COLLEGE

MATHEMATICAL METHODS

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VOLUME TWO

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INTEGRAL TRANSFORMS AND INITIAL & BOUNDARY VALUE PROBLEMS WITH APPLICATIONS

$$L\{F(t)\}=\int_0^\infty e^{-st}F(t)dt, t>0$$

$$-y=f(x), y(0)=0, y(1)=0$$

$$y=\int_0^1 G(x,s)f(s)ds$$

$$\text{where } G(x,s)=\begin{cases} s(1-x), 0\le s\le x \\ x(1-s), x\le s\le 1. \end{cases}$$

NAHAR BOOK DEPOT & PUBLICATIONS
BANGLABAZAR, DHAKA

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CHAPTER ONE

THE LAPLACE TRANSFORM

1.1 Introduction

Laplace transform (or **Laplace transformation**) is a part of operational calculus which is a powerful mathematical method (or technique) for solving ordinary and partial differential equations and also initial and boundary value problems of differential equations arising in Mathematics, Physics, Engineering and other science subjects. This subject was mainly originated in the work of **Heaviside** who found it useful to solve the equation of electromagnetic theory in the latter part of the 19th century. It reduces the problem of solving differential equation to mere algebraic manipulation.

1.2 Definition of integral transform

Let K(s, t) be a function of two variables s and t where s is a parameter (may be real or complex) independent of t. The function f(s) defined by the integral (assumed to be convergent) $f(s) = \int_{-\infty}^{\infty} K(s,t)\, F(t)dt$ is called the **integral transform** of the function F(t) and is denoted by T{F (t)}.

The function K(s,t) is sometimes called the **Kernel** of the transformation.

*[**PIERRE SIMON DE LAPLACE** (1749-1827) was a great French mathematician, who developed the foundation of potential theory and made important contributions to the celestial mechanics and probability theory]*

1.3 Definition of Laplace transform

If the **Kernel** K(s,t) is defined as

$$K(s,t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-st} & \text{for } t \geq 0 \end{cases}$$

then $f(s) = \int_0^{\infty} e^{-st} F(t)\, dt$ (1)

Method–1

The function f(s) defined by the integral (1) is called the **Laplace transform** of the function F(t) and is denoted by $\mathcal{L}\{F(t)\}$.

Alternatively, let F(t) be a function of t defined for all $t > o$. Then the **Laplace transform** of F(t), denoted by $\mathcal{L}\{F(t)\}$ or f(s), is defined by $\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)dt$ (1)

provided that the integral (1) exists. 's' is a parameter which may be real or complex number. The **Laplace transform** of F(t) is said to exist if the integral $\int_0^\infty e^{-st} F(t)dt$ converges for some value of s; otherwise it does not exist.

1.4 Laplace transforms of some elementary functions Laplace transforms of several elementary functions are determined just by the direct application of definition of the Laplace transform.

Example 1. Find the Laplace transforms of the following elementary functions :

(i) $F(t) = 1$, (ii) $F(t) = t$, (iii) $F(t) = t^n$, $n = 0, 1, 2, 3, \ldots$.

(iv) $F(t) = e^{at}$, (v) $F(t) = \sin at$, (vi) $F(t) = \cos at$, (vii) $F(t) = \sin h\, at$ and (viii) $F(t) = \cos h\, at$.

Solutions : By definition of the Laplace transform of a function F(t), we have $\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt$ (1)

(i) When $F(t) = 1$, equation (1) becomes

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1dt = \int_0^\infty e^{-st} dt = \lim_{p\to\infty} \int_0^p e^{-st} dt.$$

$$= \lim_{p\to\infty} \left[-\frac{1}{s} e^{-st}\right]_0^p = \lim_{p\to\infty} \left[-\frac{1}{s} e^{-sp} + \frac{1}{s}\right]$$

$$= \frac{1}{s} - 0 = \frac{1}{s} \text{ if } s > o.$$

(ii) When $F(t) = t$, equation (1) becomes

$$\mathcal{L}(t) = \int_0^\infty e^{-st} \cdot t\,dt = \lim_{p\to\infty} \cdot \int_0^\infty t \cdot e^{-st}\,dt$$

(Integrating by parts)

$$= \lim_{p\to\infty} \left[\frac{-t\,e^{-st}}{s}\right]_0^p + \lim_{p\to\infty} \frac{1}{s} \int_0^p e^{-st}\,dt$$

$$= \lim_{p\to\infty} \frac{-p}{se^{ps}} + 0 - \lim_{p\to\infty} \frac{1}{s^2}\left[e^{-st}\right]_0^p$$

$$= 0 - \lim_{p\to\infty} \cdot \frac{1}{s^2}\left[\frac{1}{e^{sp}} - 1\right] = \frac{1}{s^2} \text{ if } s>0.$$

(iii) When $F(t) = t^n$, equation (1) becomes

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n dt \quad (2)$$

Putting $st = y$ so that $sdt = dy \therefore dt = \frac{1}{s}dy$.

Also $t^n = \frac{y^n}{s^n}$. **Limits** $\begin{matrix} t = 0 \\ y = 0 \end{matrix}\Big\}$ $\begin{matrix} t = \infty \\ y = \infty \end{matrix}\Big\}$

Thus from (2) we have

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-y} \frac{y^n}{s^n} \cdot \frac{dy}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-y} y^n\,dy \text{ (using gamma function)}$$

$$= \frac{1}{s^{n+1}} \overline{|(n+1)} \quad \text{if } n > -1 \text{ and } s > 0$$

Again we know that if n be a positive integer, then

$\overline{|(n+1)} = \underline{|n}$

Hence $\mathcal{L}\{t^n\} = \frac{\underline{|n}}{s^{n+1}}$, when $n = 0, 1, 2, 3, \ldots\ldots\ldots$

(iv) When $F(t) = e^{at}$, equation (1) becomes

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at} dt = \int_0^\infty e^{-(s-a)t}\,dt$$

$$= -\frac{1}{(s-a)}\left[e^{-(s-a)t}\right]_0^\infty$$

$$= -\frac{1}{(s-a)}(0-1) = \frac{1}{s-a} \text{ if } s > a.$$

(v) When F(t) = sin at, equation (1) becomes.

$$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st} \sin at \, dt.$$

$$= \lim_{p\to\infty} \int_0^p e^{-st} \sin at \, dt.$$

$$= \lim_{p\to\infty} \left[\frac{e^{-st}(-s \sin at - a \cos at)}{s^2 + a^2} \right]_0^p$$

$$= \lim_{p\to\infty} \left[\frac{e^{-sp}(-s \sin ap - a \cos a p)}{s^2 + a^2} + \frac{a}{s^2 + a^2} \right]$$

$$= 0 + \frac{a}{s^2 + a^2} = \frac{a}{s^2 + a^2} \text{ if } s > 0.$$

(vi) When F(t) = cosat, equation (1) becomes

$$\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st} \cos at \, dt.$$

$$= \lim_{p\to\infty} \int_0^p e^{-st} \cos at \, dt.$$

$$= \lim_{p\to\infty} \left[\frac{e^{-st}(-s \cos at + a \sin at)}{s^2 + a^2} \right]_0^p$$

$$= \lim_{p\to\infty} \left[\frac{e^{-sp}(-s \cos ap + a \sin a p)}{s^2 + a^2} + \frac{s}{s^2 + a^2} \right]$$

$$= 0 + \frac{s}{s^2 + a^2} = \frac{s}{s^2 + a^2} \text{ if } s > 0.$$

(vii) When F(t) = sinh at, equation (1) becomes

$$\mathcal{L}\{\sinh at\} = \int_0^\infty e^{-st} \sin hat \, dt.$$

$$= \int_0^\infty e^{-st} \cdot \frac{1}{2} \left(e^{at} - e^{-at} \right) dt$$

$$= \frac{1}{2} \int_0^\infty e^{-st} \cdot e^{at} \, dt - \frac{1}{2} \int_0^\infty e^{-st} \cdot e^{-at} \, dt$$

$$= \frac{1}{2} \int_0^\infty e^{-(s-a)t} \, dt - \frac{1}{2} \int_0^\infty e^{-(s+a)t} \, dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty + \frac{1}{2}\left[\frac{e^{-(s+a)t}}{(s+a)}\right]_0^\infty$$

$$= \frac{1}{-2(s-a)}(0-1) + \frac{1}{2(s+a)}(0-1)$$

$$= \frac{1}{2(s-a)} - \frac{1}{2(s+a)} = \frac{a}{s^2-a^2} \text{ if } s > |a|$$

(viii) When $F(t) = \cosh at$, equation (1) becomes

$$\mathcal{L}\{\cosh at\} = \int_0^\infty e^{-st} \cosh at\, dt.$$

$$= \int_0^\infty e^{-st} \cdot \frac{1}{2}\left(e^{at} + e^{-at}\right) dt.$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-a)t}\, dt + \frac{1}{2}\int_0^\infty e^{-(s+a)t}\, dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty - \frac{1}{2}\left[\frac{e^{-(s+a)t}}{(s+a)}\right]_0^\infty$$

$$= \frac{1}{-2(s-a)}(0-1) - \frac{1}{2(s+a)}(0-1)$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{s}{s^2-a^2} \text{ if } s > |a|$$

Example 2. Find the Laplace transforms of the following functions:

(i) $F(t) = t \sin at$ (ii) $F(t) = t \cos at$.

Solution : By definition of the Laplace transform of a function $F(t)$, we have

$$\mathcal{L}\{F(t) = f(s) = \int_0^\infty e^{-st} F(t)\, dt \quad (1)$$

(i) When $F(t) = t \sin at$, equation (1) becomes

$$\mathcal{L}\{t \sin at\} = \int_0^\infty e^{-st} \cdot t \sin at\, dt$$

$$= \int_0^\infty t \cdot e^{-st} \sin at\, dt$$

Integrating by parts, taking t as first function and e^{-st} sin at as the second function and applying the result

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2} [a \sin bx - b \cos bx],$$

we get $\mathcal{L}\{t \sin at\} = \left[t \cdot \frac{e^{-st}}{s^2 + a^2} (-s \sin at - a \cos at) \right]_0^\infty$

$$+ \int_0^\infty \frac{e^{-st}}{s^2 + a^2} (s \sin at + a \cos at)\, dt$$

$$= 0 + \frac{s}{s^2 + a^2} \int_0^\infty e^{-st} \sin at\, dt + \frac{a}{s^2 + a^2} \int_0^\infty . e^{-st} \cos at\, dt$$

$$= \frac{s}{s^2 + a^2} \cdot \frac{a}{s^2 + a^2} + \frac{a}{s^2 + a^2} \cdot \frac{s}{s^2 + a^2} \quad \text{for } s > 0 \text{ [using examples 1 (v) \& 1 (vi)]}$$

$$= \frac{2as}{(s^2 + a^2)^2} \quad \text{for } s > o.$$

(ii) When F (t) = t cos at, equation (1) becomes

$$\mathcal{L}\{t \cos at\} = \int_0^\infty e^{-st} . t \cos at\, dt$$

$$= \int_0^\infty t . e^{-st} \cos at\, dt$$

Integrating by parts, taking t as first function and e^{-st} cos at as the second function and applying the result $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2} [a \cos bx + b \sin bx]$,

we get $\mathcal{L}\{t \cos at\} = \left[t \cdot \frac{e^{-st}}{s^2 + a^2} (-s \cos at + a \sin at) \right]_0^\infty$

$$+ \frac{s}{s^2 + a^2} \int_0^\infty e^{-st} \cos at\, dt - \frac{a}{s^2 + a^2} \int_0^\infty e^{-st} \sin at\, dt$$

$$= 0 + \frac{s}{s^2 + a^2} \cdot \frac{s}{s^2 + a^2} - \frac{a}{s^2 + a^2} \cdot \frac{a}{s^2 + a^2} \quad \text{for } s > 0$$

$$= \frac{s^2 - a^2}{(s^2 + a^2)^2} \quad \text{for } s > o.$$

Example 3. Find the Laplace transform of the function F(t)

where $F(t) = \begin{cases} t, & 0 < t < 2 \\ 3, & t > 2. \end{cases}$

Solution : Here F (t) is not defined at t = o and t = 2. By definition of the Laplace transform of a function F(t), we have

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

$$= \int_0^2 e^{-st} F(t) dt + \int_2^\infty e^{-st} F(t)\, dt$$

$$= \int_0^2 . e^{-st} . t\, dt + \int_2^\infty e^{-st} . 3\, dt$$

$$= \int_0^2 t. e^{-st}\, dt + 3\int_2^\infty e^{-st.} dt$$

$$= \left[-\frac{t}{s} e^{-st}\right]_0^2 + \int_0^2 \frac{e^{-st}}{s}\, dt - \frac{3}{s}\left[e^{-st}\right]_2^\infty$$

$$= -\frac{2}{s} e^{-2s} + 0 - \frac{1}{s^2}\left[e^{-st}\right]_0^2 - \frac{3}{s}(0 - e^{-2s})$$

$$= -\frac{2}{s} e^{-2s} - \frac{1}{s^2} e^{-2s} + \frac{1}{s^2} + \frac{3e^{-2s}}{s}$$

$$= \frac{1}{s^2} + \frac{e^{-2s}}{s} - \frac{e^{-2s}}{s^2}.$$

Definition Sectional or Piecewise continuity

A function F(t) is called **sectionally continuous** or **Piecewise continuous** in a closed interval $a \le t \le b$ if the interval can be divided into a finite number of sub intervals in each of which the function remains continuous and possesses finite right and left hand limits.

Definition Functions of exponential order.

For a given positive integer N if there exist real constants M > O and γ such that for all t > N

$|e^{-\gamma t} F(t)| < M$ or $|F(t)| < M e^{\gamma t}$

then F (t) is called a **function of exponential order** γ as $t \to \infty$ or briefly of exponential order.

1.5 Sufficient conditions for the existence of Laplace transform.

Theorem 1. Existence theorem

If the function F(t) is piecewise (sectionally) continuous over any closed interval $0 \le t \le N$ and of exponential order γ for t>N, then its Laplace transform f(s) exists for all $s > \gamma$.

Proof : The Laplace transform of F(t) exists if $f(s) = \int_0^\infty F(t)e^{-st}\,dt$ exists.

Now for N > O, we have

$$\int_0^\infty F(t)e^{-st}dt = \int_0^N e^{-st}F(t)\,dt + \int_N^\infty e^{-st}F(t)dt \qquad (1)$$

The first integral on the right of (1) is a definite integral and since F(t) is piecewise (sectionally) continuous over any closed interval $0 \le t \le N$, it exists.

Also the second integral on the right of (1) exists, since F(t) is of exponential order γ for t > N. To see this we have only to observe that in such a case

$$\left|\int_N^\infty e^{-st}F(t)dt\right| \le \int_N^\infty |e^{-st}F(t)|\,dt$$

$$\le \int_0^\infty e^{-st}|F(t)|\,dt$$

$$\le \int_0^\infty e^{-st}Me^{\gamma t}\,dt \quad \text{Since} \quad |F(t)| \le Me^{\gamma t}$$

$$= \int_0^\infty e^{-(s-\gamma)t}M\,dt$$

$$= \frac{M}{s-\gamma} \text{ for } s > \gamma.$$

Hence the Laplace transform exists for $s > \gamma$.

1.6 Some important Properties of Laplace transform

(A) Linearity property

Theorem 2. The Laplace transform is a linear transformation. i.e. if and λ_1 and λ_2 are any constants while $F_1(t)$ and $F_2(t)$ are functions with Laplace transform $f_1(s)$ and $f_2(s)$, respectively then

$$\mathcal{L}\{\lambda_1 F_1(t) + \lambda_2 F_2(t)\} = \lambda_1 \mathcal{L}\{F_1(t)\} + \lambda_2 \mathcal{L}\{F_2(t)\}.$$

Proof : Let $\mathcal{L}\{F_1(t)\} = f_1(s) = \int_0^\infty e^{-st} F_1(t)dt$

and $\mathcal{L}\{F_2(t)\} = f_2(s) = \int_0^\infty e^{-st} F_2(t)dt$

Now if λ_1 and λ_2 are any two constants then

$$\mathcal{L}\{\lambda_1 F_1(t) + \lambda_2 F_2(t)\} = \int_0^\infty e^{-st}\{\lambda_1 F_1(t) + \lambda_2 F_2(t)\}dt$$

$$= \lambda_1 \int_0^\infty e^{-st} F_1(t)dt + \lambda_2 \int_0^\infty e^{-st} F_2(t)dt$$

$$= \lambda_1 \mathcal{L}\{F_1(t)\} + \lambda_2 \mathcal{L}\{F_2(t)\}$$

$$= \lambda_1 f_1(s) + \lambda_2 f_2(s).$$

The result may be generalized for any number of functions and for the same number of arbitrary constants. i.e.

$$\mathcal{L}\left\{\sum_{r=1}^{n} \lambda_r F_r(t)\right\} = \sum_{r=1}^{n} \lambda_r \mathcal{L}\{F_r(t)\}.$$

(B) First translation (or shifting) property.

