MATH-203

02.12.24

Syllabus

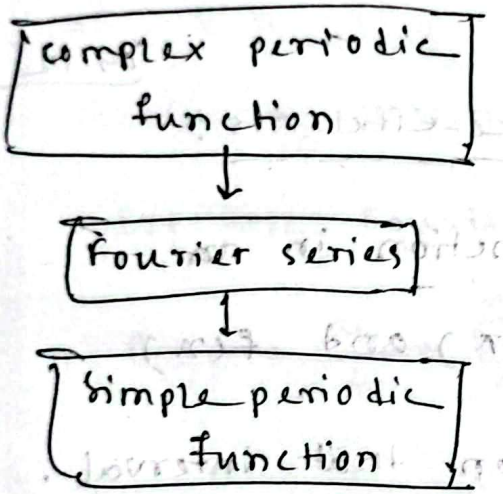
1. 50% — marks qns
2. 1.5 set qns
3. 2 set qns

-2nd book ta '

book

Mathematical method (0-07 1 & 2) abdur rahman.

Fourier series



Application

1. Image processing
2. Signal processing
3. Thermal Analysis
4. Complex Analysis
5. control system

Periodic function —

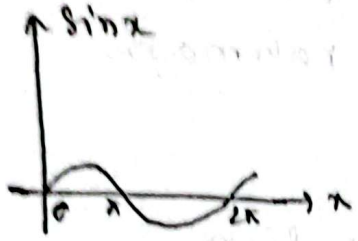
The function $f(x)$ of a real variable x is said to be periodic if there exist a non-zero number T, independent of x, such that

$$f(x) = f(x+T)$$

holds for all values of x. The least value of $T>0$ is called the period of $f(x)$.

Example: Let $f(x) = \sin x$

$$f(x+2\pi) = \sin(x+2\pi) = \sin x = f(x)$$



→ • Repeating cycle
• constant interval.

09.12

Fourier series and fourier co-efficients:

Let $f(x)$ be a periodic function in an interval of 2π say $(-\pi, \pi)$ and $f(x)$ is sectionally continuous on that interval, then the fourier series of $f(x)$ is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{---(i)}$$

where constants a_0, a_n, b_n are given by,

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{aligned}\right\} \quad \text{---(ii)}$$

Then the series (1) is called the Fourier series of $f(x)$ and the constants a_0, a_n, b_n is given by eqn (1)) are called fourier co-efficients.

uniform formula —

Let $f(x)$ be a periodic function in an interval of $b-a$ say (a,b) and $f(x)$ is sectionally continuous, on that interval.

Then the fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cdot \cos \cdot \frac{2n\pi x}{b-a} + b_n \sin \frac{2n\pi x}{b-a} \right) \quad (1)$$

where constants a_0, a_n, b_n are given by

$$\left.\begin{aligned} a_0 &= \frac{2}{b-a}\int_a^b f(x)\,dx \\ a_n &= \frac{2}{b-a}\int_a^b f(x) \cdot \cos \frac{\cdot 2n\pi x}{b-a}\,dx \\ b_n &= \frac{2}{b-a}\int_a^b f(x) \cdot \sin \frac{2n\pi x}{b-a}\,dx \end{aligned}\right\} \quad (11)$$

then same likha agar tab moto. —

- $\int u(x)\, v(x)\, dx$

$= uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots\cdots$

$u' = \frac{du}{dx}$, $v_1 = \int v\, dx$

Dirichlet condition for the convergence of fourier series —

1. If $f(x)$ is defined on $(-\pi, \pi)$
2. If $f(x)$ is periodic
3. If $f(x)$ can be extended to other values by the periodically condition

$$f(x) = f(x + 2\pi k); \quad k = 1, 2, 3, \ldots$$

4. If $f(x)$ is sectionally continuous in $(-\pi, \pi)$ with a finite number of discontinuties

then the fourier series converges to $f(x)$ of all points where $f(x)$ is continuous or $\frac{f(x-0) + f(-x+0)}{2}$ if x is a point

of discontinuity.

$(-\pi, \pi) \longrightarrow (-\pi, -\frac{\pi}{2}), (-\frac{\pi}{2}, 0), (0, \frac{\pi}{2}), (\frac{\pi}{2}, \pi)$

$\downarrow$

$-\pi < x < -\frac{\pi}{2}$

Q-1 10.12

Find the Fourier series of $f(x) = x + x^2$ for $-\pi < x < \pi$ and hence show that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$ [20 marks]

Soln - Given that, $f(x) = x + x^2$ ——— (i)

