Bessel Function – 20 marks

FAHI11-09
CE-23
Sec-A

define bessel function.

Ans: The important differential equation

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0 \quad \text{---} \quad (1)$$

is known as bessel differential equation of order n (where n be a positive number)

Solution of bessel differential equation is

$$J = A J_n + B J_{-n}(x)$$

where,

$J_n(x)$ is known

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{\lfloor r+1 \rfloor \lfloor n+r+1 \rfloor} \quad \text{-----} \quad (II)$$

where, $J_n(x)$ is known as bessel's function of first kind.

= R

Recurrence formula formula for bessel's function.

1. $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$ prove imp.
2. $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$ prove imp. } 100%
3. $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$
4. $J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$

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Q. Shows that, $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

Soln:

Soln:

by the definition of bessels function,

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{\lfloor r+1 \rfloor \lfloor r+n+1 \rfloor}$$

$$\Rightarrow \frac{d}{dx}\left[x^{-n} J_n(x)\right] = \frac{d}{dx}\left[x^{-n} \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{\lfloor r+1 \rfloor \lfloor n+r+1 \rfloor}\right]$$

$$= \frac{d}{dx}\left[\sum_{r=0}^{\infty} \frac{(-1)^r \frac{1}{2^n} \left(\frac{x}{2}\right)^{-n} \left(\frac{x}{2}\right)^{n+2r}}{\lfloor r+1 \rfloor \lfloor n+r+1 \rfloor}\right]$$

$$= \frac{d}{dx}\left[\sum_{r=0}^{\infty} \frac{(-1)^r \frac{1}{2^r} \left(\frac{x}{2}\right)^{2r}}{\lfloor r+1 \rfloor \lfloor n+r+1 \rfloor}\right]$$

$$= d \sum_{r=1}^{\infty} \frac{(-1)^r \frac{1}{2^r} \frac{1}{2} 2r \left(\frac{x}{2}\right)^{2r-1}}{\lfloor r+1 \rfloor \lfloor n+r+1 \rfloor}$$

$$= \sum_{r=1}^{\infty} \frac{(-1)^r \frac{1}{2^r} x^n \frac{1}{x^n} r \left(\frac{x}{2}\right)^{2r-1}}{r \lfloor r \rfloor \lfloor n+r+1 \rfloor} \quad \text{---- (II)}$$

γ function

$\lfloor r+1 \rfloor = r \lfloor r \rfloor$

$\lfloor r \rfloor = \int_0^{\infty} e^{-x} x^{r-1} dx$

$\lfloor r+1 \rfloor = r!$

Let, $r-1 = s$, $r = 1+s$ then $r-1 = 0$ when $s = 0$ and $r-1 = \infty$ when $s = \infty$

Therefore (II) becomes

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = x^{-n} \sum_{s=0}^{\infty} \frac{(-1)^{s+1} \left(\frac{x}{2}\right)^{n+2(s+1)}}{\lfloor s+1 \rfloor \lfloor n+s+1+1 \rfloor}$$

$$= x^{-n} \sum_{s=0}^{\infty} \frac{(-1)(-1)^s \left(\frac{x}{2}\right)^{n+1+2s}}{\lfloor s+1 \rfloor \lfloor n+1+s+1 \rfloor}$$

$$= x^{-n} J_{n+1}(x)$$

(proved)

1. show that, $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

soln: by the definition of bessel function

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{\lfloor r+1 \rfloor \lfloor n+r+1 \rfloor} \quad \text{-----(1)}$$

substituting $n = \frac{1}{2}$ in (1), we have

$$J_{\frac{1}{2}}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{\frac{1}{2}+2r}}{\lfloor r+1 \rfloor \lfloor \frac{1}{2}+r+1 \rfloor}$$

$$= \left(\frac{x}{2}\right)^{1/2} \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{2^{2r} \lfloor r+1 \rfloor \lfloor r+\frac{3}{2} \rfloor}$$

$$= \sqrt{\frac{x}{2}} \left[\frac{1}{2^0 \lfloor 3/2} - \frac{1}{2^2} \frac{x^2}{\lfloor 2 \lfloor 5/2} + \frac{x^4}{2^4 \cdot \lfloor 3 \lfloor 7/2} \cdots \right]$$

$$= \sqrt{\frac{x}{2}} \left[\frac{1}{\lfloor 3/2} - \frac{x^2}{1 \cdot 2^2 \frac{3}{2} \lfloor \frac{3}{2}} + \frac{x^4}{2^4 2 \times \frac{5}{2} \times \frac{3}{2} \lfloor \frac{3}{2}} \cdots \right]$$

$$= \sqrt{\frac{x}{2}} \times \frac{1}{\lfloor 3/2} \left[1 - \frac{x^2}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdots \right]$$

$$= \sqrt{\frac{x}{2}} \frac{1}{\lfloor 3/2} \frac{1}{x} \left[x - \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdots \right]$$

$$= \sqrt{\frac{x}{2}} \frac{2}{\sqrt{\pi}} \times \frac{1}{x} \sin x$$

$$= \sqrt{\frac{2}{\pi x}} \sin x$$

$\left[\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots \right.$

$\left[\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots \right.$

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3.2 Show that, $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right]$

Soln:

Ø from Q.1 we calculated,

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x \quad \text{---(i)}$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x \quad \text{---(II)}$$

according to the recurrence formula. we have.

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x) \quad \text{---(iii)}$$

putting $n = \frac{1}{2}$ in equation -(ii) we have.

$$\frac{2\times\frac{1}{2}}{x} J_{1/2}(x) = J_{\frac{1}{2}-1}(x) + J_{\frac{1}{2}+1}(x)$$

$$\Rightarrow \frac{1}{x} J_{1/2}(x) = J_{-1/2}(x) + J_{3/2}$$

$\Rightarrow$ substituting equation -(i) and (II) to the. equation -III we have

$$\frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cos x = J_{3/2}(x)$$

$$\Rightarrow J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right]$$

(proved)

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generating function of bessel function

$$e^{\frac{1}{2}(t-\frac{1}{t})x} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

Soln: we know,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

$$e^{\left(\frac{t}{2}-\frac{1}{2t}\right)x} = e^{\frac{xt}{2}}\, e^{-\frac{x}{2t}} = \left[\sum_{r=0}^{\infty} \frac{\left(\frac{xt}{2}\right)^r}{r!}\right]\left[\sum_{s=0}^{\infty} \frac{\left(-\frac{x}{2t}\right)^s}{s!}\right]$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{s+r} t^{r-s}}{r!\, s!}$$

find the coefficient of t^n.

Let, $r-s=n$ $\therefore$ $r = n+s$

$$\therefore e^{\frac{1}{2}(t-\frac{1}{t})x} = \sum_{n=-\infty}^{\infty}\left[\sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{n+2s}}{(n+s)!\, s!}\right] t^n$$

$$\Rightarrow e^{\frac{1}{2}(t-\frac{1}{t})x} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

(Showed)