Theorem 3. If $\mathcal{L}\{F(t)\} = f(s)$ then

$$\mathcal{L}\{e^{at}F(t)\} = f(s-a).$$

Proof : By definition of the Laplace transform we have

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)dt = f(s)$$

$$\therefore \quad \mathcal{L}\{e^{at}F(t)\} = \int_0^\infty e^{-st}\{e^{at} F(t)\}dt$$

$$= \int_0^\infty e^{-(s-a)t} F(t)\}dt = f(s-a).$$

(C) Second translation (or shifting) property

Theorem 4. If $\mathcal{L}\{F(t) = f(s)$ and $G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$

then $\mathcal{L}\{G(t) = e^{-as} f(s)$.

Proof : By definition of the Laplace transform we have

$$\mathcal{L}\{F(t) = \int_0^\infty e^{-st} F(t)dt = f(s).$$

$$\therefore \quad \mathcal{L}\{G(t) = \int_0^\infty e^{-st} G(t)dt$$

$$= \int_0^a e^{-st} G(t)dt + \int_a^\infty e^{-st} G(t)dt$$

$$= \int_0^\infty e^{-st} (O)dt + \int_a^\infty e^{-st} F(t-a)dt.$$

$$= 0 + \int_a^\infty e^{-st} F(t-a)dt \quad (1)$$

Let $t - a = u \ \therefore \ dt = du$

Limits $\left.\begin{matrix} t = a \\ u = o \end{matrix}\right\} \quad \left.\begin{matrix} t = \infty \\ u = \infty \end{matrix}\right\}$

Thus from (1), we have

$$\mathcal{L}\{G(t)\} = \int_0^\infty e^{-s(u+a)} F(u)\, du$$

$$= e^{-sa} \int_0^\infty e^{-su} F(u)du. = e^{-sa} f(s).$$

(D) The change of scale property

Theorem 5. If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$

Proof : By definition of the Laplace transform, we have

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)dt = f(s)$$

$$\therefore \quad \mathcal{L}\{F(at)\} = \int_0^\infty e^{-st} F(at)\, dt \quad (1)$$

Let $at = u$ or, $t = \frac{u}{a} \ \therefore \ dt = \frac{1}{a} du.$

Limits $\left.\begin{matrix} t = a \\ u = o \end{matrix}\right\}$ $\left.\begin{matrix} t = \infty \\ u = \infty \end{matrix}\right\}$

Thus from (1) we have

$$\therefore \quad \mathcal{L}\{F(at)\} = \int_0^\infty e^{\frac{-su}{a}} F(u).\frac{1}{a}du.$$

$$= \frac{1}{a}\int_0^\infty e^{\left(\frac{-s}{a}\right)u} F(u)du = \frac{1}{a} f\left(\frac{s}{a}\right).$$

(E) Laplace transform of derivatives.

Theorem 6. If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\{F'(t)\} = s f(s) - F(o)$.

Proof : By definition of the Laplace transform

we have $\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt = f(s)$

$$\therefore \quad \mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st} F'(t)dt \text{ (integrating by parts)}$$

$$= \left[e^{-st} F(t)\right]_0^\infty + s\int_0^\infty e^{-st} F(t)dt$$

$$= 0 - F(0) + sf(s) = sf(s) - F(0).$$

Theorem 7. If $\mathcal{L}\{F(t)\} = f(s)$ then

$$\mathcal{L}\{F''(t)\} = s^2 f(s) - sF(0) - F'(0).$$

Proof : By definition of the Laplace transform,

we have $\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)dt = f(s)$

$$\therefore \mathcal{L}\{F''(t)\} = \int_0^\infty e^{-st} F''(t)dt \text{ (integrating by parts)}$$

$$= [e^{-st} F'(t)] + \int_0^\infty s\, e^{-st} F'(t)dt$$

$$= 0 - F'(o) + s\int_0^\infty e^{-st} F'(t)dt$$

$$= -F'(o) + s\{sf(s) - F(0)\}$$

$$= s^2 f(s) - sF(0) - F'(o).$$

Similarly, we can find the Laplace transform of the third order derivative of F(t).

Theorem 8. If $\mathcal{L}\{F(t)\} = f(s)$ then

$\therefore$ $\mathcal{L}\{F'''(t)\} = s^3 f(s) - s^2F(o) - sF'(0) - F''(0)$.

Generalizing it, we find

Theorem 9. If $\mathcal{L}\{F(t)\} = f(s)$ then

$$\mathcal{L}\{F^n(t)\} = s^n f(s) - s^{n-1} F(o) - s^{n-2} F'(0) - s^{n-3} F''(o) - \ldots - F^{n-1}(o)$$

$$= s^n f(s) - \sum_{k=o}^{n-1} s^{n-k-1} F^k(o).$$

(F) Laplace transform of integrals

Theorem 10. If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\left\{\int_0^t F(u)\,du\right\} = \frac{f(s)}{s}$.

Proof : Let $G(t) = \int_0^t F(u)\,du$.

Then $G'(t) = \frac{d}{dt}\left[\int_0^t F(u)\,du\right] = F(t)$

and $G(o) = \int_0^0 F(u)\,du = 0$.

Applying theorem 6, we get

$\mathcal{L}\{G'(t)\} = s\,\mathcal{L}\{G(t)\} - G(o)$

Or, $\mathcal{L}\{F(t)\} = s\,\mathcal{L}\{G(t)\}$

Or, $f(s) = s\,\mathcal{L}\left\{\int_0^t F(u)\,du\right\}$

Hence $\mathcal{L}\left\{\int_0^t F(u)\,du\right\} = \frac{f(s)}{s}$

(G) Derivatives of Laplace transforms. (Multiplication by Powers of t)

Theorem 11. If $\mathcal{L}\{F(t)\} = f(s)$ then

$$\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s) = (-1)^n f^n(s)$$

where $n = 1, 2, 3, \ldots$

Proof : By definition of the Laplace transform we have

$$f(s) = \int_0^\infty e^{-st} F(t)\, dt.$$

Now applying **Leibnitz's rule** for differentiating under the integral sign we get

$$\frac{df}{ds} = f'(s) = \frac{d}{ds} \int_0^\infty e^{-st} F(t)\, dt$$

$$= \int_0^\infty \frac{\partial}{\partial s} e^{-st} F(t)\, dt$$

$$= \int_0^\infty - te^{-st} F(t)\, dt$$

$$= \int_0^\infty - e^{-st} \{tF(t)\}\, dt = -\mathcal{L}\{tF(t)\}.$$

Thus $\mathcal{L}\{tF(t)\} = -\frac{df}{ds} = -f'(s)$ (1)

Hence the theorem is proved for n = 1.

To prove the theorem in general, let us use the mathematical induction. Suppose that the theorem is true for n = k. Then we have

$$\int_0^\infty e^{-st} \{t^k F(t)\}\, dt = (-1)^k f^k(s) \quad (2)$$

Differentiating both sides with respect to s, we get

$$\frac{d}{ds} \int_0^\infty e^{-st} \{t^k F(t)\}\, dt = (-1)^k f^{k+1}(s)$$

Now applying **Leibnitz's rule** we get

$$-\int_0^\infty e^{-st} \{t^{k+1} F(t)\}\, dt = (-1)^k f^{k+1}(s)$$

Or, $$-\int_0^\infty e^{-st} t^{k+1} F(t)\}\, dt = (-1)^{k+1} f^{k+1}(s) \quad (3)$$

Thus it follows that if the theorem holds for n = k then it also holds for n = k + 1. But in (1) it is proved that the theorem holds for n = 1. Hence it holds for n = 1 + 1 = 2 and n = 2 + 1 = 3 etc. Thus the theorem holds for all positive integral values of n.

(H) Division by t

Theorem 12. If $\mathcal{L}\{F(t)\}=f(s)$ then $\mathcal{L}\left\{\frac{F(t)}{t}\right\}=\int_s^\infty f(u)du$

Proof : Let $G(t)=\frac{F(t)}{t}$ then $F(t)=tG(t)$.

Now $\mathcal{L}\{F(t)\}=\mathcal{L}\{tG(t)\}=(-1)\frac{d}{ds}\mathcal{L}\{G(t)\}$

Or, $f(s)=(-1)\frac{d}{ds}\mathcal{L}\{G(t)\}$

Or, $-f(s)=\frac{d}{ds}\mathcal{L}\{G(t)\}$

Integrating both sides with respect to s from s to ∞, we get

$$\left[\mathcal{L}\{G(t)\}\right]_s^\infty=-\int_s^\infty f(s)\,ds$$

or, $$-\left[\mathcal{L}\{G(t)\}\right]_s^\infty=\int_s^\infty f(s)\,ds.$$

or, $$-\lim_{s\to\infty}\mathcal{L}\{G(t)\}+\mathcal{L}\{G(t)\}=\int_s^\infty f(s)\,ds$$

Since $\lim_{s\to s}\mathcal{L}\{G(t)\}=\mathcal{L}\{G(t)\}$

or, $$0+\mathcal{L}\{G(t)\}=\int_s^\infty f(s)ds$$

Since $\lim_{s\to\infty}\mathcal{L}\{G(t)\}=\lim_{s\to\infty}\int_0^\infty e^{-st}G(t)\,dt=0$

$\therefore$ $$\mathcal{L}\left\{\frac{F(t)}{t}\right\}=\int_s^\infty f(s)\,ds=\int_s^\infty f(u)\,du.$$

(I) Periodic Functions

Theorem 13 : If F(t) has period $T>0$ then

$$\mathcal{L}\{F(t)\}=\frac{\int_0^T e^{-st}F(t)\,dt}{1-e^{-sT}}$$

Proof : If F(t) is a periodic function with period T, then F (t + T) = F(t).

Now by definition of Laplace transform, we have

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt.$$

$$= \int_0^T e^{-st} F(t)\, dt + \int_T^{2T} e^{-st} F(t)\, dt + \int_{2T}^{3T} e^{-st} F(t)\, dt + \ldots$$

In the second integral let t = u + T, in the third integral let t = u + 2T etc. Then we have

$$\mathcal{L}\{F(t)\} = \int_0^T e^{-su} F(u)\, du + \int_0^T e^{-s(u+T)} F(u+T)\, du$$

$$+ \int_0^T e^{-s(u+2T)} F(u+2T)\, du + \ldots\ldots$$

$$= \int_0^T e^{-su} F(u)\, du + \int_0^T e^{-su} e^{-sT} F(u)\, du$$

$$+ \int_0^T e^{-su} e^{-2sT} F(u)\, du + \ldots\ldots$$

$$= \int_0^T e^{-su} F(u)\, du + e^{-sT} \int_0^T e^{-su} F(u)\, du$$

$$+ e^{-2sT} \int_0^T e^{-su} F(u)\, du + \ldots$$

$$= (1 + e^{-sT} + e^{-2sT} + \ldots) \int_0^T e^{-su} F(u) du.$$

$$= \frac{1}{1 - e^{-sT}} \int_0^T e^{-su} F(u)\, du = \frac{\int_0^T e^{-st} F(t)\, dt}{1 - e^{-sT}}.$$

Thus the theorem is proved.

1.7 Initial and Final value theorems

Theorem 14. (Initial value theorem) If the indicated limit exists, then $\lim_{t \to 0} F(t) = \lim_{s \to \infty} sf(s)$.

Proof : We know that if $\mathcal{L}\{F(t)\} = f(s)$ then

$$\mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st} F'(t)\, dt = sf(s) - F(0) \quad (1)$$

But when $F'(x)$ is sectionally continuous and of exponential order, we have $\lim_{s\to\infty} \int_0^\infty e^{-st} F'(t)\, dt = 0$ (2)

Then taking the limit as $s \to \infty$ in (1) and assuming $F(t)$ to be continuous at $t = 0$, we get

$$0 = \lim_{s\to\infty} s\, f(s) - F(0)$$

Or, $$\lim_{s\to\infty} s\, f(s) = F(0) = \lim_{t\to 0} F(t)$$

Thus $$\lim_{s\to\infty} s\, f(s) = \lim_{t\to 0} F(t).$$

Theorem 15. (Final value theorem)
If the indicated limit exists then

$$\lim_{t\to\infty} F(t) = \lim_{s\to 0} s\, f(s)$$

Proof : We know that if $\mathcal{L}\{F(t)\} = f(s)$ then

$$\mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st} F'(t)dt = sf(s) - F(o) \quad (1)$$

Now $$\lim_{s\to 0} \int_0^\infty e^{-st} F'(t)dt = \int_0^\infty F'(t)dt$$

$$= \lim_{p\to\infty} \int_0^p F'(t)dt$$

$$= \lim_{p\to\infty} \{F(p) - F(o)\}$$

$$= \lim_{t\to\infty} \{F(t) - F(o)\}$$

Again the limit of the right hand side of (1) as

$s \to 0$ is $\lim_{s \to 0} sf(s) - F(0)$.

Thus we have $\lim_{t \to \infty} F(t) - F(0) = \lim_{s \to 0} sf(s) - F(0)$

$\therefore \quad \lim_{t \to \infty} F(t) = \lim_{s \to 0} sf(s)$.

WORKED OUT EXAMPLES

Example 4. Find the Laplace transform of

$e^{4t} + 4t^3 - 2\sin 3t + 3\cos 5t$.

Solution : By applying linearity property, we have

$$\mathcal{L}\{e^{4t} + 4t^3 - 2\sin 3t + 3\cos 5t.\}$$

$$= \mathcal{L}\{e^{4t}\} + 4\,\mathcal{L}\{t^3\} - 2\mathcal{L}\{\sin 3t\} + 3\mathcal{L}\{\cos 5t\}$$

$$= \frac{1}{s-4} + 4.\frac{\lfloor 3}{s^4} - 2.\frac{3}{s^2+3^2} + 3.\frac{s}{s^2+5^2}$$

$$= \frac{1}{s-4} + \frac{24}{s^4} - \frac{6}{s^2+9} + \frac{3s}{s^2+25}.$$

Example 5. Find the Laplace transform of

$3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t$.

Solution : By applying linearity property, we have

$$\mathcal{L}\{3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t\}$$

$$= 3\,\mathcal{L}\{t^4\} - 2\,\mathcal{L}\{t^3\} + 4\mathcal{L}\{e^{-3t}\} - 2\mathcal{L}\{\sin 5t\} + 3\mathcal{L}\{\cos 2t\}$$

$$= 3.\frac{\lfloor 4}{s^5} - 2.\frac{\lfloor 3}{s^4} + 4.\frac{1}{s+3} - 2.\frac{5}{s^2+5^2} + 3.\frac{s}{s^2+2^2}.$$

$$= \frac{72}{s^5} - \frac{12}{s^4} + \frac{4}{s+3} - \frac{10}{s^2+25} + \frac{3s}{s^2+4}.$$

Example 6. Find the Laplace transform of t^3e^{5t}

Solution : We have $\mathcal{L}\{t^3\} = \frac{\lfloor 3}{s^4} = \frac{6}{s^4} = f(s)$(say)

Then by using first shifting theorem, we get

$$\mathcal{L}\{t^3e^{5t}\} = f(s-5) = \frac{6}{(s-5)^4}.$$

Method–2

Example 7. Find the Laplace transform of
$e^{3t}(2\cos 5t - 3\sin 5t)$.

