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Legendre's Differention equⁿ And Legendre polynomial

Application:

1. Determination of → wave functions of electrons in the orbital an atom

→ Potential functions in the spherically symmetric geometry

Defⁿ

The differential equation:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$ is known as the legendre differential equation.

The resulting solution of legendre's equation is called the legendre polynomial of degree n (n be a positive integer) and is denoted by

$$P_n(x) = \sum_{r=0}^{N} (-1)^r \frac{(2n-2r)!}{2^n\, r!\,(n-r)!\,(n-2r)!}\, x^{n-2r}$$

$N = \frac{n}{2}$ if n be an even number

$N = \frac{n-1}{2}$ if n be an odd number

⊛ when $n=0$ then $N=0$, $r=0$ → আসতে পারে final-এ 2 marks

$$P_0(x) = (-1)^0 \frac{(0-0)!}{2^0\, 0!\,(0-0)!\,(0-0)!}\, x^0 = 1$$

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⊕ when $n=1$ (1 is odd) then $N=0$, $\pi=0$

$$P_1(x) = (-1)^0 \frac{(2-0)}{2^1\, 0!\, (1-0)!\, (1-0)!}\, x^{1-0} = \frac{2}{2}\, x = x$$

⊕ when $n=2$ (2 is even) then $N=1$,

$$P_2(x) = (-1)^0 \frac{4!}{2^2 \cdot 0! \cdot 2! \cdot 2!}\, x^{2-0} + (-1)^1 \frac{(4-2)!}{2^2 \cdot 1! \cdot 1! \cdot 1!}\, x^{2-2}$$

$$= \frac{1}{2}(3x^2-1)$$

H.W → $P_3 - P_7$ [P_4 পর্যন্ত মুখস্ত করতে হবে]

* First five Legendre's polynomials

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2-1)$$

$$P_3(x) = \frac{1}{2}(5x^3-3x)$$

$$P_4(x) = \frac{1}{4\cdot 2}(7\cdot 5x^4 - 2\cdot 5\cdot 3\cdot x^2 + 3\cdot 1)$$

⊕ Rodrigue's form for legendre polynomial's

$$P_n(x) = \frac{1}{2^n \cdot n!} \frac{d^n}{dx^n}(x^2-1)^n$$

Sol^n: By the definition of ligendre polynomial, we have

$$P_n(x) = \sum_{r=0}^{N} (-1)^r \frac{(2n-2r)!}{2^n\, r!\, (n-r)!\, (n-2r)!}\, x^{n-2r} \quad \text{---(1)}$$

where $N = \frac{n}{2}$ if n is even

$N = \frac{n-1}{2}$ if n is odd

now,

$$(x^2-1)^n = (x^2)^n - {}^nC_1 (x^2)^{n-1}\cdot 1 - {}^nC_2 (x^2)^{n-2}\cdot 1^2 - {}^nC_3 (x^2)^{n-3} + \dots$$

$$= \sum_{r=0}^{n} (-1)^r \, {}^nC_r \, (x^2)^{n-r} \cdot 1^r$$

$$= \sum_{r=0}^{n} (-1)^r \frac{n!}{r!\,(n-r)!} x^{2n-2r}$$

$$\therefore \frac{d^n}{dx^n}(x^2-1)^n = \sum_{r=0}^{n} \frac{(-1)^r\, n!}{r!\,(n-r)!} \frac{d^n}{dx^n}\left(x^{2n-2r}\right)$$

$$\left| \frac{d^n}{dx^n}(x^m) = \frac{m!}{(m-n)!} x^{m-n} \qquad m \geq n \right.$$

$$= \sum_{r=0}^{n} \frac{(-1)^r\, n!\,(2n-2r)}{r!\,(n-r)!\,(2n-2r-n)!} x^{2n-2r-n}$$

$$= \sum_{r=0}^{n} \frac{(-1)^r n!\,(2n-2r)!}{r!\,(n-r)!\,(n-2r)!} x^{n-2r} = n!\, 2^n \sum_{r=0}^{n} \frac{(-1)^r (2n-2r)!}{2^n\, r!\,(n-r)!\,(n-2r)!} x^{n-2r}$$

$$= n!\, 2^n P_n(x)$$

$$\boxed{P_n(x) = \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{2^n\, r!\,(n-r)!\,(n-2r)!} x^{n-2r}}$$

$$\boxed{P_n(x) = \frac{1}{n!\,2^n} \frac{d^n}{dx^n}(x^2-1)^n}$$

★ Find $P_0(x)$, $P_1(x)$ ... $P_5(x)$ by Rodrigue's formula (H.W)