By the definition of Fourier series, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{——— (ii)}$$

here,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2)\, dx$$

$$= \frac{1}{\pi} \left[\frac{x^2}{2} + \frac{x^3}{3} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3} \right]$$

$$= \frac{2\pi^3}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\frac{\sin nx}{n} - (1+2x)\frac{-\cos nx}{n^2} + (0+2)\frac{-\sin nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi n^2}\left[(1+2\pi)\cos n\pi - (1-2\pi)\cos n\pi\right]$$

$$= \frac{1}{\pi n^2}(1+2\pi-1+2\pi)\cos n\pi$$

$$= \frac{4}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\frac{-\cos nx}{n} - (1+2\pi)\frac{-\sin nx}{n^2} + (0+2)\cdot\frac{\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-(\pi+\pi^2)\frac{\cos n\pi}{n} + \frac{2}{n^3}\cos\pi + (-\pi+\pi^2)\cdot\frac{\cos(-n\pi)}{n}\right.$$

$$\left. - \frac{2}{n^3}\cos(-n\pi)\right]$$

$$= \frac{1}{n\pi}\cos n\pi\left[-\pi-\pi^2-\pi+\pi^2\right]$$

$$= \frac{1}{n\pi}(-1)^n\cdot(-2\pi)$$

$$= -\frac{2}{n}(-1)^n$$

Substituting the values of a_0, a_n and b_n in (1), we get,

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1} \frac{4}{n^2}(-1)^n \cos nx + \sum_{n=1} -\frac{2}{n}(-1)^n \sin nx$$

$$= \frac{\pi^2}{3} + 4\left[-\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right]$$

$$-2\left[-\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \cdots\right] \quad \text{---(iii)}$$

here,

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \quad \text{---(iv)}$$

compare (iii) and (iv), we get

$\cos x = -1$	$\cos 2x = 1$	$\cos 3x = -1$
$\Rightarrow \cos x = \cos \pi$	$\cos 2x = \cos 2\pi$	$\cos 3x = \cos 3\pi$

$$\therefore x = \pi$$

Since $f(x)$ is discontinuous at $x = \pi$.

$$\therefore f(\pi) = \frac{f(\pi - 0) + f(-\pi + 0)}{2}$$

$$= \frac{\pi + \pi^2 - \pi + \pi^2}{2}$$

$$= \pi^2$$

Now substituting $x = \pi$ in (III), we get

$$\pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] + 0$$

$$\Rightarrow \pi^2 - \frac{\pi^2}{3} = 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\therefore \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

Homework

Q-2

Find the Fourier series of $f(x) = x - x^2$ for $-\pi < x < \pi$ and hence show that

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots \quad \text{(IV)}$$

Q-3 Find the fourier series of

$$f(x) = e^x, e^{-x} \text{ for } -\pi < x < \pi$$

∴ use formula — $\int e^{ax}\cos bx\,dx$, $\int e^{ax}\sin bx\,dx$

Q-2 : Expand the func

$$f(x) = \begin{cases} -\pi & \text{when } -\pi < x < 0 \\ x & \text{when } 0 < x < \pi \end{cases}$$

in fourier series and hence show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \quad \text{— (iv)}$$

⟶ Given that,

$$f(x) = \begin{cases} -\pi & \text{when } -\pi < x < 0 \\ x & \text{when } 0 < x < \pi \end{cases} \quad \text{— (i)}$$

By the definition of fourier series, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{— (ii)}$$

where, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\,dx + \int_{0}^{\pi} x\,dx\right]$$

$$= -[x]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$= -\pi + \frac{1}{\pi}\cdot\frac{\pi^2}{2}$$

$$= -\frac{\pi}{2}$$

$$\therefore a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right]$$

$$= -\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[x\cdot\frac{\sin nx}{n} - 1\cdot\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi n^2}\left[\cos n\pi - \cos 0\right]$$

$$= \frac{1}{\pi n^2}\left[(-1)^n - 1\right]$$

and,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} -\pi \sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x \sin nx\,dx$$

$$= -\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[x\cdot\left(\frac{-\cos nx}{n}\right) - 1\cdot\left(\frac{-\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$= \frac{1}{n}\left[\cos 0 - \cos(-n\pi)\right] + \frac{1}{\pi}\left[-\pi\frac{\cos n\pi}{n}\right]$$

$$= \frac{1}{n}\left[1 - (-1)^n\right] - \frac{1}{n}(-1)^n$$

$$= \frac{1}{n}\left[1 - 2(-1)^n\right]$$

Now substituting the value of a_0, a_n and b_n in (ii), we get,

$$f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{1}{\pi n^2}\left[(-1)^n - 1\right]\cos nx + \sum_{n=1}^{\infty}\frac{1}{n}\left[1 - 2(-1)^n\right]\sin nx$$

$$= -\frac{\pi}{4} + \frac{1}{\pi}\left[-2\cos x + 0 - \frac{2}{3^2}\cos 3x + 0 - \frac{2}{5^2}\cos 5x + \ldots\right]$$

$$+ \left[3\sin x - \frac{1}{2}\sin 2x + \frac{3}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots\right]$$

$$= -\frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right]$$

$$+ \left[3\sin x - \frac{1}{2}\sin 2x + \frac{3}{3}\sin 3x - \frac{1}{4}\sin 4x\right]$$

——— (iii)

compare (III) and (IV) we get,

$\cos x = 1, \quad \cos 3x = 1, \quad \cos 5x = 1$

$\Rightarrow \cos x = \cos 0, \quad \cos 3x = \cos 0, \quad \cos 5x = \cos 0$

$\therefore x = 0$

Here $f(x)$ is discontinuous at $x = 0$

$$\therefore f(0) = \frac{f(0-0) + f(-0+0)}{2}$$

$$= \frac{-\pi + 0}{2}$$

$$= -\pi/2$$

Now substituting $x = 0$ in (III), we get

$$-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\therefore \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Homework