Solution : We have

$$\mathcal{L}\{2\cos 5t - 3\sin 5t\} = 2\,\mathcal{L}\{\cos 5t\} - 3\,\mathcal{L}\{\sin 5t\}$$

$$= 2.\frac{s}{s^2+5^2} - 3.\frac{5}{s^2+5^2}$$

$$= \frac{2s-15}{s^2+25} = f(s) \text{ (say)}$$

Then by applying the first shifting theorem we have

$$\mathcal{L}\{e^{3t}(2\cos 5t - 3\sin 5t)\}$$

$$= f(s-3) = \frac{2(s-3)-16}{(s-3)^2+25}$$

$$= \frac{2s-21}{s^2-6s+34}.$$

Example 8. By applying first shifting theorem prove that

(i) $\mathcal{L}\{t\sin at\} = \dfrac{2as}{(s^2+a^2)^2}$

(ii) $\mathcal{L}\{t\cos at\} = \dfrac{s^2-a^2}{(s^2+a^2)^2}$.

Proof : We have $\mathcal{L}\{t\} = \dfrac{1}{s^2} = f(s)$ (say)

Then by applying the first shifting theorem we get

$$\mathcal{L}\{t\,e^{iat}\} = f(s-ia) = \frac{i}{(s-ia)^2}$$

Or, $\mathcal{L}\{t\cos at + it\sin at\} = \dfrac{(s+ia)^2}{\{(s+ia)(s-ia)\}^2}$

Or, $\mathcal{L}\{t\cos at\} + i\,\mathcal{L}\{t\sin at\} = \dfrac{(s^2-a^2)+i(2as)}{(s^2+a^2)^2}$

Equating real and imaginary parts from both sides, we get

$\mathcal{L}\{t\cos at\} = \dfrac{s^2-a^2}{(s^2+a^2)^2}$ and

$\mathcal{L}\{t\sin at\} = \dfrac{2as}{(s^2+a^2)^2}$.

Similar applications of the first shifting property leads us to the following results :

(1) $\mathcal{L}\{e^{at} t^n\} = \frac{\lfloor n}{(s-a)^{n+1}}$

(2) $\mathcal{L}\{e^{at} \sin bt\} = \frac{b}{(s-a)^2 + b^2}$

(3) $\mathcal{L}\{e^{at} \cos bt\} = \frac{s-a}{(s-a)^2 + b^2}$

(4) $\mathcal{L}\{e^{at} \sinh bt\} = \frac{b}{(s-a)^2 - b^2}$

(5) $\mathcal{L}\{e^{at} \cosh bt\} = \frac{s-a}{(s-a)^2 - b^2}$.

Example 9. Find the Laplace transform of F(t), where

$$F(t) = \begin{cases} \sin(t - \pi/3), & t > \pi/3 \\ 0 & , t < \pi/3 \end{cases}$$

Solution : Let $\varphi(t) = \sin t$

$\therefore \varphi(t - \pi/3) = \sin(t - \pi/3)$

$$F(t) = \begin{cases} \varphi(t - \frac{\pi}{3}), & t > \frac{\pi}{3} \\ 0 & , t < \pi/3 \end{cases}$$

Now $\mathcal{L}\{\varphi(t)\} = \mathcal{L}\{\sin t\} = \frac{1}{s^2+1} = f(s)$ (say)

Then applying the second shifting theorem we get

$$\mathcal{L}\{F(t)\} = e^{-as} f(s)$$

$$= e^{-\frac{\pi s}{3}} \frac{1}{s^2+1} = \frac{e^{-\frac{\pi s}{3}}}{s^2+1}.$$

Example 10. Find the Laplace transforms of cosat and sinhat using the change of scale property.

Solution : (i) We know that

$$\mathcal{L}\{\cos t\} = \frac{s}{s^2+1} = f(s) \text{ (say)}$$

$$\mathcal{L}\{\cos at\} = \frac{1}{a} f\left(\frac{s}{a}\right) = \frac{1}{a} \cdot \frac{\frac{s}{a}}{(s/a)^2 + 1}.$$

$$= \frac{s}{a^2} \cdot \frac{a^2}{s^2 + a^2} = \frac{s}{s^2 + a^2}.$$

(ii) We know that $\mathcal{L}\{\sinh t\} = \frac{1}{s^2 - 1} = f(s)$ (say)

$$\therefore \mathcal{L}\{\sinh at\} = \frac{1}{a} f\left(\frac{s}{a}\right)$$

$$= \frac{1}{a} \cdot \frac{1}{\left(\frac{s}{a}\right)^2 - 1} = \frac{a}{s^2 - a^2}.$$

Example 11. Find the Laplace transform of cosat using theorem 7.

Solution : Let $F(t) = \cos at$

Then $\left.\begin{matrix} F'(t) = -a \sin at \\ F''(t) = -a^2 \cos at \end{matrix}\right\}$ $\left.\begin{matrix} F(o) = 1 \\ F'(o) = o \end{matrix}\right\}$

Now by theorem 11, we have

$\mathcal{L}\{F''(t)\} = s^2 \mathcal{L}\{F(t)\} - s F(o) - F'(o)$.

$\mathcal{L}\{-a^2 \cos at\} = s^2 \mathcal{L}\{\cos at\} - s.1$—o.

Or, $-a^2 \mathcal{L}\{\cos at\} = s^2 \mathcal{L}\{\cos at\} - s$.

Or, $(s^2 + a^2) \mathcal{L}\{\cos at\} = s$

Or, $\mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}$.

Example 12. Find the Laplace trasfrom of $t^2 \cos at$ using theorem 11.

Solution : Since $\mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}$, $s > o$

$$\therefore \mathcal{L}\{t^2 \cos at\} = (-1)^2 \frac{d^2}{ds^2} \mathcal{L}\{\cos at\}$$

$$= \frac{d^2}{ds^2}\left\{\frac{s}{s^2 + a^2}\right\}$$

$$= \frac{d}{ds}\left\{\frac{-2s^2}{(s^2+a^2)^2} + \frac{1}{s^2+a^2}\right\}$$

$$= \frac{8s^3}{(s^2+a^2)^3} - \frac{6s}{(s^2+a^2)^2}$$

$$= \frac{8s^3 - 6s^3 - 6a^2s}{(s^2+a^2)^3} = \frac{2s(s^2-3a^2)}{(s^2+a^2)^3}.$$

Example 13. Find the Laplace transform of $\int_0^t \frac{\sin t}{t}\,dt$.

Solution : We know that $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1} = f(s)$ (say).

Now $\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty F(u)\,du = \int_s^\infty \frac{1}{u^2+1}\,du$

$$= [\tan^{-1} u]_s^\infty = \tan^{-1}\infty - \tan^{-1}s$$

$$= \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s = \tan^{-1}\frac{1}{s}$$

Since $\tan^{-1}s + \cot^{-1}s = \frac{\pi}{2}$.

Now we also know that $\mathcal{L}\left\{\int_0^t F(u)\,du\right\} = \frac{f(s)}{s}$.

$$\therefore \mathcal{L}\left\{\int_0^t \frac{\sin t}{t}\,dt\right\} = \frac{1}{s}f(s) = \frac{1}{s}\cdot\tan^{-1}\frac{1}{s}.$$

Example 14. Find the Laplace transform of the function

$$F(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$$

Solution : If F (t) has period T > 0 then.

$$\mathcal{L}\{F(t)\} = \frac{\int_0^T e^{-st} F(t)\,dt}{1 - e^{-sT}}$$

In the given problem F(t) has period $T = 2\pi$.

$$\therefore \text{Here we have } \mathcal{L}\{F(t)\} = \frac{1}{1 - e^{-2\pi s}} \int_0^{2\pi} e^{-st} F(t)\,dt.$$

$$= \frac{1}{1 - e^{-2\pi s}} \left[\int_0^{\pi} e^{-st} F(t)\, dt + \int_{\pi}^{2\pi} e^{-st} F(t)\, dt \right]$$

$$= \frac{1}{1 - e^{-2\pi s}} \left[\int_0^{\pi} e^{-st} \sin t\, dt + \int_{\pi}^{2\pi} e^{-st} \cdot 0\, dt \right]$$

$$= \frac{1}{1 - e^{-2\pi s}} \left[\frac{e^{-st} (-s \sin t - \cos t)}{s^2 + 1} \right]_0^{\pi} + 0$$

$$= \frac{1}{1 - e^{-2\pi s}} \left[\frac{e^{-\pi s} + 1}{s^2 + 1} \right] = \frac{1}{(1 - e^{-\pi s})(s^2 + 1)}.$$

1.8 Some special functions

1. The sine and cosine integrals

The sine and cosine integrals, denoted by $Si(t)$ and $C_i(t)$ respectively, are defined by

$$S_i(t) = \int_0^t \frac{\sin u}{u}\, du \text{ and}$$

$$C_i(t) = \int_t^{\infty} \frac{\cos u}{u}\, du.$$

2. The exponential integral

The exponential integral, denoted by Ei (t), is defined by $E_i(t) = \int_t^{\infty} \frac{e^{-u}}{u}\, du.$

3. The error function

The error function, denoted by erf (t), is defined by

$$\text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2}\, du.$$

4. Heaviside's unit step function.

The **Heaviside's unit step function** about a point a, denoted by H (t – a), is defined by

$$H(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a. \end{cases}$$

5. The gamma function

If $n > 0$, the gamma function is defined by

$$\overline{|(n)} = \int_0^\infty t^{n-1} e^{-t} dt.$$

6. The Bessel Function

The Bessel Function of order n is defined by

$$J_n(t) = \frac{t^n}{2^n \overline{|(n+1)}} \left[1 - \frac{t^2}{2.(2n+2)} + \frac{t^4}{2.4\,(2n+2)\,(2n+4)} - \cdots\cdots\right]$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{\underline{|r}\ \overline{|(n+r+1)}} \left(\frac{t}{2}\right)^{n+2r}$$

7. The unit impulse Function (or **Dirac delta function**)

The unit impulse function is defined by

$$F_\epsilon(t) = \begin{cases} \frac{1}{\epsilon}, & 0 \leq t \leq \epsilon \\ 0, & t > \epsilon \end{cases}$$

8. Laguerre plynomial

Laguerre polynomial is defined by

$$L_n(t) = \frac{e^t}{\underline{|n}} \frac{d^n}{dt^n} \left(e^{-t} \cdot t^n\right), \quad n = 0, 1, 2, \ldots\ldots$$

9. Null function

If N (t) is a function of t such that for all $t > 0$

$$\int_0^t N(u)\, du = 0.$$

Then we call N (t) a null function.

1.9 Laplace transforms of some special functions.

(i) Laplace transform of sine integral

we know that

$$S_i(t) = \int_0^t \frac{\sin u}{u}\,du$$

$$= \int_0^t \frac{1}{u}\left(u - \frac{u^3}{\lfloor 3} + \frac{u^5}{\lfloor 5} - \frac{u^7}{\lfloor 7} + \cdots\right) du$$

$$= \int_0^t \left(1 - \frac{u^2}{\lfloor 3} + \frac{u^4}{\lfloor 5} - \frac{u^6}{\lfloor 7} + \cdots\right) du$$

$$= \left[u - \frac{u^3}{3\lfloor 3} + \frac{u^5}{5\lfloor 5} - \frac{u^7}{7\lfloor 7} + \cdots\right]_0^t$$

$$= t - \frac{t^3}{3\lfloor 3} + \frac{t^5}{5\lfloor 5} - \frac{t^7}{7\lfloor 7} + \cdots\cdots\cdots (1)$$

Taking the Laplace transform of both sides of (1) we get

$$\mathcal{L}\{S_i(t)\} = \mathcal{L}\left\{t - \frac{t^3}{3\lfloor 3} + \frac{t^5}{5\lfloor 5} - \frac{t^7}{7\lfloor 7} + \cdots\right\}$$

$$= \frac{1}{s^2} - \frac{1}{3\lfloor 3}\cdot\frac{\lfloor 3}{s^4} + \frac{1}{5.\lfloor 5}\cdot\frac{\lfloor 5}{s^6} - \frac{1}{7\lfloor 7}\cdot\frac{\lfloor 7}{s^8} + \cdots$$

$$= \frac{1}{s^2} - \frac{1}{3s^4} + \frac{1}{5s^6} - \frac{1}{7s^8} + \cdots$$

$$= \frac{1}{s}\left\{\frac{1}{s} - \frac{1}{3s^3} + \frac{1}{5s^5} - \frac{1}{7s^7} + \cdots\cdots\right\}$$

$$= \frac{1}{s}\left[\frac{\frac{1}{s}}{1} - \frac{\left(\frac{1}{s}\right)^3}{3} + \frac{\left(\frac{1}{s}\right)^5}{5} - \frac{\left(\frac{1}{s}\right)^7}{7} + \cdots\right]$$

$$= \frac{1}{s}\tan^{-1}\frac{1}{s}.$$ (Using the Gregory's series

$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$)

This problem can be solved by other methods also.

(ii) Laplace transform of cosine integral.

We know that $C_i(t) = \int_t^\infty \frac{\cos u}{u} du$ (1)

Taking the Laplace transform of both sides of (1), we get

$$\mathcal{L}\{C_i(t)\} = \mathcal{L}\left\{\int_t^\infty \frac{\cos u}{u} du\right\} \quad (2)$$

Let $F(t) = \int_t^\infty \frac{\cos u}{u} du = -\int_\infty^t \frac{\cos u}{u} du$

so that $F'(t) = -\frac{\cos t}{t}$

Or, $tF'(t) = -\cos t$ (3)

Taking Laplace transform of both sides of (3) we get

$\mathcal{L}\{t F'(t)\} = \mathcal{L}\{-\cos t\}$

Or, $-\frac{d}{ds}\mathcal{L}\{F'(t)\} = -\frac{s}{s^2+1}$

Or, $-\frac{d}{ds}\{sf(s) - F(0)\} = -\frac{s}{s^2+1}$

where $f(s) = \mathcal{L}\{F(t)\}$

Or, $-\frac{d}{ds}\{s f(s)\} = -\frac{s}{s^2+1}$

since $F(0)$ is constant.

Or, $\frac{d}{ds}\{s f(s)\} = \frac{s}{s^2+1}$ (4)

Integrating both sides of (4), we get

$s f(s) = \frac{1}{2}\log(s^2+1) + C$ (5)

By the Final value theorem we have

$\lim_{s\to 0} s f(s) = \lim_{t\to\infty} F(t) = 0.$

As $s \to 0$ from (5), we have

$0 = \frac{1}{2}\log 1 + c$

or, $0 = 0 + c$ Since $\log 1 = 0$

$\therefore c = 0.$

Putting c = 0 in (5), we get

$$s f(s) = \frac{1}{2}\log(s^2 + 1)$$

$$\text{Or, } f(s) = \frac{\log(s^2 + 1)}{2s}.$$

Hence $f(s) = \mathcal{L}\{F(t)\} = \mathcal{L}\{C_1(t)\} = \frac{\log(s^2 + 1)}{2s}$.

2. Laplace transform of the exponential integral.

we know that $E_1(t) = \int_t^\infty \frac{e^{-u}}{u}\,du$ (1)

Taking the Laplace transform of both sides of (1), we get

$$\mathcal{L}\{E_1(t)\} = \mathcal{L}\left\{\int_t^\infty \frac{e^{-u}}{u}\,du\right\} \quad (2)$$

Putting $u = tv$ so that $du = tdv$

and limits $\left.\begin{matrix} u = t \\ v = 1 \end{matrix}\right\}$ $\left.\begin{matrix} u = \infty \\ v = \infty \end{matrix}\right\}$

$$\therefore \quad \mathcal{L}\{E_1(t)\} = \mathcal{L}\left\{\int_1^\infty \frac{e^{-tv}}{v}\,dv\right\}$$

$$= \int_0^\infty e^{-st}\left\{\int_1^\infty \frac{e^{-tv}}{v}\,dv\right\}dt$$

$$= \int_1^\infty \frac{1}{v}\left\{\int_0^\infty e^{-st}\cdot e^{-tv}\,dt\right\}dv$$

$$= \int_1^\infty \frac{1}{v}\left\{\int_0^\infty e^{-(s+v)t}\,dt\right\}dv$$

$$= \int_1^\infty \frac{1}{v}\cdot\left[\frac{e^{-(s+v)t}}{-(s+v)}\right]_0^\infty dv$$

$$= \int_1^\infty \frac{1}{v(s+v)}\,dv$$

$$= \int_1^\infty \frac{1}{s}\left(\frac{1}{v} - \frac{1}{s+v}\right) dv$$

$$= \frac{1}{s}[\log v - \log (s+v)]_1^\infty$$

$$= \frac{1}{s}\left[-\{\log (s+v) - \log v\}\right]_1^\infty$$

$$= \frac{1}{s}\left\{-\log\left(\frac{s+v}{v}\right)\right\}_1^\infty$$

$$= \frac{1}{s}\left[-\log\left(\frac{s}{v}+1\right)\right]_1^\infty$$

$$= \frac{1}{s}\left[-\log 1 + \log (s+1)\right]$$

$$= \frac{1}{s}\log (s+1) \text{ Since } \log 1 = 0.$$

Hence $\mathcal{L}\{E_1(t)\} = \mathcal{L}\left\{\int_t^\infty \frac{e^{-u}}{u} du\right\} = \frac{1}{s}\log (s+1)$.