$$P_n(x) = \frac{1}{2^n\, n!} \frac{d^n}{dx^n}(x^2-1)^n$$

$$P_2(x) = \frac{1}{2^2 \cdot 2!} \cdot \frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8} \frac{d^2}{dx^2}(x^4 - 2x^2 + 1)$$

$$= \frac{1}{8} \frac{d}{dx}\left\{4x^3 - 4x + 0\right)$$

$$= \frac{1}{8}(12x^2 - 4) = \frac{1}{2}(3x^2 - 1)$$

Q. Using Rodrigues formula, show that,

$$\int_{-1}^{1} x^n P_n(x) = \frac{\sqrt{1/2} \cdot \sqrt{n+1}}{2^n \cdot \sqrt{n+3/2}}$$

Sol^n: By the definition of Rodrigues formula for legendry polynomial we have,

$$P_n(x) = \frac{1}{2^n . n!} \frac{d^n}{dx^n} (x^2-1)^n$$

$$\therefore \int_{-1}^{1} x^n P_n(x)\, dx$$

$$= \int_{-1}^{1} \underbrace{x^n}_{u} \frac{1}{2^n . n!} \underbrace{\frac{d^n}{dx^n} (x^2-1)^n dx}_{v}$$

$$= \frac{1}{2^n . n!} \left[x^n \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n - nx^{n-1} \frac{d^{n-2}}{dx^{n-2}} (x^2-1)^n + (n)(n-1) x^{n-2} \frac{d^{n-3}}{dx^{n-3}} (x^2-1)^n \right.$$

(when limits applied)

$$\left. - \cdots\cdots + (-1)^n n.(n-1) \cdots 3.2 \frac{d^0}{dx^0} (x^2-1)^n + (-1)^n n! \int (x^2-1)^n dx \right]_{-1}^{1}$$

$$= \frac{(-1)^n}{2^n} \int_{-1}^{1} (x^2-1)^n dx$$

$$= \frac{(-1)^n (-1)^n}{2^n} \int_{-1}^{1} (1-x^2)^n dx$$

$(1-x^2)^n \rightarrow$ even number

$$= \frac{1}{2^n} \int_{-1}^{1} (1-x^2)^n dx$$

$$= \frac{1}{2^n} \cdot 2 \int_{0}^{1} (1-x^2) dx \quad [\because (1-x^2) = \text{even function}]$$

$$= \frac{1}{2^{n-1}} \int_{0}^{1} (1-x^2)^n dx$$

let
$x = \sin\theta$
$dx = \cos\theta\, d\theta$
$x = 0 \rightarrow \theta = 0$
$x = 1 \rightarrow \theta = \pi/2$

$$= \frac{1}{2^{n-1}} \int_{0}^{\pi/2} (1-\sin^2\theta)^n \cos\theta\, d\theta$$

$$= \frac{1}{2^{n-1}} \int_0^{\pi/2} \cos^{2n}\theta \cdot \cos\theta \, d\theta$$

$$= \frac{1}{2^{n-1}} \int_0^{\pi/2} \cos^{2n+1}\theta \cdot \sin^0\theta \, d\theta$$

$$= \frac{1}{2^{n-1}} \frac{\sqrt{\frac{2n+1+1}{2}} \cdot \sqrt{\frac{0+1}{2}}}{2\sqrt{\frac{2n+1+0+2}{2}}}$$

$$= \frac{1}{2^n} \frac{\sqrt{1/2} \cdot \sqrt{n+1}}{\sqrt{n+3/2}}$$

$$\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \, d\theta = \frac{\sqrt{\frac{p+1}{2}} \sqrt{\frac{q+1}{2}}}{2\sqrt{\frac{p+q+2}{2}}}$$

★ Generating function of the legendres polynomial $(1 - 2xh + h^2)^{-1/2}$

Hence, $(1 - 2xh + h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$

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Q. Express $f(x) = x^4 + 2x^3 + 2x^2 - x - 3$ in terms of legendre polynomial.

Soln: Given function,

$$f(x) = x^4 + 2x^3 + 2x^2 - x - 3 \quad ---(1)$$

we know that,

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

now,

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$\Rightarrow 2P_2(x) = 3x^2 - 1$$

$$\Rightarrow x^2 = \frac{2P_2(x) + 1}{3}$$

$$\therefore x^2 = \frac{1}{3}\{2P_2(x) + P_0(x)\}$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$\Rightarrow 2P_3(x) = 5x^3 - 3x$$

$$\Rightarrow x^3 = \frac{1}{5}\{2P_3(x) + 3x\}$$

$$\therefore x^3 = \frac{1}{5}\{2P_3(x) + 3P_1(x)\}$$

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$$P_4(x) = \frac{1}{8}\left[35x^4 - 30x^2 + 3\right]$$

