARY INSTITUTE OF SCIENCE AND TECHNOLOGY

ASSIGNMENT | MATH 203

GAMMA FUNCTION

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Theorem 8

$$\int_0^1 x^m \left(\ln \frac{1}{x}\right)^n dx = \frac{\Gamma(n+1)}{(1+m)^{n+1}}$$

Proof:

Let, $I = \int_0^1 x^m \left(\ln \frac{1}{x}\right)^n dx$

$= \int_\infty^0 (e^{-t})^m t^n (-e^{-t})\, dt$

$= \int_0^\infty (e^{-t})^{m+1} t^n \, dt$

$= \int_0^\infty t^n e^{-(1+m)t} \, dt$ — (1)

Let,
$x = e^{-t} \Rightarrow dx = -e^{-t} dt$
or, $\frac{1}{x} = e^t$
or, $\ln\left(\frac{1}{x}\right) = t$

Limits:
when $x = 0$, $t = \infty$
$x = 1$, $t = 0$

Again,
Let, $(1+m)t = \theta$

$\Rightarrow t = \frac{\theta}{1+m}$

$\Rightarrow dt = \frac{d\theta}{1+m}$

Limits:
when $t = 0$, $\theta = 0$
$t = \infty$, $\theta = \infty$

(1) $\Rightarrow I = \int_0^\infty \left(\frac{\theta}{1+m}\right)^n e^{-\theta} \frac{d\theta}{1+m}$

$= \frac{1}{(1+m)^{n+1}} \int_0^\infty e^{-\theta} \theta^n \, d\theta$

$= \frac{1}{(1+m)^{n+1}} \int_0^\infty e^{-\theta} \theta^{n+1-1} \, d\theta$ $\Rightarrow I = \frac{\Gamma(n+1)}{(1+m)^{n+1}}$

(Proved)

Theorem 9

$$\int_0^1 x^m (\ln x)^n \, dx = \frac{(-1)^n \lfloor (n+1)}{(1+m)^{n+1}}$$

Proof:

Let, $I = \int_0^1 x^m (\ln x)^n \, dx$

$= -\int_\infty^0 (e^{-t})^m (-t)^n e^{-t} \, dt$

$= (-1)^n \int_0^\infty e^{-(1+m)t} t^n \, dt$ — (1)

Let,
$x = e^{-t} \Rightarrow dx = -e^{-t} dt$

$\Rightarrow \ln x = -t$

Limits:
when $x = 0$, $t = \infty$
$x = 1$, $t = 0$

Again,
Let, $(1+m)t = y$

$\Rightarrow t = \frac{y}{1+m}$

$\Rightarrow dt = \frac{dy}{1+m}$

Limits:
when, $t = 0$, $y = 0$
$t = \infty$, $y = \infty$

(1) $\Rightarrow I = (-1)^n \int_0^\infty e^{-y} \left(\frac{y}{1+m}\right)^n \frac{dy}{1+m}$

$= (-1)^n \int_0^\infty e^{-y} \frac{y^n}{(1+m)^{n+1}} \, dy$

$= \frac{(-1)^n}{(1+m)^{n+1}} \int_0^\infty e^{-y} y^n \, dy \Rightarrow I = \frac{(-1)^n}{(1+m)^{n+1}} \lfloor (n+1)$

(Proved)

Theorem 19

$$\int_0^\infty \frac{\sin x}{x^n}\,dx = \frac{\pi}{2\Gamma n \sin\frac{n\pi}{2}}, \quad \text{where } 0 < n < 1$$

Proof:

$$\frac{\Gamma(n)}{x^n} = \int_0^\infty e^{-xt} t^{n-1}\,dt$$

$$\Rightarrow \frac{1}{x^n} = \frac{1}{\Gamma n}\int_0^\infty e^{-xt} t^{n-1}\,dt$$

$$\Rightarrow \frac{\sin x}{x^n} = \frac{\sin x}{\Gamma n}\int_0^\infty e^{-xt} t^{n-1}\,dt$$

$$\Rightarrow \int_0^\infty \frac{\sin x}{x^n}\,dx = \frac{1}{\Gamma n}\int_0^\infty\int_0^\infty e^{-xt} t^{n-1} \sin x\,dx\,dt$$

$$= \frac{1}{\Gamma n}\int_0^\infty \left[\frac{-t\sin x e^{-xt} - \cos x\, e^{-xt}}{t^2+1}\right]_0^\infty t^{n-1}\,dt$$