3. Laplace transform of the error function

We know that

$$\text{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-u^2} du \quad (1)$$

Writing $\sqrt{t}$ for t in (1), we get $\text{erf}\left(\sqrt{t}\right) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2} du$

$$= \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} \left(1 - \frac{u^2}{\underline{|1}} + \frac{u^4}{\underline{|2}} - \frac{u^6}{\underline{|3}} + \text{------}\right) du$$

$$= \frac{2}{\sqrt{\pi}}\left[u - \frac{u^3}{3} + \frac{u^5}{5\underline{|2}} - \frac{u^7}{7\underline{|3}} + \text{------}\right]_0^{\sqrt{t}}$$

$$= \frac{2}{\sqrt{\pi}}\left[t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5\underline{|2}} - \frac{t^{7/2}}{7\underline{|3}} + \text{-------}\right] \quad (2)$$

Taking the Laplace transform of both sides of (2), we get

$$\mathcal{L}\{\operatorname{erf}\sqrt{t}\}$$

$$=\frac{2}{\sqrt{\pi}}\left[\mathcal{L}\left\{t^{1/2}\right\}-\frac{1}{3}\mathcal{L}\left\{t^{3/2}\right\}+\frac{1}{5\lfloor 2}\mathcal{L}\left\{t^{5/2}\right\}-\frac{1}{7\lfloor 3}\mathcal{L}\left\{t^{7/2}\right\}+\cdots\cdots\right]$$

$$=\frac{2}{\sqrt{\pi}}\left[\frac{\lfloor 3/2}{s^{3/2}}-\frac{1}{3}\cdot\frac{\lfloor 5/2}{s^{5/2}}+\frac{1}{5\lfloor 2}\cdot\frac{\lfloor 7/2}{s^{7/2}}-\frac{1}{7\lfloor 3}\cdot+\frac{\lfloor 9/2}{s^{9/2}}+\cdots\cdots\right]$$

$$=\frac{2}{\sqrt{\pi}}\left[\frac{1/2\lfloor 1/2}{s^{3/2}}-\frac{1}{3}\cdot\frac{3/2\,\frac{1}{2}\lfloor 1/2}{s^{5/2}}+\frac{1}{5\lfloor 2}\frac{5/2\cdot 3/2\cdot 1/2\lfloor 1/2}{s^{7/2}}\cdot\frac{1}{7\lfloor 3}\cdot\frac{7/2\cdot 5/2\cdot 3/2/1/2\lfloor 1/2}{s^{9/2}}+\right.$$

$$=\frac{1}{s^{3/2}}-\frac{1}{2}\cdot\frac{1}{s^{5/2}}+\frac{1.3}{2.4}\cdot\frac{1}{s^{7/2}}-\frac{1.3.5}{2.4.6}\cdot\frac{1}{s^{9/2}}+\cdots\cdots$$

$$=\frac{1}{s^{3/2}}\left[1-\frac{1}{2}\cdot\frac{1}{s}+\frac{1.3}{2.4}\cdot\frac{1}{s^2}-\frac{1.3.5}{2.4.6}\cdot\frac{1}{s^3}+\cdots\cdots\right]$$

$$=\frac{1}{s^{3/2}}\left[\left(1+\frac{1}{s}\right)^{-\frac{1}{2}}\right]$$ (Using binomial theorem)

$$=\frac{1}{s^{3/2}}\left(\frac{s+1}{s}\right)^{-1/2}$$

$$=\frac{1}{s^{3/2}}\frac{s^{1/2}}{(s+1)^{1/2}}$$

$$=\frac{1}{s\sqrt{s+1}}$$

Hence $\mathcal{L}\{\operatorname{erf}\sqrt{t}\}=\dfrac{1}{s\sqrt{s+1}}$.

4. Laplace transform of Heaviside's unit step function

We know that $H(t-a)=\begin{cases}0, t<a\\ 1, t\geq a.\end{cases}$

Now $\mathcal{L}\{H(t-a)\}=\int_0^a e^{-st}(0)dt+\int_a^\infty e^{-st}(1)dt$

$$=0+\int_a^\infty e^{-st}dt$$

$$= \lim_{p\to\infty} \int_a^p e^{-st}\,dt$$

$$= \lim_{p\to\infty} -\frac{1}{s}[e^{-st}]_a^p$$

$$= \lim_{p\to\infty} \left[\frac{-e^{-sp} + e^{-sa}}{s}\right]$$

$$= \frac{e^{-sa}}{s} \text{ Since } \lim_{p\to\infty} e^{-sp} = 0$$

Hence $\mathcal{L}\{H(t-a)\} = \frac{e^{-as}}{s}$.

5. Laplace transform for gamma function .

By definition of Laplace transform, we have

$$\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\,dt \quad (1)$$

Putting $F(t) = t^n$ in (1), we get

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n dt \quad (2)$$

Letting $st = u$ so that $dt = \frac{1}{s}du$.

limits $\left.\begin{matrix} t = 0 \\ u = 0 \end{matrix}\right\}$ $\left.\begin{matrix} t = \infty \\ u = \infty \end{matrix}\right\}$

Assuming $s > 0$, (2) becomes

$$\mathcal{L}\{t^n\} = \frac{1}{s}\int_0^\infty e^{-u}.\frac{u^n}{s^n}du$$

$$= \frac{1}{s^{n+1}}\int_0^\infty e^{-u}u^n du$$

$$= \frac{1}{s^{n+1}}\int_0^\infty u^{(n+1)-1}e^{-u}du$$

$$= \frac{1}{s^{n+1}}.\overline{(n+1)} = \frac{\overline{(n+1)}}{s^{n+1}}.$$

Hence $\mathcal{L}\{t^n\} = \frac{\lceil(n+1)}{s^{n+1}}$ if $n > -1$, $s > 0$.

Putting $n = -\frac{1}{2}$ in the above result,

we get $\mathcal{L}\left\{t^{-\frac{1}{2}}\right\} = \frac{\lceil(-\frac{1}{2}+1}{s^{-\frac{1}{2}+1}} = \frac{\lceil(\frac{1}{2})}{s^{1/2}} = \frac{\sqrt{\pi}}{s^{1/2}}$

$\therefore$ we get $\mathcal{L}\left\{t^{-\frac{1}{2}}\right\} = \sqrt{\frac{\pi}{s}}$.

6. Laplace transform of Bessel functions.

(i) Prove that $\mathcal{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$.

Proof : By definition of Bessel function of order n, we have

$$J_n(t) = \frac{t^n}{2^n\lceil(n+1)}$$

$$\left[1 - \frac{t^2}{2.(2n+2)} + \frac{t^4}{2.4.(2n+2)(2n+4)} - \text{----------}\right] \quad (1)$$

Putting $n = 0$ in (1), we get

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2.4^2} - \frac{t^6}{2^2.4^2.6^2} + \text{----------} \quad (2)$$

Then $\mathcal{L}\{J_0(t)\} = \mathcal{L}\{1\} - \frac{1}{2^2}\mathcal{L}\{t^2\} + \frac{1}{2^2.4^2}\mathcal{L}\{t^4\}$

$$- \frac{1}{2^2.4^2.6^2}.\mathcal{L}\{t^6\} + \text{---}$$

$$= \frac{1}{s} - \frac{1}{2^2}\frac{\lceil 3}{s^3} + \frac{1}{2^2.4^2}.\frac{\lceil 5}{s^5} - \frac{1}{2^2.4^2.6^2}.\frac{\lceil 7}{s^7} + \text{----}$$

$$= \frac{1}{s} - \frac{1}{2^2}\frac{\lfloor 2}{s^3} + \frac{1}{2^2.4^2}.\frac{\lceil 4}{s^5} - \frac{1}{2^2.4^2.6^2}.\frac{\lfloor 6}{s^7} + \text{-----})$$

$$= \frac{1}{s}\left[1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1.3}{2.4}\left(\frac{1}{s^4}\right) - \frac{1.3.5}{2.4.6}\left(\frac{1}{s^6}\right) + \text{-----}\right]$$

$$= \frac{1}{s}\left(1 + \frac{1}{s^2}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{s}\left(\frac{s^2+1}{s^2}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{s}\cdot\frac{s}{(s^2+1)^{1/2}} = \frac{1}{\sqrt{s^2+1}}.$$

Hence $\mathcal{L}\{J_o(t)\} = \frac{1}{\sqrt{s^2+1}}$.

(ii) Prove that $\mathcal{L}\{J_o(at)\} = \frac{1}{\sqrt{s^2+a^2}}$

Proof : We Know that $\mathcal{L}\{J_o(t)\} = \frac{1}{\sqrt{s^2+1}}$

By the change of scale property, we have if $\mathcal{L}\{F(t)\} = f(s)$, then

$$\mathcal{L}\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right).$$

$$\therefore \mathcal{L}\{J_o(at)\} = \frac{1}{a}\frac{1}{\sqrt{\left(\frac{s}{a}\right)^2+1}} = \frac{1}{a}\cdot\frac{a}{\sqrt{s^2+a^2}}$$

Or, $\mathcal{L}\{J_o(at)\} = \frac{1}{\sqrt{s^2+a^2}}$.

(iii) Prove that $\mathcal{L}\{J_1(t)\} = \frac{\sqrt{s^2+1}-s}{\sqrt{s^2+1}}$

Proof : From the property of Bessel function for n = 0, we have

$$J_o'(t) = -J_1(t). \quad J_1(t) = -j_o'(t)$$

Now $\mathcal{L}\{J_1(t)\} = -\mathcal{L}\{J_o'(t)\}$

$$= -\{s\,\mathcal{L}\{J_o(t)\} - J_o(0)\}$$

$$= -\left[s.\frac{1}{\sqrt{s^2+1}} - 1\right] \text{ Since } J_o(0) = 1$$

$$= -\frac{s}{\sqrt{s^2+1}} + 1 = \frac{\sqrt{s^2+1}-s}{\sqrt{s^2+1}}$$

Hence $\mathcal{L}\{J_1(t)\} = \frac{\sqrt{s^2+1}-s}{\sqrt{s^2+1}}$.

7. Laplace transform of unit impulse function

By definition of the unit impulse function, we have

$$F_\epsilon(t) = \begin{cases} \frac{1}{\epsilon}, & 0 \le t \le \epsilon \\ 0, & t > \epsilon \end{cases}$$

Then $\mathcal{L}\{F_\epsilon(t)\} = \int_0^\infty e^{-st} F_\epsilon(t)\, dt$

$$= \int_0^\infty e^{-st} \frac{1}{\epsilon}\, dt + \int_\epsilon^\infty e^{-st}(0)\, dt$$

$$= \frac{1}{\epsilon}\int_0^\epsilon e^{-st}\, dt$$

$$= \frac{1}{\epsilon} \cdot \left[-\frac{1}{s} e^{-st}\right]_0^\epsilon$$

$$= -\frac{1}{\epsilon s}(e^{-s\epsilon} - 1).$$

$$= \frac{1 - e^{-s\epsilon}}{\epsilon s}$$

Hence $\mathcal{L}\{F_\epsilon(t)\} = \frac{1 - e^{-s\epsilon}}{\epsilon s}$.

8. Laplace transform of Laguerre polynomial

By definition we have

$$L_n(t) = \frac{e^t}{\lfloor n} \frac{d^n}{dt^n}(e^{-t}.\, t^n),\ n = 0, 1, 2, \text{----}$$

$$\therefore \mathcal{L}\{L_n(t)\} = \int_0^\infty e^{-st} \frac{e^t}{\lfloor n} \frac{d^n}{d\, t^n}(e^{-t}\, t^n)\, dt$$

$$= \frac{1}{\lfloor n} \int_0^\infty e^{-(s-1)t} \frac{d^n}{d\, t^n}\left(e^{-t}.\, t^n\right) dt$$

$$= \frac{1}{\lfloor n}\left[e^{-(s-1)t} \cdot \frac{d^{n-1}}{dt^{n-1}}\left(e^{-t}.\, t^n\right)\right]_0^\infty$$

$$+\frac{s-1}{\lfloor n}\int_0^\infty e^{-(s-1)t}\frac{d^{n-1}}{dt^{n-1}}\left(e^{-t}\cdot t^n\right)dt,\ s>1$$

$$=0+\frac{s-1}{\lfloor n}\int_0^\infty e^{-(s-1)t}\frac{d^{n-1}}{dt^{n-1}}\left(e^{-t}\cdot t^n\right)dt,\ s>1$$

[Each term in $\frac{d^{n-1}}{dt^{n-1}}\left(e^{-t}t^n\right)$ contains some integral power of t.]

Proceeding in this way again and again, we have .

$$\mathcal{L}\{L_n(t)\}=\frac{(s-1)^n}{\lfloor n}\cdot\int_0^\infty e^{-(s-1)t}\cdot\left(e^{-t}.t^n\right)dt;$$

$$=\frac{(s-1)^n}{\lfloor n}\int_0^\infty e^{-st}\,t^n\,dt$$

$$=\frac{(s-1)^n}{\lfloor n}\mathcal{L}\{t^n\}$$

$$=\frac{(s-1)^n}{\lfloor n}\cdot\frac{\lfloor n}{s^{n+1}}=\frac{(s-1)^n}{s^{n+1}}.$$

Hence $\mathcal{L}\{L_n(t)\}=\frac{(s-1)^n}{s^{n+1}}$

Example 15. Prove that

$$\int_0^\infty t\,e^{-st}\cos at\,dt=\frac{s^2-a^2}{(s^2+a^2)^2}.$$

Proof : By definition of Laplace transform, we get

$$\mathcal{L}\{F(t)\}=\int_0^\infty e^{-st}F(t)dt.$$

$$\text{L.H.S}=\int_0^\infty te^{-st}\cos at\,dt$$

$$=\int_0^\infty e^{-st}t\cos at\,dt$$

$$=\mathcal{L}\{t\cos at\}$$

Method–3

$$= -\frac{d}{ds}\mathcal{L}\{\cos at\}$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2 + a^2}\right)$$

$$= -\left[\frac{(s^2 + a^2)1 - s.2s}{(s^2 + a^2)^2}\right]$$

$$= -\left[\frac{s^2 + a^2 - 2s^2}{(s^2 + a^2)^2}\right]$$

$$= \frac{s^2 - a^2}{(s^2 + a^2)^2} = \text{R.H.S.}$$

Example 16. Prove that $\int_0^\infty t\, e^{-st} \sin at\, dt = \frac{2as}{(s^2 + a^2)^2}$.