$$\Rightarrow 8P_4(x) = 35x^4 - 30x^2 + 3$$

$$\Rightarrow x^4 = \frac{1}{35}\left\{8P_4(x) + 30x^2 - 3\right\}$$

$$= \frac{1}{35}\left\{8P_4(x) + 30\times\frac{2}{3}P_2(x) + 30\times\frac{1}{3}P_0(x) - 3P_0(x)\right\}$$

$$= \frac{1}{35}\left\{8P_4(x) + 20P_2(x) + 7P_0(x)\right\}$$

Substituting this values in ①, we have

$$f(x) = \frac{8}{35}P_4'(x) + \frac{4}{5}P_3(x) + \frac{40}{21}P_2(x) + \frac{1}{5}P_1(x) - \frac{32}{15}P_0(x)$$

Formula

⊛ Orthogonal property of legendre polynomial

$$\int_{-1}^{1} P_m(x)\cdot P_n(x)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{2}{2n+1} & \text{if } m = n \end{cases}$$

⊛ Recurrence relation for legendre polynomial

1. $(2n+1)\,xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$

2. $xP_n'(x) = nP_n(x) + P_{n-1}'(x)$

3. $(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$

4. $(n+1)P_n(x) = P_{n+1}'(x) - xP_n'(x)$

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Q. Prove that $\int_{-1}^{1} x P_n(x) P_{n-1}(x)\,dx = \frac{2n}{4n^2-1}$

Soln: For the first recurrence relation, we have

$$(2n+1)\,x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x) \quad \ldots\ldots (1)$$

Multiplying both sides of (1) by P_{n-1}, we have

$$(2n+1)\,x P_n(x) P_{n-1}(x) = (n+1) P_{n+1}(x) P_{n-1}(x) + n P_{n-1}(x) P_{n-1}(x) \quad \ldots\ldots (2)$$

Now integrating (2) with respect to x $(-1, 1)$, we have

$$(2n+1)\int_{-1}^{1} x P_n(x) \cdot P_{n-1}(x)\,dx = (n+1)\int_{-1}^{1} P_{n+1}(x) P_{n-1}(x)\,dx + n\int_{-1}^{1} [P_{n-1}(x)]^2\,dx$$

(from orthogonal property)

$$\Rightarrow (2n+1)\int_{-1}^{1} x P_n(x) P_{n-1}(x)\,dx = (n+1)\cdot 0 + n \cdot \frac{2}{2(n-1)+1}$$

$$\Rightarrow \int_{-1}^{1} x P_n(x) P_{n-1}(x)\,dx = \frac{2n}{(2n-1)(2n+1)} = \frac{2n}{4n^2-1}$$

Q. $P_n(x) = (-1)^n P_n(x)$

Soln: By the definition of Generating function for legendre polynomial -

$$(1 - 2xh + h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x) \quad \text{---} (1)$$

Substituting $h = -h$ in (1), we have

$$(1 + 2xh + h^2)^{-1/2} = \sum_{n=0}^{\infty} (-h)^n P_n(x) = \sum_{n=0}^{\infty} (-1)^n (h)^n P_n(x) \quad \text{---} (11)$$

Again substituting $x = -x$ in (1) we have,

$$(1 + 2xh + h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(-x) \quad \text{---} (111)$$

From (11) and (111), we have

$$\sum_{n=0}^{\infty} (-1)^n h^n P_n(x) = \sum_{n=0}^{\infty} h^n P_n(-x)$$

Equating the coefficien of h^n, we have,

$$P_n(-x) = (-1)^n P_n(x)$$

H.W. → $P_{2n}(\ ) = (-1)^n \dfrac{(2n)!}{2^{2n} \cdot (n!)^2}$

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⊛ Legendre series:

General form of legendre series is

$$f(x) = \sum_{n=0}^{n} c_n P_n(x)$$

where

$$c_n = \left(n+\frac{1}{2}\right) \int_{-1}^{1} f(x) P_n(x) dx$$

Q $f(x) = \begin{cases} 0 ; & -1 < x < 0 \\ x ; & 0 < x < 1 \end{cases}$

$$c_n = \left(n+\frac{1}{2}\right) \int_{+0}^{1} x P_n(x) dx$$

$$c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) + c_3 P_3(x) + \cdots$$

$$c_0 = \frac{1}{2} \int_0^1 x \cdot 1 \, dx$$

$$c_1 = \left(1+\frac{1}{2}\right) \int_0^1 x \cdot x \, dx = \frac{3}{2} \int_0^1 x^2 dx$$