Q-3 · Expand the function

$$f(x) = \begin{cases} x-\pi & \text{when } -\pi < x < 0 \\ \pi - x & \text{when } 0 < x < \pi \end{cases}$$

is fourier series and hence show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Q-4 · Expand the function $= \begin{cases} 0 & \text{when } -\pi < x < 0 \\ \frac{\pi x}{4} & \text{when } 0 < x < \pi \end{cases}$

in F.S and $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$

Even func^{l}

$f(-x) = f(x)$

Example: $f(x) = x^2$

$\therefore f(-x) = (-x)^2 = x^2 = f(x)$

odd func^{l}

$f(-x) = -f(x)$

Example: $f(x) = x^3$

$\therefore f(-x) = -x^3 = -f(x)$

fourier series $(-\pi, \pi)$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx,$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx$$

$$b_n = 0$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx\, dx$$

$$a_0 = a_n = 0$$

Q Expand in fourier series the func $f(x) = x \sin x$ in the interval $-\pi < x < \pi$ and hence show that

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\times 3} - \frac{1}{3\times 5} + \dots \quad \text{---(IV)}$$

⟶ Given that,

$f(x) = x \cdot \sin x$ ⌜Even func kmne

$\therefore f(-x) = x \cdot \sin(-x)$ bujhlam. bole to

$= x \sin x = f(x)$ dei nai

Hence $f(x)$ be an even func and by the definition of Fourier series of even func is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{---(1)}$$

Here, $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x \sin x\,dx$

$= \frac{2}{\pi}\left[x \cdot (-\cos x) - 1 \cdot (-\sin x)\right]_0^{\pi}$

$= \frac{2}{\pi} \cdot \pi \cdot (-\cos \pi)$

$= 2$

$a_n = \frac{2}{\pi}\int_0^{\pi} x \sin x \cdot \cos nx\,dx$

1 kmne ⌜ $= \frac{2}{\pi}\int_0^{\pi} \frac{x}{2}\left[\sin(1+n)x + \sin(1-n)x\right]dx$

$= \frac{1}{\pi}\left[x \cdot \frac{-\cos(1+n)x}{1+n} - 1 \cdot \frac{-\sin(1+n)x}{(1+n)^2} + x \cdot \frac{-\cos(1-n)x}{1-n} - 1 \cdot \frac{-\sin(1-n)x}{(1-n)^2}\right]_0^{\pi}$

$= \frac{1}{\pi}\left[-\frac{\pi}{1+n}\cos(1+n)\pi - \frac{\pi}{1-n}\cos(1-n)\pi\right]$ $\quad (-1)^{n+1}$

$= \frac{-\pi}{\pi}(-1)^{n+1}\left[\frac{1}{1+n} + \frac{1}{1-n}\right]$

$= (-1)^{n+1}\frac{1-n+1+n}{1-n^2}$

$= (-1)^{n}\frac{2}{1-n^2}$

Now, $a_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x\cdot\cos x\,dx$

$= \frac{1}{\pi}\int_0^{\pi} x\cdot\sin 2x\cdot dx$

$= \frac{1}{\pi}\left[x\cdot\frac{-\cos 2x}{2} - 1\cdot\frac{-\sin 2x}{4}\right]_0^{\pi}$

$= \frac{1}{\pi}\left[\pi\cdot\frac{-\cos 2\pi}{2}\right]$

$= -\frac{1}{2}$

From (ii), we have

$f(x) = \frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n\cos nx$

$= \frac{2}{2} - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}(-1)^n\frac{2}{1-n^2}\cos nx$

$= 1 - \frac{1}{2}\cos x + 2\left[-\frac{1}{3}\cos 2x + \frac{1}{8}\cos 3x - \frac{1}{15}\cos 4x + \cdots\right]$

$\therefore f(x) = 1 + 2\left[-\frac{1}{4}\cos x - \frac{1}{3}\cos 2x + \frac{1}{8}\cos 3x - \frac{1}{15}\cos 4x + \cdots\right]$ ——— (iii)

compare (iii) and (iv). we get,

$\cos 2x = -1$, $\cos 4x = 1$

$\cos 2x = \cos\pi$ $\quad \cos 4x = \cos 2\pi$

$\therefore x = \frac{\pi}{2}$

Substituting $x = \frac{\pi}{2}$ in (iii), we get.

$\frac{\pi}{2}\cdot\sin\frac{\pi}{2} = 1 + 2\left[-0 + \frac{1}{3} + 0 - \frac{1}{15} + \cdots\right]$

$\Rightarrow \frac{\pi}{2} = 2\left[\frac{1}{2} + \frac{1}{1\times 3} - \frac{1}{3\times 5} + \cdots\right]$

$\therefore \frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\times 3} - \frac{1}{3\times 5} + \cdots$

Homework

Qus Expand in fourier series the func $f(x) = x\cdot\cos x$ in the interval $-\pi < x < \pi$

06.01.25

Complex form of Fourier Series

If $f(x)$ is defined on $(-c, c)$ with period $2c$ then the Fourier series is given by..