$$= \frac{1}{\Gamma n}\int_0^\infty \frac{1}{t^2+1} t^{n-1}\,dt$$

$$= \frac{1}{\Gamma n}\int_0^\infty \frac{t^{n-1}}{t^2+1}\,dt$$

Let, $t^2 = u$

$\Rightarrow 2t\,dt = 1\,du$

$\Rightarrow dt = \frac{du}{2t} = \frac{du}{2\sqrt{u}}$

when $t = 0$, $u = 0$

$t = \infty$, $u = \infty$

$$= \frac{1}{2\Gamma n}\int_0^\infty \frac{u^{\frac{n-1}{2}}}{u+1}\frac{u^{-1/2}}{}\,du$$

$$= \frac{1}{2\Gamma n}\int_0^\infty \frac{u^{n/2}\, u^{-1/2}\, u^{-1/2}}{u+1}\,du$$

$$= \frac{1}{2\sqrt{n}} \int_0^\infty \frac{u^{n/2-1}}{u+1}\, du$$

$$= \frac{1}{2\sqrt{n}} \frac{\pi}{\sin\frac{n\pi}{2}} \qquad \left[\because \int_0^\infty \frac{y^{n-1}}{1+y}\, dy = \frac{\pi}{\sin n\pi}\right]$$

$$\therefore \int_0^\infty \frac{\sin x}{x^n}\, dx = \frac{\pi}{2\sqrt{n}\sin\frac{n\pi}{2}}$$

(Proved)

Theorem 20:

$$\int_0^\infty \frac{\cos nx}{x^n}\,dx = \frac{\pi}{2\sqrt{n}\cos\frac{n\pi}{2}} \quad \text{where } 0<n<1$$

Proof:

$$\frac{\sqrt{n}}{k^n} = \int_0^\infty e^{-kt} t^{n-1}\,dt$$

$$\Rightarrow \frac{1}{x^n} = \frac{1}{\sqrt{n}} \int_0^\infty e^{-xt} t^{n-1}\,dt$$

$$\Rightarrow \frac{\cos nx}{x^n} = \frac{\cos nx}{\sqrt{n}} \int_0^\infty e^{-xt} t^{n-1}\,dt$$

$$\Rightarrow \int_0^\infty \frac{\cos nx\,dx}{x^n} = \frac{1}{\sqrt{n}} \int_0^\infty \int_0^\infty e^{-xt} t^{n-1} \cos nx\,dx\,dt$$

$$= \frac{1}{\sqrt{n}} \int_0^\infty \left[\frac{-t e^{-xt}\cos nx + e^{-kt}\sin nx}{t^2+1} \right]_0^\infty t^{n-1}\,dt$$

$$= \frac{1}{\sqrt{n}} \int_0^\infty \frac{t \cdot t^{n-1}}{t^2+1}\,dt$$

$$= \frac{1}{\sqrt{n}} \int_0^\infty \frac{t^n}{t^2+1}\,dt$$

$$= \frac{1}{2\sqrt{n}} \int_0^\infty \frac{u^{n/2} u^{-1/2}}{1+u}\,du$$

$$= \frac{1}{2\sqrt{n}} \int_0^\infty \frac{u^{\frac{n-1}{2}}}{1+u}\,du$$

Let, $t^2 = u$

$\Rightarrow 2t\,dt = du$

$\Rightarrow dt = \frac{du}{2\sqrt{u}}$

when $t=0$, $u=0$

$t=\infty$, $u=\infty$

$$= \frac{1}{2\sqrt{n}} \int_0^\infty \frac{u^{\frac{n+1}{2}-1}}{1+u}\, du$$

$$= \frac{1}{2\sqrt{n}} \frac{\pi}{\sin\left(\frac{n+1}{2}\right)\pi} \qquad \left[\because \int_0^\infty \frac{y^{n-1}}{1+y}\, dy = \frac{\pi}{\sin n\pi}\right]$$

$$\Rightarrow \int_0^\infty \frac{\cos x}{x^n}\, dx = \frac{\pi}{2\sqrt{n}\sin\left(\frac{n+1}{2}\right)\pi}$$

$$= \frac{\pi}{2\sqrt{n}\sin\left(\frac{\pi}{2} + \frac{n\pi}{2}\right)}$$

$$\therefore \int_0^\infty \frac{\cos x}{x^n}\, dx = \frac{\pi}{2\sqrt{n}\cos\frac{n\pi}{2}}$$

(Proved)