Proof : By definition of Laplace transform, we have

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t) dt.$$

$$\therefore \text{L.H.S} = \int_0^\infty te^{-st} \sin at\, dt$$

$$= \int_0^\infty e^{-st}\, t \sin at\, dt$$

$$= \mathcal{L}\{t \sin at\}$$

$$= -\frac{d}{ds}\left(\frac{a}{s^2 + a^2}\right)$$

$$= -\left[-\frac{2as}{(s^2 + a^2)^2}\right]$$

$$= \frac{2as}{(s^2 + a^2)^2} = \text{R.H.S.}$$

Example 17. Using Laplace transform prove that

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt = \log \frac{b}{a}.$$

Proof : Let $F(t) = e^{-at} - e^{-bt}$

$$\therefore \quad f(s) = \mathcal{L}\{F(t)\} = \mathcal{L}\left\{e^{-at}\right\} - \mathcal{L}\left\{e^{-bt}\right\}$$

$$= \frac{1}{s+a} - \frac{1}{s+b}$$

By theorem 12, we have

$$\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)du$$

or, $$\int_0^\infty e^{-st} \cdot \frac{F(t)}{t}\, dt = \int_s^\infty \left(\frac{1}{u+a} - \frac{1}{u+b}\right) du$$

or, $$\int_0^\infty e^{-st} \cdot \frac{\left(e^{-at} - e^{-bt}\right)}{t}\, dt = \int_0^\infty \left(\frac{1}{u+a} - \frac{1}{u+b}\right) du$$

$$= \Big[\log(u+a) - \log(u+b)\Big]_s^\infty$$

$$= \log(\infty + a) - \log(s+a) - \log(\infty + b) + \log(s+b)$$

$$= \log \infty - \log \infty + \log \frac{s+b}{s+a}.$$

$$= \log \frac{s+b}{s+a}.$$

Now taking limit as $s \to 0$, we have

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t}\, dt = \log \frac{b}{a}.$$

Example 18. Prove that

$$\mathcal{L}\left\{\frac{\cos at - \cos bt}{t}\right\} = \frac{1}{2} \log\left(\frac{s^2 + b^2}{s^2 + a^2}\right)$$

Proof : Let $F(t) = \cos at - \cos bt$

Then $f(s) = \mathcal{L}\{F(t)\} = \mathcal{L}\{\cos at - \cos bt\}$

$$= \frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2}$$

By Theorem 12, we have

$$\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)\,du$$

$$\therefore \quad \mathcal{L}\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty \left(\frac{u}{u^2+a^2} - \frac{u}{u^2+b^2}\right) du$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{2u}{u^2+a^2} - \frac{2u}{u^2+b^2}\right) du$$

$$= \frac{1}{2}\left[\log(u^2+a^2) - \log(u^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\infty - \log(s^2+a^2) - \log\infty + \log(s^2+b^2)\right]$$

$$= \frac{1}{2}\left[\log(s^2+b^2) - \log(s^2+a^2)\right]$$

$$= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right).$$

Hence $\mathcal{L}\left\{\dfrac{\cos at - \cos bt}{t}\right\} = \dfrac{1}{2}\log\left(\dfrac{s^2+b^2}{s^2+a^2}\right)$.

Example 19. Prove that

$$\mathcal{L}\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}} \cdot e^{-\frac{1}{4s}}$$

Proof : From trigonometric series, we have

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \ldots\ldots\ldots (1)$$

Putting $x = \sqrt{t}$ in (1), we get

$$\sin\sqrt{t} = \sqrt{t} - \frac{(\sqrt{t})^3}{\lfloor 3} + \frac{(\sqrt{t})^5}{\lfloor 5} - \frac{(\sqrt{t})^7}{\lfloor 7} + \ldots\ldots\ldots$$

$$= t^{1/2} - \frac{t^{3/2}}{\lfloor 3} + \frac{t^{5/2}}{\lfloor 5} - \frac{t^{7/2}}{\lfloor 7} + \ldots\ldots\ldots$$

$$\therefore \mathcal{L}\{\sin\sqrt{t}\} = \mathcal{L}\{t^{1/2}\} - \frac{1}{\lfloor 3}\mathcal{L}\{t^{3/2}\} + \frac{1}{\lfloor 5}\mathcal{L}\{t^{5/2}\} - \frac{1}{\lfloor 7}\mathcal{L}\{t^{7/2}\} + \ldots.$$

$$= \frac{\lceil 3/2}{s^{3/2}} - \frac{1}{\lfloor 3}\cdot\frac{\lceil 5/2}{s^{5/2}} + \frac{1}{\lfloor 5}\frac{\lceil 7/2}{s^{7/2}} - \frac{1}{\lfloor 7}\frac{\lceil 9/2}{s^{9/2}} + \ldots\ldots$$

$$= \frac{\frac{1}{2}\lceil 1/2}{s^{3/2}} - \frac{1}{\lfloor 3}\cdot\frac{3/2.\,1/2\lceil 1/2}{s^{5/2}} + \frac{1}{\lfloor 5}\cdot\frac{5/2.\,3/2.\,1/2\lceil 1/2}{s^{7/2}} - \frac{1}{\lfloor 7}\cdot\frac{7/2.\,5/2.\,3/2.\,1/2\lceil 1/2}{s^{9/2}} + \ldots\ldots$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} - \frac{1}{\lfloor 3}\cdot\frac{3/2.\,1/2\sqrt{\pi}}{s^{5/2}} + \frac{1}{\lfloor 5}\cdot\frac{5/2.\,3/2.\,1/2\sqrt{\pi}}{s^{7/2}} - \frac{1}{\lfloor 7}\cdot\frac{7/2.\,5/2.\,3/2.\,1/2\sqrt{\pi}}{s^{9/2}} + \ldots\ldots\ldots$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{2^2 s} + \frac{1}{\lfloor 2}\cdot\left(\frac{1}{2^2 s}\right)^2 - \frac{1}{\lfloor 3}\left(\frac{1}{2^2 s}\right)^3 + \ldots\ldots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[e^{-\frac{1}{2^2 s}}\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\, e^{-\frac{1}{4s}}$$

$$\therefore \mathcal{L}\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}}\, e^{-\frac{1}{4s}}.$$

This can be proved by other methods also.

Example 20. Prove that

$$\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{4s}}.$$

Proof . Let $F(t) = \sin\sqrt{t}$.

Then $F'(t) = \dfrac{\cos\sqrt{t}}{2\sqrt{t}}$ and $F(o) = \sin 0 = 0$.

From the Laplace transform of the derivatives.

we have $\mathcal{L}\{F'(t)\} = s\mathcal{L}\{F(t)\} - F(0)$

$$\therefore \mathcal{L}\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = s\,\mathcal{L}\{\sin\sqrt{t}\} - 0$$

$$= s\,\frac{\sqrt{\pi}}{2s^{3/2}}\, e^{-\frac{1}{4s}}$$

Therefore, $\mathcal{L}\left\{\dfrac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \dfrac{\sqrt{\pi}}{s^{1/2}}\, e^{-\frac{1}{4s}} = \sqrt{\dfrac{\pi}{s}}\, e^{-\frac{1}{4s}}$.

This can be proved by other methods also.

Example 21. Prove that

$$\mathcal{L}\{e^{-at} J_0(bt)\} = \frac{1}{\sqrt{(s+a)^2 + b^2}}.$$

Prof : By formula, we have **[D. U. M. Sc. (F) 1986]**

if $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{F(at)\} = \dfrac{1}{a} f\left(\dfrac{s}{a}\right)$

$$\mathcal{L}\{J_0(t)\} = \frac{1}{s^2 + 1}$$

$$\therefore \mathcal{L}\{J_0(at)\} = \frac{1}{a} \cdot \frac{1}{\sqrt{\left(\frac{s}{a}\right)^2 + 1}} = \frac{1}{\sqrt{s^2 + a^2}}$$

Similarly, $\mathcal{L}\{J_0(bt)\} = \frac{1}{\sqrt{s^2+b^2}}$.

Again by formula, we have

$\mathcal{L}\{e^{-at}F(t)\} = f(s+a)$ where $f(s) = \mathcal{L}\{F(t)\}$

$\therefore \mathcal{L}\{e^{-at}J_0(bt)\} = \frac{1}{\sqrt{(s+a)^2+b^2}}$.

Example 22. Evaluate : $\mathcal{L}\left\{\frac{\sinh t}{t}\right\}$ **[D. U. M. Sc. (F) 1989]**

Solution : By theorem we have

If $\mathcal{L}\{F(t)\} = F(s)$, then $\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_0^\infty f(u)du$

Let $F(t) = \sinh t$. Then $\mathcal{L}\{\sinh t\} = \mathcal{L}\left\{\frac{1}{2}(e^t - e^{-t})\right\}$

$= \frac{1}{2}\left[\frac{1}{s-1} - \frac{1}{s+1}\right] = \frac{1}{s^2-1}$

$\therefore f(s) = \frac{1}{s^2-1}$. Therefore, $f(u) = \frac{1}{u^2-1}$

Now by theorem $\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)du$

$\mathcal{L}\left\{\frac{\sinh t}{t}\right\} = \int_s^\infty \frac{1}{u^2-1}du$

$= \frac{1}{2}\int_s^\infty \left(\frac{1}{u-1} - \frac{1}{u+1}\right)du$

$= \frac{1}{2}[\log(u-1) - \log(u+1)]_s^\infty$

$= \frac{1}{2}[\log(\infty-1) - \log(s-1) - \log(\infty+1) + \log(s+1)$

$= \frac{1}{2}\log\left(\frac{s+1}{s-1}\right)$

Hence $\mathcal{L}\left\{\frac{\sin ht}{t}\right\} = \frac{1}{2}\log\left(\frac{s+1}{s-1}\right)$.

Example 23. If $F(t) = t^2$, $0 < t < 2$ and $F(t+2) = F(t)$. Then find $\mathcal{L}\{F(t)\}$.

Solution : By Theorem we have if $F(t)$ is a periodic function with period $T > 0$, then

$$\mathcal{L}\{F(t)\} = \frac{\int_0^T e^{-st} F(t)\, dt}{1 - e^{-sT}}.$$

Here $F(t)$ is a periodic function with period $T = 2$.

$$\text{Thus } \mathcal{L}\{F(t)\} = \frac{\int_0^2 e^{-st} t^2\, dt}{1 - e^{-2s}}$$

$$= \frac{\left[-\frac{t^2}{s} e^{-st}\right]_0^2 + \frac{2}{s}\int_0^2 t\, e^{-st}\, dt}{1 - e^{-2s}}$$

$$= \frac{-\frac{4}{s} e^{-2s} + \frac{2}{s}\left[-\frac{t}{s} e^{-st}\right]_0^2 + \frac{2}{s^2}\int_0^2 e^{-st} dt}{1 - e^{-2s}}$$

$$= \frac{-\frac{4}{s} e^{-2s} - \frac{4}{s^2} e^{-2s} + \frac{2}{s^2}\left[-\frac{e^{-st}}{s}\right]_0^2}{1 - e^{-2s}}$$

$$= \frac{-\frac{4}{s} e^{-2s} - \frac{4}{s^2} e^{-2s} - \frac{2}{s^3} e^{-2s} + \frac{2}{s^3}}{1 - e^{-2s}}$$

$$= \frac{-4s^2 e^{-2s} - 4s\, e^{-2s} - 2e^{-2s} + 2}{s^3 (1 - e^{-2s})}$$

$$= \frac{-(4s^2 + 4s + 2)\, e^{-2s} + 2}{s^3 (1 - e^{-2s})}.$$

Example 24. Find the Laplace transform of $F(t)$ where

$$F(t) = \begin{cases} t/a\,, & 0 < t < a \\ 1\,, & t > a. \end{cases}$$

Solution : By definition of Laplace transform.

we have $\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st}F(t)\,dt.$

$$\therefore \mathcal{L}\{F(t)\} = \int_0^a e^{-st}.\, t/a\, dt. + \int_a^\infty e^{-st}.\, 1dt$$

$$= -\frac{1}{sa}[t\, e^{-st}]_0^a + \frac{1}{sa}\int_0^a e^{-st}dt - \frac{1}{s}[e^{-st}]_a^\infty$$

$$= -\frac{e^{-sa}}{s} + 0 - \frac{1}{s^2a}[e^{-st}]_0^a - \frac{1}{s}(0 - e^{-sa})$$

$$= -\frac{e^{-sa}}{s} - \frac{e^{-sa}}{s^2a} + \frac{1}{s^2a} + \frac{e^{-sa}}{s}$$

$$= \frac{1-e^{-sa}}{s^2a}. \text{ Hence } \mathcal{L}\{F(t)\} = \frac{1-e^{-sa}}{s^2a}.$$

Example 25. Find $\mathcal{L}\{t^2(\sin at + e^{at})\}$.

Solustion : Since $\mathcal{L}\{\sin at + e^{at}\} = \frac{a}{s^2 + a^2} + \frac{1}{s-a}$

$$\therefore \mathcal{L}\{t^2(\sin at + e^{at})\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2} + \frac{1}{s-a}\right)$$

$$= \frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2} + \frac{1}{s-a}\right)$$

$$= \frac{d}{ds}\left\{\frac{-2as}{(s^2+a^2)^2} - \frac{1}{(s-a)^2}\right\}$$

$$= \frac{(s^2 + a^2)^2(-2a) + 2as.\, 2(s^2+a^2)2s}{(s^2 + a^2)^4} + \frac{2}{(s-a)^3}$$

$$= \frac{-2a(s^2 + a^2) + 8as^2}{(s^2 + a^2)^3} + \frac{2}{(s-a)^3}$$

$$= \frac{-2as^2 - 2a^3 + 8as^2}{(s^2 + a^2)^3} + \frac{2}{(s-a)^3}$$

$$= \frac{6as^2 - 2a^3}{(s^2 + a^2)^3} + \frac{2}{(s-a)^3} = \frac{2a(3s^2-a^2)}{(s^2 + a^2)^3} + \frac{2}{(s-a)^3}$$

Hence $\mathcal{L}\{t^2(\sin at + e^{at})\} = \frac{2a(3s^2 - a^2)}{(s^2 + a^2)^3} + \frac{2}{(s-a)^3}$.

1.10. Table of Laplace transform theorems

No	Operation	F (t)	$\mathcal{L}\{F(t)\} = f(s)$
1.	Linearity property	$a_1F_1(t) + a_2F_2(t)$	$a_1\mathcal{L}\{F_1(t)\} + a_2\mathcal{L}\{F_2(t)\}$
2.	First translation or Shifting property	$e^{at}F(t)$	$f(s-a)$
3.	Second translation or Shifting property	$G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$	$e^{-as}f(s)$
4.	Change of scale property	$F(at)$	$\frac{1}{a}f\left(\frac{s}{a}\right)$
5.	Differentiation theorems	$F'(t)$ $F^n(t)$	$s\,f(s) - F(0)$ $s^n f(s) - \sum_{r=0}^{n-1} s^{n-r-1} F^r(0)$
6.	Multiplication theorems	$tF(t)$ $t^nF(t)$	$-\frac{d}{ds}f(s)$ $(-1)^n \frac{d^n}{ds^n}f(s)$
7.	Division theorem	$\frac{1}{t}F(t)$	$\int_s^\infty f(u)\,du$
8.	Integral theorem	$\int_0^t F(u)\,du$	$\frac{1}{s}f(s)$
9.	Initial-value theorem	$\lim_{t\to 0} F(t)$	$= \lim_{s\to\infty} s\,\mathcal{L}\{F(t)\}$ $= \lim_{s\to\infty} s\,f(s)$
10.	Final-value theorem	$\lim_{t\to\infty} F(t)$	$= \lim_{s\to 0} s\,\mathcal{L}\{F(t)\}$ $= \lim_{s\to 0} s\,f(s)$
11.	Fundamental theorem for periodic function	$\mathcal{L}\{F(t)\} = \frac{\int_0^T e^{-st}F(t)\,dt}{1-e^{-sT}}$ F (t) is a periodic function of period T	

1. 11 Table of Laplace transforms of some elementary functions.