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi x}{c}\right) \quad \text{---(1)}$$

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$i^2 = -1$ ehh? not like that

From (1), we get,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \frac{e^{\frac{in\pi x}{c}} + e^{\frac{-in\pi x}{c}}}{2} - (+i) b_n \frac{e^{\frac{in\pi x}{c}} - e^{\frac{-in\pi x}{c}}}{2}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[e^{\frac{in\pi x}{c}}\left(\frac{a_n}{2} - \frac{ib_n}{2}\right) + e^{\frac{-in\pi x}{c}}\left(\frac{a_n}{2} + \frac{ib_n}{2}\right)\right]$$

$$= c_0 + \sum_{n=1}^{\infty}\left(c_n e^{\frac{in\pi x}{c}} + c_{-n} e^{\frac{-in\pi x}{c}}\right)$$

where,

$$c_0 = \frac{a_0}{2} = \frac{1}{2c}\int_{-c}^{c} f(x)\,dx$$

$$c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2c}\int_{-c}^{c} f(x)\, e^{\frac{-in\pi x}{c}}\,dx$$

$$c_{-n} = \frac{1}{2}(a_n + ib_n) = \frac{1}{2c}\int_{-c}^{c} f(x)\, e^{\frac{in\pi x}{c}}\,dx$$

Q. Find the complex form of the Fourier series of the periodic function whose definition in one period is $f(x) = e^{-x}$, $-1 < x < 1$

$a = -1$, $b = 1$

→ By the definition of Fourier series in complex, we have,

$$f(x) = c_0 + \sum_{n=1}^{\infty}\left(c_n e^{in\pi x} + c_{-n} e^{-in\pi x}\right)$$

here,

$$c_0 = \frac{1}{2c}\int_{-c}^{c} f(x)\,dx = \frac{1}{2}\int_{-1}^{1} e^{-x}\,dx = \frac{1}{2}\left[-e^{-x}\right]_{-1}^{1} = \frac{e^1 - e^{-1}}{2}$$

$c = \frac{b-a}{2} = \frac{1-(-1)}{2} = 1$ $\quad = \sinh 1$

Now,

$$c_n = \frac{1}{2}\int_{-1}^{1} e^{-x}\cdot e^{\pm in\pi x}\,dx$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-(1\mp in\pi)x}\,dx$$

$$= \frac{1}{2}\left[-\frac{e^{-(1\mp in\pi)x}}{1\mp in\pi}\right]_{-1}^{1}$$

$$= \frac{1}{(1\mp in\pi)}\left[\frac{e^{1\mp in\pi} - e^{-(1\mp in\pi)}}{2}\right]$$

$$= \frac{(-1)^n}{1\mp in\pi}\left[\frac{e^1 - e^{-1}}{2}\right]$$

$$= \frac{(-1)^n}{1\mp in\pi}\sinh 1$$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$\sinh x = \frac{e^x - e^{-x}}{2}$

$e^{\pm in\pi} = \cos n\pi \pm i\sin n\pi$

$= (-1)^n \pm i\times 0$

$e^{in\pi} = \cos n\pi + i\sin n\pi$

$e^{-in\pi} = \cos n\pi - i\sin n\pi$

$\left[e(-1)^n - e^{-1}(-1)^n\right]$

$\left[e\left(\cos n\pi + i\sin n\pi\right) - e^{-1}\left(\cos n\pi - i\sin n\pi\right)\right]$

previous line

From (1), we get.

$$f(x) = \sinh 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{1+in\pi} \sinh 1 \cdot e^{in\pi x} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1-in\pi} \sinh 1 \cdot e^{-in\pi x}$$

13.01.25

Fourier Integral

Statement: If a function satisfied the Dirichlet's condition in the interval $-l < x < l$ and $\int_{-\infty}^{\infty} f(x)dx$ is absolute convergent then for every continuous point.

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cdot \underbrace{\cos u(x-t)}_{\text{single term}} du\,dt$$

Different form of fourier integral

1. General form : $\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cos u(x-t)\, du\,dt$
2. For even function : $\frac{2}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} f(t) \cos ux \cdot \cos ut \cdot du\,dt$
3. For odd function : $\frac{2}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} f(t) \cdot \sin ux \cdot \sin ut \cdot du\,dt$

Complex form of fourier Integral

By the definition of fourier Integral, we get

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cdot \cos u(x-t) \cdot du \cdot dt \quad \text{---(1)}$$

Here $\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \sin u(x-t) \cdot du \cdot dt$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \left\{ \int_{-\infty}^{\infty} \sin u(x-t)\, du \right\} dt$$

we know,

$\int_{-a}^{a} \varphi(u)\, du = 0$ when $\varphi(-u) = -\varphi(u)$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \cdot 0 \, du\,dt$$

$$= 0$$

From (1), we get,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cos u(x-t)\, du\,dt + \frac{0i}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \sin u(x-t) \cdot du \cdot dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \left[\cos u(x-t) + i \sin u(x-t) \right] du\,dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{iu(x-t)}\, du \cdot dt$$

$\cos\theta + i\sin\theta = e^{i\theta}$

if even
$\int_{-a}^{a} \varphi(u)\, du = 2\int_{a}^{a} \varphi(u)\, du$

Evaluate the fourier integral of the function

$$f(x) = \begin{cases} 0 & \text{when } x<0 \\ \frac{1}{2} & \text{when } x=0 \\ e^{-x} & \text{when } x>0 \end{cases}$$