No	F (t)	$\mathcal{L}[F(t)] = f(s)$
1.	1	$\frac{1}{s}$ $s>0$
2.	t	$\frac{1}{s^2}$ $s>0$
3.	t^n $n=0, 1, 2, 3, \ldots$	$\frac{\lfloor n}{s^{n+1}}$ $s>0$ where $\lfloor n = 1.2.3.\ldots, n$ and $\lfloor 0 = 1$.
4.	e^{at}	$\frac{1}{s-a}$ $s>a$
5.	sin at	$\frac{a}{s^2+a^2}$ $s>0$
6.	cos at	$\frac{s}{s^2+a^2}$ $s>0$
7.	sin h at	$\frac{a}{s^2-a^2}$ $s>\|a\|$
8.	cos h at	$\frac{s}{s^2-a^2}$ $s>\|a\|$
9.	t sin at	$\frac{2as}{(s^2+a^2)^2}$ $s>0$
10.	t cos at	$\frac{s^2-a^2}{(s^2+a^2)^2}$ $s>0$
11.	$t^n e^{at}$ $(n = 1, 2, 3, \ldots)$	$\frac{\lfloor n}{(s-a)^{n+1}}$ $s>a$
12.	e^{at} sin bt	$\frac{b}{(s-a)^2+b^2}$
13.	e^{at} cos bt	$\frac{s-a}{(s-a)^2+b^2}$
14.	e^{at} sinh bt	$\frac{b}{(s-a)^2-b^2}$
15.	e^{at} cos h bt	$\frac{s-a}{(s-a)^2-b^2}$

1.12 Table of Laplace transforms of some special functions

No	F (t)	$\mathcal{L}\{F(t)\} = f(s)$
1.	$J_o(at)$	$\frac{1}{\sqrt{s^2 + a^2}}$
2.	$J_n(at)$	$\frac{(\sqrt{s^2 + a^2} - s)^n}{a^n \sqrt{s^2 + a^2}}$
3.	$\sin\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}} e^{-\frac{1}{4s}}$
4.	$\frac{\cos\sqrt{t}}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}} e^{-\frac{1}{4s}}$
5.	$\text{erf}(\sqrt{t})$	$\frac{1}{s\sqrt{s+1}}$
6.	erf(t)	$\frac{e^{s^2/4}}{s}$ erfc (s/2)
7.	Si (t)	$\frac{1}{s} \tan^{-1}\frac{1}{s}$
8.	Ci (t)	$\frac{\log(s^2 + 1)}{2s}$
9.	Ei (t)	$\frac{\log(s + 1)}{s}$
10.	U (t – a)	$\frac{e^{-as}}{s}$

EXERCISES 1

1. Find the Laplace transforms of

(i) $t^2 - 3t + 5$ (ii) $e^{-4t} + 3e^{-2t}$

(iii) cos hat + sin h at (iv) 6 sin 2t–5cos 2t

(v) $e^{2t} + 4t^3 - 2 \sin 3t + 3 \cos 3t$ (vi) t^2 sin h 4t.

(vii) sin h at sin at (viii) cos at cos h at

(ix) $7e^{2t} + 9e^{-2t} + 5 \cos t + 7t^3 + 5 \sin 3t + 2$

(x) $3t^4 - 2t^3 + 4e^{-3t} - 2 \sin 5t + 3 \cos 2t$.

Answers :

(i) $\frac{2}{s^3} - \frac{3}{s^2} + \frac{5}{s}$, $s > 0$ (ii) $\frac{2(2s + 7)}{(s + 2)(s + 4)}$, $s > -2$.

(iii) $\frac{s + a}{s^2 - a^2}$, $s > |a|$

(iv) $\frac{12 - 5s}{s^2 + 4}$ (v) $\frac{1}{s - 2} + \frac{24}{s^4} + \frac{3(s - 2)}{s^2 + 9}$ (vi) $\frac{24s^2 + 128}{(s^2 - 16)^3}$

(vii) $\frac{2a^2 s}{s^4 + 4a^4}$ (viii) $\frac{s^3}{s^4 + 4a^4}$ (ix) $\frac{16s - 4}{s^2 - 4} + \frac{5s}{s^2 + 1} + \frac{42 + 2s^3}{s^4} + \frac{15}{s^2 + 9}$

(x) $\frac{72}{s^5} - \frac{12}{s^4} + \frac{4}{s + 3} - \frac{10}{s^2 + 25} + \frac{3s}{s^2 + 4}$

2. Find the Laplace transforms of

(i) $(\sin t - \cos t)^2$ (ii) $4 \cos^2 2t$ (iii) $\sin^3 2t$

(iv) $\cosh^2 4t$ (v) $(5e^{2t} - 3)^2$ (vi) $(t + 2)^2 e^t$

(vii) $e^{-t} \sin^2 t$ (viii) $(1 + te^{-t})^3$ (ix) $\sinh^3 2t$

(x) $\cos^3 2t$.

Answers : (i) $\frac{s^2 - 2s + 4}{s(s^2 + 4)}$, $s > 0$ (ii) $\frac{2}{s} + \frac{2s}{s^2 + 16}$, $s > 0$

(iii) $\frac{48}{(s + 4)(s^2 + 36)}$ (iv) $\frac{s^2 - 32}{s(s^2 - 64)}$ (v) $\frac{25}{s - 4} - \frac{30}{s - 2} + \frac{9}{s}$, $s > 4$

(vi) $\frac{4s^2 - 4s + 2}{(s - 1)^3}$ (vii) $\frac{2}{(s + 1)(s^2 + 2s + 5)}$ (viii) $\frac{1}{s} + \frac{3}{(s + 1)^2} + \frac{6}{(s + 2)^3} + \frac{6}{(s + 3)^4}$. (ix) $\frac{48}{(s^2 - 4)(s^2 - 36)}$ (x) $\frac{s(s^2 + 28)}{(s^2 + 4)(s^2 + 36)}$

3. Find the Laplace transforms of

(i) t^2e^{2t} (ii) $2e^{3t}\sin 4t$ (iii) $e^t \sin^2 t$ (iv) $e^{2t}(3\sin 4t - 4\cos 4t)$
(v) $e^{-t}(3\sinh 2t - 5\cosh 2t)$ (vi) $e^{-2t}(3\cos 6t - 5\sin 6t)$
(vii) $t^2 \sin at$ (viii) $t^2 \cos at$ (ix) $te^{-4t}\sin 3t$ (x) $e^{-4t}\cosh 2t$.

Answers : (i) $\frac{2}{(s-2)^3}$ (ii) $\frac{8}{s^2-6s+25}$ (iii) $\frac{2}{(s-1)(s^2-2s+5)}$

(iv) $\frac{20-4s}{s^2-4s+20}$ (v) $\frac{1-5s}{s^2+2s-3}$ (vi) $\frac{3(s-8)}{s^2+4s+40}$

(vii) $\frac{2a(3s^2-a^2)}{(s^2+a^2)^3}$ (viii) $\frac{2s^3-6a^2s}{(s^2+a^2)^3}$

(ix) $\frac{6(s+4)}{(s^2+8s+25)^2}$ (x) $\frac{s+4}{s^2+8s+12}$.

4. Find $\mathcal{L}\{F(t)\}$ where $F(t) = \begin{cases} \cos\left(t-\frac{2\pi}{3}\right), & t > \frac{2\pi}{3} \\ 0, & t < \frac{2\pi}{3} \end{cases}$

Answer : $\frac{se^{-\frac{2\pi s}{3}}}{s^2+1}$.

5. Find $\mathcal{L}\{F(t)\}$ where $F(t) = \begin{cases} (t-1)^2, & t > 1 \\ 0, & t < 1 \end{cases}$

Answer : $\frac{2e^{-s}}{s^3}$.

6. Find $\mathcal{L}\{F(t)\}$ where $F(t) = \begin{cases} \sin\left(t-\frac{2\pi}{3}\right), & t > \frac{2\pi}{3} \\ 0, & t < \frac{2\pi}{3} \end{cases}$

Answer : $\frac{e^{-\frac{2\pi s}{3}}}{s^2+1}$.

7. Using theorem 7 find the Laplace transforms of

(i) e^{at} (ii) $-a\sin at$ (iii) $\frac{\cos\sqrt{t}}{\sqrt{t}}$

Answer : (i) $\frac{1}{s-a}$ (ii) $-\frac{a^2}{s^2+a^2}$ (iii) $\sqrt{\frac{\pi}{s}}e^{-\frac{1}{4s}}$.

8. Find the Laplace transform of the function

$$F(t) = \begin{cases} 3t, & 0 < t < 2 \\ 6, & 2 < t < 4 \end{cases}$$

where F (t) has period 4.

Answer : $\frac{3 - 3e^{-2s} - 6se^{-4s}}{s^2(1 - e^{-4s})}$.

9. Find $\mathcal{L}\{F(t)\}$ where $F(t) = \begin{cases} t, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases}$

and $F(t + 2) = F(t)$ for $t > 0$.

Answer : $\frac{1 - e^{-s}(s + 1)}{s^2(1 - e^{-2s})}$

10. Using Laplace transform prove that

$$\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}.$$

11. Using Laplace transform prove that $\int_0^\infty J_0(t)\,dt = 1$

12. Prove that $\mathcal{L}\{t J_0(at)\} = \frac{s}{(s^2 + a^2)^{3/2}}$

13. Prove that $\mathcal{L}\{tj_1(t)\} = \frac{1}{(s^2 + 1)^{3/2}}$

14. Prove that $\mathcal{L}\{t\,\text{erf}(2\sqrt{t})\} = \frac{3s + 8}{s^2(s + 4)^{3/2}}$

15. Prove that $\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dt = \log\frac{3}{2}$.

16. Prove that $\int_0^\infty te^{-3t}\sin t\,dt = \frac{3}{50}$

17. Prove that $\int_0^\infty te^{-2t}\cos t\,dt = \frac{3}{25}$

18. Prove that $\int_0^\infty t^3 e^{-t} \sin t \, dt = 0$

19. Prove that $\int_0^\infty \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$

20. Prove that $\int_0^\infty \frac{1 - \cos t}{t^2} dt = \frac{\pi}{2}$

21. Prove that $\int_0^\infty t^2 J_0(t) \, dt = -1$.

22. Prove that $\int_0^\infty u e^{-u^2} J_0(au) \, du = \frac{1}{2} e^{-a^2/4}$

23. Prove that $\int_0^\infty \frac{e^{-\sqrt{2}t} \sinh t \sin t}{t} dt = \frac{\pi}{8}$.

24. If $F(t) = \begin{cases} 5 & , 0 < t < 3 \\ 0 & , t > 3. \end{cases}$

Then show that $\mathcal{L}\{F(t)\} = \frac{5(1 - e^{-3s})}{s}$.

25. Show That $\int_0^\infty t e^{-3t} J_0(4t) \, dt = \frac{3}{125}$.

26. Prove that (i) $\mathcal{L}\{t^3 \cos t\} = \frac{6s^4 - 36s^2 + 6}{(s^2 + 1)^4}$

(ii) $\mathcal{L}\{t^n e^{at}\} = \frac{\lfloor n}{(s - a)^{n+1}}$, $s > a$.

CHAPTER TWO

THE INVERSE LAPLACE TRANSFORM

2.1 Definition of inverse Laplace transform

If $f(s)$ is the Laplace transform of a function F(t), i. e. if $\mathcal{L}\{F(t)\} = f(s)$, then F(t) is called the **inverse Laplace transform** of the function $f(s)$ and is symbolically written as $F(t) = \mathcal{L}^{-1}\{f(s)\}$ where $\mathcal{L}^{-1}$ is called the **inverse Laplace trans-formation operator.**

Definition of Null function.

If N (t) is a function of t such that

$$\int_0^\infty N(t)\,dt = 0 \text{ for all } t > 0.$$

Then N (t) is called a **null function.**

2.2 Lerch's theorem (without proof)

Statement : if $\mathcal{L}\{F_1(t)\} = \mathcal{L}\{F_2(t)\}$, then $F_1(t) - F_2(t) = N(t)$. That is, an inverse Laplace transform is unique except for the addition of an arbitrary null function.

Having found the Laplace transforms of the following simple functions, let us now determine directly the inverse Laplace transform of given functions of s.

Since $\mathcal{L}\{t\} = \frac{1}{s^2}$, $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ and $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$,

we can write respectively

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t, \quad \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

and $$\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at.$$

Method—4

WORKED OUT EXAMPLES

Example 1. Prove that (i) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\}=\frac{\sin at}{a}$

(ii) $\mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\}=\frac{t^n}{\lfloor(n+1)}$, $n>1$.

Proof : (i) $\mathcal{L}\left\{\frac{\sin at}{a}\right\}=\frac{1}{a}\mathcal{L}\{\sin at\}=\frac{1}{a}\cdot\frac{a}{s^2+a^2}=\frac{1}{s^2+a^2}$

$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\}=\frac{\sin at}{a}$.

(ii) $\mathcal{L}\left\{\frac{t^n}{\lfloor(n+1)}\right\}=\frac{1}{\lfloor(n+1)}\mathcal{L}\{t^n\}$

$=\frac{1}{\lfloor(n+1)}\cdot\frac{\lfloor(n+1)}{s^{n+1}}=\frac{1}{s^{n+1}}$

$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\}=\frac{t^n}{\lfloor(n+1)}$.

Example 2. Prove that (i) $\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\}=\frac{1}{b}e^{at}\sin bt$

(ii) $\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\}=e^{at}\cos bt$.

Proof : (i) $\mathcal{L}\left\{\frac{1}{b}e^{at}\sin bt\right\}=\frac{1}{b}\mathcal{L}\{e^{at}\sin bt\}$

$=\frac{1}{b}\cdot\frac{b}{(s-a)^2+b^2}=\frac{1}{(s-a)^2+b^2}$

$\therefore \mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\}=\frac{1}{b}e^{at}\sin bt$.

(ii) $\mathcal{L}\{e^{at}\cos bt\}=\frac{s-a}{(s-a)^2+b^2}$

$\therefore \mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\}=e^{at}\cos bt$.

2.3 Some important properties of the inverse Laplace transform

1. Linearity property

Theorem 1. If $\mathcal{L}\{F_1(t)\} = f_1(s)$ and $\mathcal{L}\{F_2(t)\} = f_2(s)$ and c_1 and c_2 are any two constants, then

$$\mathcal{L}^{-1}\{c_1f_1(s) + c_2f_2(s)\} = c_1\mathcal{L}^{-1}\{f_1(s)\} + c_2\mathcal{L}^{-1}\{f_2(s)\}$$
$$= c_1F_1(t) + c_2F_2(t).$$

Proof : Given $\mathcal{L}\{F_1(t)\} = f_1(s)$ and $\mathcal{L}\{F_2(t)\} = f_2(s)$.

$\therefore$ by the definition of inverse Laplace transform, we have

$F_1(t) = \mathcal{L}^{-1}\{f_1(s)\}$ and $F_2(t) = \mathcal{L}^{-1}\{f_2(s)\}$.

Now $\mathcal{L}\{c_1F_1(t) + c_2F_2(t)\} = c_1\mathcal{L}\{F(t)\} + c_2\mathcal{L}\{F_2(t)\}$
$= c_1f_1(s) + c_2f_2(s)$.

$\therefore$ $\mathcal{L}^{-1}\{c_1f_1(s) + c_2f_2(s)\} = c_1F_1(t) + c_2F_2(t)$

$= c_1\mathcal{L}^{-1}\{f_1(s)\} + c_2\mathcal{L}^{-1}\{f_2(s)\}$

This result can be easily generalised.

The above theorem is illustrated by the following examples :

$$\mathcal{L}^{-1}\left\{\frac{2}{s-a} + \frac{3}{s^2} + \frac{4a}{s^2+a^2} + \frac{5s}{s^2-a^2}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + 4\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\}$$
$$+ 5\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\}$$

$$= 2e^{at} + 3t + 4\sin at + 5\cosh at.$$

$$\mathcal{L}^{-1}\left\{\frac{5}{(s-2)^2} + 2\tan^{-1}\frac{1}{s} + \frac{s+2}{s^2+2s+13}\right\}$$

$$= 5\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\} + 2\mathcal{L}^{-1}\left\{\tan^{-1}\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+3^2}\right\}$$

$$= 5t\ e^{2t} + \frac{2\sin t}{t} + e^{-2t}\cos 3t.$$

2. First translation (or shifting) property

Theorem 2. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\{f(s-a)\} = e^{at} F(t).$$

Proof : By definition of Laplace transform we have

$$f(s) = \mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)dt$$

$$\therefore \quad f(s-a) = \int_0^\infty e^{-(s-a)t} F(t)dt$$

$$= \int_0^\infty e^{-st} . \{e^{at} F(t)dt$$

$$= \mathcal{L}\{e^{at} F(t)\}$$

$$\therefore \quad \mathcal{L}^{-1}\{f(s-a)\} = e^{at} F(t).$$

The above theorem is illustrated by the following examples :

Since $\mathcal{L}^{-1}\left\{\frac{s}{s^2+4^2}\right\} = \cos 4t.$

$$\mathcal{L}^{-1}\left\{\frac{s-2}{s^2-4s+20}\right\} = \mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+4^2}\right\}$$

$$= e^{2t}\cos 4t .$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+8s+16}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+4)^2}\right\}$$

$$= e^{-4t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = e^{-4t}.\, t = te^{-4t}.$$

3. Second translation (or shifting) property

Theorem 3. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\{e^{-as} F(s)\} = G(t) \text{ where } G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$$

Proof : By definition of Laplace transform, we have

$$f(s) = \int_0^\infty e^{-st} F(t)dt.$$

$$\therefore \quad e^{-as}f(s) = \int_0^\infty e^{-as}.\, e^{-st}\, F(t)dt$$

$$= \int_0^\infty e^{-s(t+a)}\, F(t)dt$$

Let $t + a = u$

$\therefore$ $dt = du$ and $t = u - a$

Limits $\left.\begin{matrix} t = 0 \\ u = a \end{matrix}\right\} \left.\begin{matrix} t = \infty \\ u = \infty \end{matrix}\right\}$

$$\text{Thus } e^{-as}f(s) = \int_a^\infty e^{-su}\, F(u-a)du = \int_a^\infty e^{-st}\, F(t-a)dt$$

$$= \int_0^a e^{-st}(0)\, dt + \int_a^\infty e^{-st}\, F(t-a)\, dt$$

$$= \int_0^\infty e^{-st}\, G(t)dt = \mathcal{L}\{G(t)\}$$

$\therefore$ $\mathcal{L}^{-1}\{e^{-as}f(s)\} = G(t)$.