→ By the definition of fourier integral

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos u(x-t)\,du\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\left[\int_{-\infty}^{\infty}\cos u(x-t)\,du\right]dt$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\left[\int_{0}^{\infty}\cos u(x-t)\,du\right]dt$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\left[\int_{0}^{\infty}(\cos ux\cdot\cos ut + \sin ux\cdot\sin ut)\,du\right]dt$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} f(t)\cdot\cos ux\cdot\cos ut\,du\,dt + \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} f(t)\cdot\sin ux\sin ut\,du\,dt$$

$$\therefore f(x) = \frac{1}{\pi}\int_{0}^{\infty}\left\{\int_{-\infty}^{\infty} f(t)\cos ut\,dt\right\}\cos ux\,du + \frac{1}{\pi}\int_{0}^{\infty}\left\{\int_{-\infty}^{\infty} f(t)\sin ut\,dt\right\}\sin ux\,du \quad \text{——(I)}$$

Now, $\int_{-\infty}^{\infty} f(t)\sin ut\,dt$

$$= \int_{-\infty}^{0} 0\cdot\sin ut\,dt + \int_{0}^{\infty} e^{-t}\sin ut\,dt$$

$$= \left[\frac{e^{-t}}{1+u^2}(-\sin ut - u\cos ut)\right]_0^{\infty}$$

$$= \frac{u}{1+u^2}$$

$$\int_{-\infty}^{\infty} f(t)\cos ut\,dt$$

$$= \int_{-\infty}^{0} 0\cdot\cos ut\,dt + \int_{0}^{\infty} e^{-t}\cos ut\,dt$$

$$= \left[\frac{e^{-t}(-\cos ut + u\sin ut)}{1+u^2}\right]_0^{\infty}$$

$$= \frac{1}{1+u^2}$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

27.01.

Rest part

From (1), we have,

$$f(x) = \frac{1}{\pi}\left[\int_0^{\infty}\frac{\cos ux}{1+u^2}\,du + \frac{u\sin ux}{1+u^2}\,dx\right]$$

$$= \frac{1}{\pi}\int_0^{\infty}\frac{\cos ux + u\sin ux}{1+u^2}\,du \quad \text{——(II)}$$

putting $x=0$ in (II) we have,

$$f(0) = \frac{1}{\pi}\int_0^{\infty}\frac{1}{1+u^2}\,du$$

$$= \frac{1}{\pi}\left[\tan^{-1}u\right]_0^{\infty}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - 0\right] = \frac{1}{2}$$

hence eqn (II) is our required fourier integral.

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$$

Q2 Find the fourier integral of the func $f(x) = e^{-kx}$, $x>0$, $k>0$ and $f(-x) = f(x)$, hence show that

$$\int_0^\infty \frac{u \sin ux}{k^2+u^2}\,du = \frac{\pi}{2} e^{-kx}$$

⟶ here $f(-x) = f(x)$, hence $f(x)$ be an odd func

by the definition of fourier Integral for odd func, we have

$$f(x) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(t) \sin ux \cdot \sin ut \cdot du\,dt$$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_0^\infty \left\{ \int_0^\infty f(t) \sin ut\,dt \right\} \sin ux\,dx$$

Now, $\int_0^\infty f(t) \sin ut\,dt$

$$= \int_0^\infty e^{-kt} \sin ut\,dt$$

$$= \left[\frac{e^{-kt}(-k\sin ut - u\cos ut)}{k^2+u^2} \right]_0^\infty$$

$$= \frac{u}{k^2+u^2}$$

From (1), we get.

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{u \sin ux}{k^2+u^2}\,du$$

Now, $e^{-kx} = \frac{2}{\pi}\int_0^\infty \frac{u \sin ux}{k^2+u^2}\,du$

$$\therefore \int_0^\infty \frac{u \sin ux}{k^2+u^2}\,du = \frac{\pi}{2} e^{-kx}$$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_0^\infty \left\{ \int_0^\infty f(t) \sin ut\,dt \right\} \sin ux\,dx$$

Q3

Find the fourier Integral of the func $f(x) = e^{-kx}$, $x>0$, $k>0$, and $f(-x) = f(x)$. hence show that

$$\int_0^\infty \frac{\cos ux}{k^2+u^2}\,du = \frac{\pi}{2k} e^{-kx}$$

Fourier Transformation (compulsory)

If the func $f(x)$ be a continuous, piece-wise smooth and absolutely Integrable over $(-\infty, \infty)$ the Fourier Transformation of $f(x)$ is denoted by $f(\omega)$ and defined by.

$$\mathcal{F}[f(x)] = F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\,dx$$

The inverse of $F(\omega)$ is denoted by $\mathcal{F}^{-1}$

$\mathcal{F}^{-1}[F(\omega)]$ and defined by.

$\mathcal{F}[F(\omega)] = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega x}\, d\omega \rightarrow$

Diff with series ⟶ periodic

Piecewise continuous

$f(x)$ — $(-\infty, \infty)$

1. continuous except somepoint
2. point — limit exist

$\frac{df}{dx}$ piecewise smooth.