The above theorem is illustrated by the following examples :

$$\text{Since } \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^4}\right\} = e^{2t}\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = e^{2t}.\frac{t^3}{\lfloor 3} = \frac{1}{6}t^3e^{2t}.$$

$$\therefore \quad \mathcal{L}^{-1}\left\{\frac{e^{-5s}}{(s-2)^4}\right\} = \begin{cases} \frac{1}{6}(t-5)\, e^{2(t-5)}, & t>5 \\ 0, & t<5. \end{cases}$$

4. Change of scale property

Theorem 4. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\{f(ks)\} = \frac{1}{k}F\left(\frac{t}{k}\right).$$

Proof : By the definition of Laplace transform we have

$$f(s) = \mathcal{L}\{F(t)\} = \int_0^\infty e^{-st}\, F(t)\, dt.$$

$$\therefore \quad f(\mathbf{k}s) = \int_0^\infty e^{-kst} F(t)dt$$

Let $u = kt \therefore du = kdt$ and $t = \frac{u}{k}$

Limits $\left.\begin{matrix} t = 0 \\ u = 0 \end{matrix}\right\} \left.\begin{matrix} t = \infty \\ u = \infty \end{matrix}\right\}$

$$\text{Thus } f(ks) = \int_0^\infty e^{-su} F\left(\frac{u}{k}\right) \cdot \frac{1}{k} du$$

$$= \frac{1}{k} \int_0^\infty e^{-su} F\left(\frac{u}{k}\right) du$$

$$= \frac{1}{k} \int_0^\infty e^{-st} F\left(\frac{t}{k}\right) dt$$

$$= \frac{1}{k} \mathcal{L}\left\{F\left(\frac{t}{k}\right)\right\}$$

$$= \mathcal{L}\left\{\frac{1}{k} F\left(\frac{t}{k}\right)\right\}$$

$$\therefore \quad \mathcal{L}^{-1}\{f(ks)\} = \frac{1}{k} F\left(\frac{t}{k}\right).$$

The above theorem is illustrated by the following examples

Since $\mathcal{L}^{-1}\left\{\frac{s}{s^2 - 3^2}\right\} = \cosh 3t$

we have $\mathcal{L}^{-1}\left\{\frac{5s}{25s^2 - 9}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{5s}{(5s)^2 - 3^2}\right\}$$

$$= \frac{1}{5} \cosh \frac{3t}{5}.$$

Also since $\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 2^2}\right\} = \cos 2t,$

we have

$$\mathcal{L}^{-1}\left\{\frac{3s}{9s^2 + 4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{3s}{(3s)^2 + 2^2}\right\}$$

$$= \frac{1}{3}\cos\frac{2t}{3}.$$

5. Inverse Laplace transform of derivatives

Theorem 5. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ then

$$\mathcal{L}^{-1}\{f^n(s)\} = \mathcal{L}^{-1}\left\{\frac{d^n}{ds^n} f(s)\right\} = (-1)^n t^n F(t).$$

where $n = 1, 2, 3, \ldots\ldots\ldots\ldots\ldots$

Proof : From Laplace transform we have

if $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\{t^n F(t)\} = (-1)^n f^n(s)$

where $f^n(s) = \frac{d^n}{ds^n} f(s)$.

$\therefore \quad \mathcal{L}^{-1}\{(-1)^n f^n(s)\} = t^n F(t)$

Or, $(-1)^n \mathcal{L}^{-1}\{f^n(s)\} = t^n F(t)$.

Or, $\{(-1)^n\}^2 \mathcal{L}^{-1}\{f^n(s)\} = (-1)\, t^n F(t)$

Or, $(-1)^{2n} \mathcal{L}^{-1}\{f^n(s)\} = (-1)^n t^n F(t)$

Or, $\mathcal{L}^{-1}\{f^n(s)\} = (-1)^n t^n F(t)$ since $(-1)^{2n} = 1$

The above theorem is illustrated by the following examples :

Since $\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 4}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2 + 2^2}\right\} = \cos 2t.$

and $\frac{d}{ds}\left(\frac{s}{s^2 + 4}\right) = \frac{(s^2 + 4) - 2s^2}{(s^2 + 4)^2} = \frac{4 - s^2}{(s^2 + 4)^2}$

$\therefore \quad \mathcal{L}^{-1}\left\{\frac{4 - s^2}{(s^2 + 4)^2}\right\} = (-1)t\cos 2t = -t\cos 2t$

or, $\mathcal{L}^{-1}\left\{\frac{s^2 - 4}{(s^2 + 4)^2}\right\} = t\cos 2t.$

6. Inverse Laplace transform of integrals

Theorem 6. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, Then

$$\mathcal{L}^{-1}\left\{\int_s^\infty f(u)\,du\right\} = \frac{F(t)}{t}$$

Proof : From Laplace transform, we have

if $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)du$

$$\therefore \quad \mathcal{L}^{-1}\left\{\int_s^\infty f(u)du\right\} = \frac{F(t)}{t}.$$

The above theorem is illustrated by the following example :

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+1}\right\} = t - \sin t$$

$$\mathcal{L}^{-1}\left\{\int_s^\infty \left(\frac{1}{u^2} - \frac{1}{u^2+1}\right) du\right\} = \mathcal{L}^{-1}\left\{\left[-\frac{1}{u} - \tan^{-1}\frac{1}{u}\right]_s^\infty\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{\pi}{2} + \tan^{-1}s\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s} - \cot^{-1}s\right\} \text{ since } \cot^{-1}s + \tan^{-1}s = \frac{\pi}{2}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s} - \tan^{-1}\frac{1}{s}\right\}$$

$$= 1 - \frac{\sin t}{t}$$

$$= \frac{t}{t} - \frac{\sin t}{t} = \frac{t - \sin t}{t}$$

Hence $\mathcal{L}^{-1}\left\{\int_s^\infty \left(\frac{1}{u^2} - \frac{1}{u^2+1}\right) du\right\} = \frac{t - \sin t}{t}$.

7. Multiplication by s^n

Theorem 7. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ and $F(o) = o$, then

$$\mathcal{L}^{-1}\{sf(s)\} = F'(t)$$

Proof : From Laplace transform of derivatives we have

$$\mathcal{L}^{-1}\{F'(t)\} = sf(s) - F(0)$$
$$= sf(s) - 0 = sf(s), \text{ since } F(0) = 0$$

$$\therefore \quad \mathcal{L}^{-1}\{sf(s)\} = F'(t).$$

we have to note that if $F(0) \neq 0$, then

$$\mathcal{L}^{-1}\{sf(s) - F(0)\} = F'(t)$$

or, $$\mathcal{L}^{-1}\{sf(s)\} = F'(t) + F(0)\,\delta(t)$$

where $\delta(t)$ is the **Dirac delta function** or **unit impulse function.**

Theorem (7) can be generalised to

if $\mathcal{L}^{-1}\{f(s)\} = F(t)$ and $F(0) = 0$, then

$$\mathcal{L}^{-1}\{s^n f(s)\} = F^n(t).$$

The above theorem is illustrated by the following example :

Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2 - 1}\right\} = \sinh t$ and $\sinh 0 = 0$,

then $\mathcal{L}^{-1}\left\{\frac{s}{s^2-1}\right\} = \frac{d}{dt}(\sinh t) = \cosh t.$

8. Division by s

Theorem 8. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)\,du.$$

Proof : Let $G(t) = \int_0^t F(u)\,du$

Then $G'(t) = F(t)$ and $G(0) = 0$.

Now by the Laplace transform of derivatives we have,

$$\mathcal{L}\{G'(t)\} = s\mathcal{L}\{G(t)\} - G(0)$$

or, $\mathcal{L}\{G'(t)\} = s\mathcal{L}\{G(t)\}$ since $G(0) = 0$

Again since $\mathcal{L}\{G'(t)\} = \mathcal{L}\{F(t) = f(s)$

$\therefore \quad f(s) = s\mathcal{L}\{G(t)\}$

or, $\mathcal{L}\{G(t)\} = \frac{f(s)}{s}$..

Hence $\mathcal{L}^{-1}\left\{\frac{f(s)}{s}\right\} = G(t) = \int_0^t F(u)du.$

Theorem 9. $\mathcal{L}^{-1}\left\{\frac{f(s)}{s^2}\right\} = G(t) = \int_0^t \int_0^v F(u)dudv.$

Proof : Let $G(t) = \int_0^t \int_0^v F(u)\, dudv.$

Then $G'(t) = \int_0^t F(u)du$ and $G''(t) = F(t)$

Also we have $G(0) = 0$, $G'(0) = 0$.

Now $\mathcal{L}\{G''(t)\} = s^2\mathcal{L}\{G(t)\} - sG(0) - G'(0)$

$= s^2\mathcal{L}\{G(t)\}$ since $G(0) = G'(0) = 0$

Again since $\mathcal{L}\{G''(t)\} = \mathcal{L}\{F(t)\} = f(s)$

$\therefore \quad f(s) = s^2\,\mathcal{L}\{G(t)\}$

or, $\mathcal{L}\{G(t)\} = \frac{f(s)}{s^2}$

Thus $\mathcal{L}^{-1}\left\{\frac{f(s)}{s^2}\right\} = G(t) = \int_0^t \int_0^v F(u)dudv.$

This theorem can be written as

$$\mathcal{L}^{-1}\left\{\frac{f(s)}{s^2}\right\} = \int_0^t \int_0^t F(t)dt^2.$$

which can be generalized to, $\mathcal{L}^{-1}\left\{\frac{f(s)}{s^n}\right\} = \int_0^t \int_0^t \ldots \int_0^t F(t)\, dt^n.$

The above theorems 8 and 9 are illustrated by the following examples:

Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = \sinh t$

$\therefore$ (i) $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2-1)}\right\} = \int_0^t \sinh u\, du = [\cosh u]_0^t = \cosh t - 1.$

(ii) $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-1)}\right\} = \int_0^t (\cosh u - 1)\, du = [\sinh u - u]_0^t$

$= \sinh t - t - 0$

$= \sinh t - t.$

(iii) $\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2-1)}\right\} = \int_0^t (\sinh u - u)\, du = \left[\cosh u - \frac{u^2}{2}\right]_0^t$

$= \cosh t - 1 - \frac{t^2}{2} - 0$

$= \cosh t - 1 - \frac{t^2}{2}.$

WORKED OUT EXAMPLES

Example 3. Evaluate $\mathcal{L}^{-1}\left\{\frac{5s-6}{s^2+9} - \frac{s-15}{s^2-25}\right\}$

Solution : $\mathcal{L}^{-1}\left\{\frac{5s-6}{s^2+9} - \frac{s-15}{s^2-25}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{5s}{s^2+3^2} - \frac{6}{s^2+3^2} - \frac{s}{s^2-5^2} + \frac{15}{s^2-5^2}\right\}$

$$= 5\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2-5^2}\right\} + 3\mathcal{L}^{-1}\left\{\frac{5}{s^2-5^2}\right\}$$

$$= 5\cos 3t - 2\sin 3t - \cosh 5t + 3\sinh 5t.$$

Example 4. Prove that

$$\mathcal{L}^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\} = 2e^{2t}(3\cos 4t + \sin 4t)$$

D. U. H. (T) 1989

Proof. $\mathcal{L}^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\} = \mathcal{L}^{-1}\left\{\frac{6s-12+8}{s^2-4s+20}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{6(s-2)}{(s-2)^2+16}\right\} + \mathcal{L}^{-1}\left\{\frac{8}{(s-2)^2+16}\right\}$$

$$= 6\mathcal{L}^{-1}\left\{\frac{(s-2)}{(s-2)^2+4^2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{4}{(s-2)^2+4^2}\right\}$$

$$= 6e^{2t}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+4^2}\right\} + 2e^{2t}\,\mathcal{L}^{-1}\left\{\frac{4}{s^2+4^2}\right\}$$

$$= 6e^{2t}\cos 4t + 2e^{2t}\sin 4t$$

$$= 2e^{2t}(3\cos 4t + \sin 4t).$$

Example 5. Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+4)}\right\}$

Solution : By theorem we have **D. U, M. Sc. (F) 1986**

$$\mathcal{L}^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)\,du$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\} = \frac{\sin 2t}{2}$$

$$\therefore \quad \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \int_0^t \frac{\sin 2u}{2}\,du$$

$$= \frac{1}{2}\left[-\frac{1}{2}\cos 2u\right]_0^t$$

$$= -\frac{1}{4}(\cos 2t - 1)$$

$$= \frac{1}{4}(1 - \cos 2t).$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = \int_0^t \frac{1}{4}(1 - \cos 2u)\, du$$

$$= \frac{1}{4}\left[u - \frac{1}{2}\sin 2u\right]_0^t$$

$$= \frac{1}{4}(t - 0) - \frac{1}{8}(\sin 2t - 0)$$

$$= \frac{1}{4}t - \frac{1}{8}\sin 2t.$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+4)}\right\} = \int_0^t \left(\frac{1}{4}u - \frac{1}{8}\sin 2u\right) du$$

$$= \left[\frac{1}{8}u^2 + \frac{1}{16}\cos 2u\right]_0^t$$

$$= \frac{1}{8}(t^2 - 0) + \frac{1}{16}(\cos 2t - 1)$$

$$= \frac{1}{8}t^2 + \frac{1}{16}\cos 2t - \frac{1}{16}.$$

Example 6. By expanding $e^{-k\sqrt{s}}$ where k is a positive constant, show that,

$$\mathcal{L}^{-1}\left\{e^{-k\sqrt{s}}\right\} = \frac{k}{2\sqrt{\pi}\, t^{\frac{3}{2}}} e^{-\frac{k^2}{4t}}.$$ **D. U. M. Sc. (F) 1984**

Proof :

$$e^{-k\sqrt{s}} = 1 - \frac{k\sqrt{s}}{\lfloor 1} + \frac{k^2(\sqrt{s})^2}{\lfloor 2} - \frac{k^3(\sqrt{s})^3}{\lfloor 3} + \frac{k^4(\sqrt{s})^4}{\lfloor 4} - \frac{k^5(\sqrt{s})^5}{\lfloor 5} + \ldots.$$

$$\text{or, } e^{-k\sqrt{s}} = 1 - \frac{ks^{1/2}}{\lfloor 1} + \frac{k^2 s}{\lfloor 2} - \frac{k^3}{\lfloor 3} s^{3/2} + \frac{k^4}{\lfloor 4} s^2 - \frac{k^5}{\lfloor 5} s^{5/2} + \ldots.$$

$$\therefore \quad \mathcal{L}^{-1}\left\{e^{K\sqrt{s}}\right\} = \mathcal{L}^{-1}\left\{1 - \frac{K}{\lfloor 1}s^{\frac{1}{2}} + \frac{K^2}{\lfloor 2}s - \frac{K^3}{\lfloor 3}s^{\frac{3}{2}} + \frac{K^4}{\lfloor 4}s^2 - \frac{K^5}{\lfloor 5}s^{\frac{5}{2}} + \ldots\right\} \quad (1)$$

By Theorem we have

$$\mathcal{L}^{-1}\left\{s^{n+\frac{1}{2}}\right\} = \frac{t^{-n-\frac{3}{2}}}{\lceil -n-\frac{1}{2}}$$

$$= \frac{(-1)^{n+1}}{\sqrt{\pi}} \cdot \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\frac{2n+1}{2} \cdot t^{-n-\frac{3}{2}}$$

Also $\mathcal{L}^{-1}\{s^n\} = 0$ if n is a positive integer. Using the above theorems on the right hand side of (1), we get

$$\mathcal{L}^{-1}\left\{e^{-k\sqrt{s}}\right\} =$$

$$0 - \frac{k}{\lfloor 1}(-1)\frac{1}{2\sqrt{\pi}}t^{\frac{-3}{2}} + \frac{k^2}{\lfloor 2}\cdot 0 - \frac{k^3}{\lfloor 3}\cdot\frac{(-1)^2}{\sqrt{\pi}}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot t^{-\frac{5}{2}}$$

$$+\frac{k^4}{\lfloor 4}\cdot 0 - \frac{k^5}{\lfloor 5}\cdot\frac{(-1)^3}{\sqrt{\pi}}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}t^{-\frac{7}{2}} + \ldots.$$

$$= \frac{k}{2\sqrt{\pi}}t^{-\frac{3}{2}} - \frac{k^3}{\lfloor 3}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{t^{-5/2}}{\sqrt{\pi}} + \frac{k^5}{\lfloor 5}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{t^{-7/2}}{\sqrt{\pi}} - \ldots.$$

$$= \frac{k}{2\sqrt{\pi}\,t^{3/2}}\left[1 - \frac{1}{\lfloor 1}\cdot\frac{k^2}{4t} + \frac{1}{\lfloor 2}\left(\frac{k^2}{4t}\right)^2 - \frac{1}{\lfloor 3}\left(\frac{k^2}{4t}\right)^3 + \ldots\right]$$

$$= \frac{k}{2\sqrt{\pi}\,t^{3/2}}e^{-\frac{k^2}{4t}}.$$

Hence $\mathcal{L}^{-1}\left\{e^{-k\sqrt{s}}\right\} = \frac{k}{2\sqrt{\pi}\,t^{3/2}}e^{\frac{-k^2}{4t}}$.