Fourier Integral — ~~ft~~ math + complex func / form

this tuesday-class time

40 page lec — 2 set qns

xm mid or sunday

Q-1

Find the fourier Transformation of the function.

$$f(x) = \begin{cases} 1-x^2 & \text{when } |x|<1 \\ 0 & \text{when } |x|>1 \end{cases}$$

also find its inversion and hence show that.

$$\int_{-\infty}^{\infty} \left(\frac{1}{\alpha^3} \sin\alpha - \frac{1}{\alpha^2}\cos\alpha \right) d\alpha = \frac{\pi}{2}$$

⟶ By the definition of fourier transformation, we have,

$$\mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx \quad \text{——— (1)}$$

Here,

$$f(x) = \begin{cases} 1-x^2 & \text{when } |x|<1 \\ 0 & \text{when } |x|>1 \end{cases}$$

$$= \begin{cases} 1-x^2 & \text{when } -1<x<1 \\ 0 & \text{when } x>1 \text{ or } x<-1 \end{cases} \quad \text{——— (11)}$$

from (1), we get.

$$\mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-1} 0 \cdot e^{i\alpha x} dx + \int_{-1}^{1} (1-x^2) e^{i\alpha x} dx + \int_{1}^{\infty} 0 \cdot e^{i\alpha x} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[(1-x^2) \frac{e^{i\alpha x}}{i\alpha} - (0-2x) \frac{e^{i\alpha x}}{(i\alpha)^2} + (-2) \frac{e^{i\alpha x}}{(i\alpha)^3} \right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(\frac{2e^{i\alpha}}{i^2\alpha^2} + \frac{2e^{-i\alpha}}{i^2\alpha^2} \right) - \left(\frac{2e^{i\alpha}}{i^3\alpha^3} - \frac{2e^{-i\alpha}}{i^3\alpha^3} \right) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{-4}{\alpha^2} \left(\frac{e^{i\alpha} + e^{-i\alpha}}{2} \right) + \frac{4}{\alpha^3} \left(\frac{e^{i\alpha} - e^{-i\alpha}}{2i} \right) \right].$$

$\therefore F[f(x)] = F(\alpha) = \frac{1}{\sqrt{2\pi}}\left(-\frac{4}{\alpha^2}\cos\alpha + \frac{4}{\alpha^3}\sin\alpha\right)$ ——— (iii)

Now. taking fourier Inverse Transformation on (iii). we get.

$\left|\sin\alpha = \frac{e^{\alpha} - e^{-\alpha}}{2i}\right.$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\alpha)\, e^{-i\alpha x} d\alpha$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{4}{\sqrt{2\pi}}\left[\frac{1}{\alpha^3}\sin\alpha - \frac{1}{\alpha^2}\cos\alpha\right] e^{-i\alpha x}\, d\alpha$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty}\left[\frac{1}{\alpha^3}\sin\alpha - \frac{1}{\alpha^2}\cos\alpha\right](\cos\alpha x - i\sin\alpha x)\, d\alpha$$ ——— (iv)

Now equating real part and imaginary part, from (iv), we get.

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\alpha^3}\sin\alpha - \frac{1}{\alpha^2}\cos\alpha\right)\cos\alpha x\, d\alpha = \begin{cases} 1-x^2 & ; -1 < x < 1 \\ 0 & ; x > 1 \text{ or } x < -1 \end{cases}$$ ——— (v)

and $\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\alpha^3}\sin\alpha - \frac{1}{\alpha^2}\cos\alpha\right)\sin\alpha x\, d\alpha = 0$ ——— (vi)

Substituting $x = 0$ in (v) we get,

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\left[\frac{1}{\alpha^3}\sin\alpha - \frac{1}{\alpha^2}\cos\alpha\right] d\alpha = 1$$

$$\therefore \int_{-\infty}^{\infty}\left(\frac{1}{\alpha^2}\sin\alpha - \frac{1}{\alpha^2}\cos\alpha\right) d\alpha = \frac{\pi}{2}$$

Find the fourier Transformation

$$f(x) = \begin{cases} 1 & ; |x| < a \\ 0 & ; |x| > a \end{cases}$$

also find its inverse hence show that

$$\int_{-\infty}^{\infty} \frac{\sin\alpha}{\alpha}\, d\alpha = \frac{\pi}{2}$$

Fourier Series Transformation

10.02

If the func. $f(x)$ is defined $(0, \infty)$ and it is piece-wise smooth and absolutely integrable then

$$f_s[f(x)] = F_s(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin\alpha x\, dx$$

with inversion,

$$f_s^{-1}[F_s(\alpha)] = f(x) = \frac{\sqrt{2}}{\pi}\int_0^{\infty} F_s(\alpha)\sin\alpha x\, dx$$

Fourier cosine Transformation

If the function $f(x)$ is defined on $(0, \infty)$ and it is piece-wise smooth and absolutely integrable then.

$$f_c[f(x)] = F_c(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos\alpha x\, dx$$

with inversion,

$$f_c^{-1}[F_c(\alpha)] = f(x) = \frac{\sqrt{2}}{\pi}\int_0^{\infty} F_s(\alpha)\cos\alpha x\, dx$$

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Finite Fourier Sine Transformation

If the function $f(x)$ is defined on $(0,l)$ and it is piecewise smooth and absolutely integrable then.

$$f_s[f(x)] = F_s(\alpha) = \int_0^l f(x) \sin\frac{\alpha\pi x}{l}\, dx$$

with inversion.

$$f_s^{-1}[F_s(\alpha)] = f(x) = \frac{2}{l}\sum_{\alpha=1}^{\infty} F_s(\alpha)\sin\alpha x$$

Finite Fourier cosine Transformation

$$f_c[f(x)] = F_c(\alpha) = \int_0^l f(x)\cos\frac{\alpha\pi x}{l}\, dx$$

$$f_c^{-1}[F_c(\alpha)] = f(x) = \frac{1}{l}F_c(0) + \sum_{\alpha=1}^{\infty} F_c(\alpha)\cos\alpha x$$

where, $F_c(0) = \int_0^l f(x)\,dx$

$u(x,t)$ — boundary — initial
- $u(0,t)$, $u(2,t)$ — Sine trans. use
- $u_x(0,t)$, $u_x(2,t)$ — cosine "

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$$

Fourier Complex Transformation

Notes - compulsory — 20 marks

If the fun^c $f(x)$ is defined on $(-\infty,\infty)$ and it is piecewise smooth and absolutely integrable then.

$$f[f(x)] = F(\alpha) = \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$$

with inversion.

$$f^{-1}[F(\alpha)] = f(x) = \frac{2}{\pi}\int_0^{\infty} F(\alpha)\, e^{i\alpha x}\, d\alpha$$

Q3. Find the Fourier sine transformation of $f(x) = e^{-x}$.

⟶ By the definition of Fourier sine transformation we get,

$$f_s[f(x)] = F_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin \alpha x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}(-\sin \alpha x - \alpha \cos \alpha x)}{1+\alpha^2} \right]_0^\infty$$

$$= \sqrt{2/\pi} \, \frac{\alpha}{1+\alpha^2}$$

I.C $= U(x,0) = x$

B.C $= U(0,t) = U(l,t) = 0$

17.02

Q1 use the method of fourier transform to determine the temperature $U(x,t)$ at any point pt x at an instant of time t in a solid bounded by the plane $x=0$ and $x=2$. At the boundary ends the temperature of solid is zero while the initial temperature is $2x$.

⟶ According to this problem we have to solve one dimensional heat eqn —

$$\frac{\delta U}{\delta t} = c^2 \frac{\delta^2 U}{\delta x^2} \quad \text{—— (1)}$$

Boundary condition : $U(0,t) = U(2,t) = 0$

Initial condition : $U(x,0) = 2x$

Now taking finite fourier sine transformational in both sides of (1) we have,

$$f_s\left[\frac{\delta u}{\delta t}\right] = f_s\left[c^2 \frac{\delta^2 u}{\delta x^2}\right]$$

$$\Rightarrow \int_0^2 \frac{\delta u}{\delta t} \sin \frac{2\pi x}{2} \, dx = c^2 \int_0^2 \frac{\delta^2 u}{\delta x^2} \sin \frac{2\pi x}{2} \, dx$$

$$\Rightarrow \frac{d}{dt} \int_0^2 U(x,t) \sin \frac{2\pi x}{2} \, dx = c^2 \int_0^2 \frac{\delta^2 u}{\delta x^2} \sin \frac{2\pi x}{2} \, dx \quad \text{—— (11)}$$

Let, $\int_0^2 u(x,t) \sin\frac{\alpha\pi x}{2}\,dx = v(\alpha,t)$ ——— (iii)

From (ii), we have,

$$\frac{dv(\alpha,t)}{dt} = c^2\int_0^2 \frac{\delta^2 u}{\delta x^2}\sin\frac{\alpha\pi x}{2}\,dx$$

$$= c^2\left[\sin\frac{\alpha\pi x}{2}\cdot\frac{\delta u}{\delta x} - \frac{\alpha\pi}{2}\int\cos\frac{\alpha\pi x}{2}\cdot\frac{\delta u}{\delta x}\,dx\right]_0^2$$

$$= -\frac{c^2\alpha\pi}{2}\left[\cos\frac{\alpha\pi x}{2}\,u(x,t) + \frac{\alpha\pi}{2}\int\sin\frac{\alpha\pi x}{2}\,u(x,t)\,dx\right]_0^2$$

$$\therefore \frac{dv}{dt} = -\frac{c^2\alpha^2\pi^2}{4}\int_0^2 u(x,t)\sin\frac{\alpha\pi x}{2}\,dx$$

$$\Rightarrow \frac{dv}{dt} = -\frac{c^2\alpha^2\pi^2}{4}v$$

$$\Rightarrow \int\frac{dv}{v} = -\frac{c^2\alpha^2\pi^2}{4}\int dt$$

$$\Rightarrow \ln v = -\frac{c^2\alpha^2\pi^2}{4}t + \ln A$$

$$\Rightarrow v(\alpha,t) = Ae^{-\frac{c^2\alpha^2\pi^2}{4}t} \quad \text{——— (iv)}$$

Now substitution $t=0$ in (iv), we have

$$v(\alpha,0) = A$$

$$\therefore A = \int_0^2 u(x,0)\sin\frac{\alpha\pi x}{2}\,dx$$

$$= \int_0^2 2x\cdot\sin\frac{\alpha\pi x}{2}\,dx$$

$$= 2\left[x\cdot\frac{-\cos\frac{\alpha\pi x}{2}}{\frac{\alpha\pi}{2}} - 1\cdot\frac{-\sin\frac{\alpha\pi x}{2}}{\left(\frac{\alpha\pi}{2}\right)^2}\right]_0^2$$

$$= -2\left[2\cdot\frac{2}{\alpha\pi}\cos\alpha\pi\right].$$

$$= -\frac{8}{\alpha\pi}(-1)^\alpha$$

$$= \frac{8}{\alpha\pi}(-1)^{\alpha+1}$$

From (v),

$$v(\alpha,t) = \frac{8}{\alpha\pi}(-1)^{\alpha+1}\cdot e^{-\frac{c^2\alpha^2\pi^2}{4}t} \quad \text{——— (v)}$$

Now taking finite Inverse Fourier sine transformation in (v), we have,

$$u(x,t) = \frac{2}{2}\sum_{\alpha=1}^{\infty} v(\alpha,t)\sin\frac{\alpha\pi x}{2}$$

$$= \sum_{\alpha=1}^{\infty}\frac{8}{\alpha\pi}(-1)^{\alpha+1}e^{-\frac{c^2\alpha^2\pi^2}{4}t}\cdot\sin\frac{\alpha\pi x}{2}$$

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Q-2

use the method of fourier transform. to determine the temperature $u(x,t)$ at any point x at an instant of time t in a solid bounded by the plane $x=0$ and $x=2$.

$u_x(0,t)$ and $u_x(2,t)$

At the boundary ends the temperature of solid zero while the initial temperature is x^2

⇒ Differential property,

$$f[f^n(x)] = (-i\alpha)^n f[f(x)]$$

$$f^n(x) = \frac{\partial^n f}{\partial x^n}$$

Q3

use the method of fourier transformation to determine the displacement $y(x,t)$ of an infinite string, given that the string is initially at rest and that the initial displacement is $f(x)$, $-\infty < x < \infty$

⟶ According to this problem, we have to solve one dimension wave eqn,

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{—— (i)}$$

Subject to the following condition,

$$y(x,0) = f(x) \quad \text{—— (ii)}$$

$$y_t(x,0) = 0 \quad \text{—— (iii)}$$

Now taking fourier complex transform on both sides of (i), we have,

$$F\left[\frac{\partial^2 y}{\partial t^2}\right] = F\left[c^2 \frac{\partial^2 y}{\partial x^2}\right]$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\partial^2 y}{\partial t^2} e^{-i\alpha x} dx = c^2 \int_{-\infty}^{\infty} \frac{\partial^2 y}{\partial x^2} e^{i\alpha x} dx$$

$$\Rightarrow \frac{d^2}{dt^2} \int_{-\infty}^{\infty} y(x,t) e^{-i\alpha x} dx = c^2 (-i\alpha)^2 \int_{-\infty}^{\infty} y(x,t) e^{-i\alpha x} dx$$

—— (iv)

$\therefore f[f^n(x)] = (-i\alpha)^n f[f(x)]$

let, $\int_{-\infty}^{\infty} y(x,t) e^{-i\alpha x} dx = Y(\alpha,t)$ ——— (v)

From (iv), we get,

$\frac{d^2Y}{dt^2} = -c^2\alpha^2 Y$

$\Rightarrow \frac{d^2Y}{dt^2} + c^2\alpha^2 Y = 0$ ——— (vi)

Here Auxilary eqn.

$D^2 + c^2\alpha^2 = 0$

$\Rightarrow D = \pm ic\alpha$

Hence the soln of (vi) is

$Y(\alpha,t) = A\cos c\alpha + B\sin c\alpha$ ——— (vii)

Now differentiating (vi) w.r to t,

$Y_t(\alpha,t) = -A\sin c\alpha t + B\cos c\alpha \cos c\alpha t$ ——— (viii)

from (v).

$Y(\alpha,0) = \int_{-\infty}^{\infty} y(x,0) e^{-i\alpha x} dx$

$= \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx$

$= F(\alpha)$ (let)

and $Y_t(\alpha,0) = \int_{-\infty}^{\infty} y_t(x,0) e^{-i\alpha x} dx$

$= \int_{-\infty}^{\infty} 0 \cdot e^{-i\alpha x} dx = 0$

Now substituting $t = 0$, in (viii) & (ix)

$Y(\alpha,0) = A$
$\therefore A = F(\alpha)$

$Y_t(\alpha,0) = B c\alpha$
$\Rightarrow 0 = B\alpha$
$\therefore B = 0$ $[c\alpha \neq 0]$

from (vii), we have

$Y(\alpha,t) = F(\alpha)\cos c\alpha t$ ——— (ix)

Now taking Inverse Fourier complex transformation in both sides of (ix).

we have,

$y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)[\cos c\alpha t] e^{i\alpha x} d\alpha$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\left(\frac{e^{ic\alpha t} + e^{-ic\alpha t}}{2}\right) e^{i\alpha x} d\alpha$

$= \frac{1}{2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha) e^{i\alpha(x+ct)} d\alpha + \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha) e^{i\alpha(x-ct)} d\alpha\right]$

$= \frac{1}{2} f(x+ct) + \frac{1}{2} f(x-ct)$

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