Example 7. (i) Prove that $\mathcal{L}^{-1}\left\{\frac{1}{s^{3/2}}\right\} = 2\sqrt{\frac{t}{\pi}}$.

D. U. H. 1992

Proof : we know that

$$\mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{\lfloor(n+1)} \text{ where } n > -1.$$

$$\therefore \quad \mathcal{L}^{-1}\left\{\frac{1}{s^{3/2}}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^{\frac{1}{2}+1}}\right\} = \frac{t^{1/2}}{\lfloor(\frac{1}{2}+1)} = \frac{t^{1/2}}{\frac{1}{2}\lfloor\frac{1}{2}}$$

$$= \frac{2t^{\frac{1}{2}}}{\sqrt{\pi}} \text{ since } \lfloor\frac{1}{2} = \sqrt{\pi}.$$

$$\therefore \quad \mathcal{L}^{-1}\left\{\frac{1}{s^{3/2}}\right\} = 2\sqrt{\frac{t}{\pi}}.$$

(ii) Prove that $\mathcal{L}^{-1}\left\{\frac{1}{s^{7/2}}\right\} = \frac{8}{15\sqrt{\pi}}t^{5/2}$.

$$\mathcal{L}^{-1}\left\{\frac{1}{s^{7/2}}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^{\frac{5}{2}+1}}\right\} = \frac{t^{5/2}}{\lfloor(\frac{5}{2}+1)}$$

$$= \frac{t^{5/2}}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\lfloor\frac{1}{2}} = \frac{8}{15\sqrt{\pi}}t^{5/2}.$$

Example 8. Prove that $\mathcal{L}^{-1}\left\{\log\left(1+\frac{1}{s^2}\right)\right\} = \frac{2}{t}(1-\cos t)$.

Proof : We know that

$$\mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{\lfloor n} \text{ for } n = 0, 1, 2, 3, \ldots\ldots$$

and $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots\ldots$

$$\therefore \quad \log\left(1+\frac{1}{s^2}\right)=\frac{1}{s^2}-\frac{1}{2}\cdot\frac{1}{s^4}+\frac{1}{3}\cdot\frac{1}{s^6}-\frac{1}{4}\cdot\frac{1}{s^8}+ \ \ldots\ldots\ldots \quad (1)$$

Taking the inverse Laplace transform of both sides of (1), we get

$$\mathcal{L}^{-1}\left\{\log\left(1+\frac{1}{s^2}\right)\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}-\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}+\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^6}\right\}$$

$$-\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^8}\right\}+\ldots\ldots\ldots$$

$$=t-\frac{1}{2}\cdot\frac{t^3}{\lfloor 3}+\frac{1}{3}\cdot\frac{t^5}{\lfloor 5}-\frac{1}{4}\cdot\frac{t^7}{\lfloor 7}+\ldots\ldots\ldots$$

$$=\frac{2t^2}{2t}-\frac{2}{t}\cdot\frac{t^4}{4\lfloor 3}+\frac{2}{t}\cdot\frac{t^6}{6\lfloor 5}-\frac{2}{t}\cdot\frac{t^8}{8\lfloor 7}+\ldots\ldots\ldots$$

$$=\frac{2}{t}\left[\frac{t^2}{\lfloor 2}-\frac{t^4}{\lfloor 4}+\frac{t^6}{\lfloor 6}-\frac{t^8}{\lfloor 8}+\ldots\ldots\ldots\right]$$

$$=\frac{2}{t}\left[1-1+\frac{t^2}{\lfloor 2}-\frac{t^4}{\lfloor 4}+\frac{t^6}{\lfloor 6}-\frac{t^8}{\lfloor 8}+\ldots\ldots\right]$$

$$=\frac{2}{t}\left[1-\left(1-\frac{t^2}{\lfloor 2}+\frac{t^4}{\lfloor 4}-\frac{t^6}{\lfloor 6}+\frac{t^8}{\lfloor 8}-\ldots\ldots\ldots\right)\right]$$

$$=\frac{2}{t}[1-\cos t] \text{ since } \cos\alpha=1-\frac{\alpha^2}{\lfloor 2}+\frac{\alpha^4}{\lfloor 4}-\frac{\alpha^6}{\lfloor 6}+\ldots\ldots$$

Hence $\mathcal{L}^{-1}\left\{\log\left(1+\frac{1}{s^2}\right)\right\}=\frac{2}{t}(1-\cos t)$.

Example 9. Prove that

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\log\left(1+\frac{1}{s^2}\right)\right\}=2\int_0^t\frac{1-\cos u}{u}\,du.$$

Proof : By theorem we have

$$\mathcal{L}^{-1}\left\{\frac{f(s)}{s}\right\}=\int_0^t F(u)du \quad (1)$$

Let $f(s) = \log\left(1 + \frac{1}{s^2}\right) = \mathcal{L}\{F(t)\}$ (2)

Now $\mathcal{L}^{-1}\left\{\log\left(1 + \frac{1}{s^2}\right)\right\} = \frac{2}{t}(1 - \cos t)$ (3)

Or, $F(t) = \frac{2}{t}(1 - \cos t)$

$\therefore \quad F(u) = \frac{2}{u}(1 - \cos u)$

Thus from (1), we have

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\log\left(1 + \frac{1}{s^2}\right)\right\} = 2\int\frac{1 - \cos u}{u}du.$$

Example 10. Evaluate $\mathcal{L}^{-1}\left\{\frac{8s + 20}{s^2 - 12s + 32}\right\}$

Solution : $\mathcal{L}^{-1}\left\{\frac{8s + 20}{s^2 - 12s + 32}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{8(s - 6) + 68}{(s - 6)^2 - 4}\right\}$

$= 8\mathcal{L}^{-1}\left\{\frac{s - 6}{(s - 6)^2 - 2^2}\right\} + 34\mathcal{L}^{-1}\left\{\frac{2}{(s - 6)^2 - 2^2}\right\}$

$= 8e^{6t}\mathcal{L}^{-1}\left\{\frac{s}{s^2 - 2^2}\right\} + 34e^{6t}\mathcal{L}^{-1}\left\{\frac{2}{s^2 - 2^2}\right\}$

$= 8e^{6t}\cosh 2t + 34e^{6t}\sinh 2t$

$= 8e^{6t}\left[\frac{1}{2}(e^{2t} + e^{-2t})\right] + 34e^{6t}\left[\frac{1}{2}(e^{2t} - e^{-2t})\right]$

$= 4e^{8t} + 4e^{-4t} + 17e^{8t} - 17e^{-4t}$

$= 21e^{8t} - 13e^{-4t}.$

Thus $\mathcal{L}^{-1}\left\{\frac{8s + 20}{s^2 - 12s + 32}\right\} = 21e^{8t} - 13e^{-4t}.$

Method–5

Example 11. Find $\mathcal{L}^{-1}\left\{\frac{3s+2}{(s-1)^5}\right\}$ as a function of t.

Solution : $\mathcal{L}^{-1}\left\{\frac{3s+2}{(s-1)^5}\right\} = \mathcal{L}^{-1}\left\{\frac{3s-3+5}{(s-1)^5}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{3(s-1)}{(s-1)^5} + \frac{5}{(s-1)^5}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{\lfloor 3}{(s-1)^4}\right\} + \frac{5}{\lfloor 4}\mathcal{L}^{-1}\left\{\frac{\lfloor 4}{(s-1)^5}\right\}$$

$$= \frac{1}{2}t^3e^t + \frac{5}{24}t^4e^t$$

Hence $\mathcal{L}^{-1}\left\{\frac{3s+2}{(s-1)^5}\right\} = \left(\frac{1}{2}t^3 + \frac{5}{24}t^4\right)e^t$

2.4 PARTIAL FRACTION DECOMPOSITIONS

Example 12. Evaluate $\mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$

Solution : Let $\frac{2s^2-4}{(s+1)(s-2)(s-3)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s-3}$ (1)

$\therefore \quad 2s^2-4 = A(s-2)(s-3) + B(s+1)(s-3) + C(s+1)(s-2)$ (2)

Putting $s = -1$ in (2), we get

$$2-4 = 12A \therefore \boxed{A = -\frac{1}{6}}$$

Putting $s = 2$ in (2), we get

$$8-4 = -3B \therefore \boxed{B = -\frac{4}{3}}$$

Putting $s = 3$ in (2), we get

$$14 = 4C \therefore \boxed{C = \frac{7}{2}}$$

Thus from (1), we have

$$\frac{2s^2-4}{(s+1)(s-2)(s-3)} = \frac{-\frac{1}{6}}{s+1} + \frac{-\frac{4}{3}}{s-2} + \frac{\frac{7}{2}}{s-3}$$

$$\therefore \quad \mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$$

$$= -\frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{7}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= -\frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}.$$

Example 13. Find the inverse Laplace transform of $\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}$.

Solution : Let $\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}$

$$= \frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}$$

$$= \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3} \quad (1)$$

$$\therefore \quad 2s^2-6s+5 = A(s-2)(s-3) + B(s-3)(s-1) + C(s-1)(s-2) \quad (2)$$

Putting $s = 1$ in (2), we get

$$2-6+5 = 2A, \ \therefore \ \boxed{A = \frac{1}{2}}$$

Putting $s = 2$ in (2), we get

$$8-12+5 = -B, \ \therefore \ \boxed{B = -1}$$

Putting $s = 3$ in (2), we get

$$18-18+5 = 2C \ \therefore \ \boxed{C = \frac{5}{2}}$$

Thus from (1), we have

$$\frac{2s^2-6s+5}{s^3-6s^2+11s-6} = \frac{1}{2}\cdot\frac{1}{s-1} - \frac{1}{s-2} + \frac{5}{2}\cdot\frac{1}{s-3}$$

$$\therefore \quad \mathcal{L}^{-1}\left\{\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= \frac{1}{2}e^{t} - e^{2t} + \frac{5}{2}e^{3t}.$$

Example 14. Evaluate $\mathcal{L}^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\}$

Solution : Let $\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5}$ (1)

$\therefore \quad 5s+3 = A(s^2+2s+5) + (Bs+C)(s-1)$ (2)

Putting $s = 1$ in (2), we get

$8 = 8A \therefore \boxed{A = 1}$

Equating coefficients of s^2 from both sides of (2), we get

$0 = A + B \therefore 1 + B = 0$, Or, $\boxed{B = -1}$

Putting $s = 0$ in (2), we get

$3 = 5A - C \therefore C = 5 - 3 = 2$ or, $\boxed{C = 2}$

Thus from (1), we have

$$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{1}{s-1} + \frac{-s+2}{s^2+2s+5}$$

$$= \frac{1}{s-1} + \frac{-s+2}{(s+1)^2+2^2}$$

$$= \frac{1}{s-1} + \frac{-(s+1)+3}{(s+1)^2+2^2}$$

$$= \frac{1}{s-1} - \frac{(s+1)}{(s+1)^2+2^2} + \frac{3}{2}\cdot\frac{2}{(s+1)^2+2^2}$$

$$\therefore \quad \mathcal{L}^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\} + \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\}$$

$$= e^t - e^{-t}\cdot\cos 2t + \frac{3}{2}e^{-t}\sin 2t.$$

Example 15. Prove that $\mathcal{L}^{-1}\left\{\frac{2s+1}{(s+2)^2(s-1)^2}\right\} = \frac{1}{3}t(e^t - e^{-2t})$

Proof : Let $\frac{2s+1}{(s+2)^2(s-1)^2} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s-1)} + \frac{D}{(s-1)^2}$ (1)

or, $2s + 1 = A(s + 2)(s - 1)^2 + B(s - 1)^2 + C(s - 1)(s + 2)^2 + D(s + 2)^2$ (2)

Putting $s = -2$ in (2), we get

$$-4 + 1 = 0 + 9B + 0 \therefore \boxed{B = -\frac{1}{3}}$$

Putting $s = 1$ in (2), we get

$$2 + 1 = 0 + 0 + 0 + 9D \therefore \boxed{D = \frac{1}{3}}$$

Equating the coefficients of s^3 from both sides of (2) we get

$$0 = A + C \quad \text{(i)}$$

Putting $s = o$ in (2), we get

$$1 = 2A + B - 4C + 4D$$

$$1 = 2A - \frac{1}{3} - 4C + \frac{4}{3}$$

or, $2A - 4C = 0$ (ii)

Solving (i) & (ii), we have $A = 0$, $C = 0$.

Thus from (1), we have

$$\frac{2s + 1}{(s + 2)^2(s - 1)^2} = -\frac{1}{3}\cdot\frac{1}{(s + 2)^2} + \frac{1}{3}\cdot\frac{1}{(s - 1)^2}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{2s + 1}{(s + 2)^2 (s - 1)^2}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s + 2)^2}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s - 1)^2}\right\}$$

$$= -\frac{1}{3}te^{-2t} + \frac{1}{3}te^{t}$$

$$= \frac{1}{3}t(e^{t} - e^{-2t}).$$

Example 16. Prove that

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2 - 2s + 5)(s^2 + 2s + 5)}\right\} = \frac{1}{4}\sin 2t \sinh t.$$

Proof : $\mathcal{L}^{-1}\left\{\frac{s}{(s^2 - 2s + 5)(s^2 + 2s + 5)}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{4}\,\frac{(s^2 + 2s + 5) - (s^2 - 2s + 5)}{(s^2 - 2s + 5)(s^2 + 2s + 5)}\right\}$$

$$= \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+2^2} - \frac{1}{(s+1)^2+2^2}\right\}$$

$$= \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+2^2}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}$$

$$= \frac{1}{4}\cdot\frac{1}{2}e^{t}\sin 2t - \frac{1}{4}\cdot\frac{1}{2}e^{-t}\sin 2t$$

$$= \frac{1}{4}\sin 2t.\ \frac{1}{2}.(e^{t}-e^{-t}) = \frac{1}{4}\sin 2t \sinh t.$$

2.5 Table of some important properties of inverse Laplace transform

No	f(s)	$\mathcal{L}^{-1}\{f(s)\} = F(t)$
1.	$\frac{1}{s}$	1
2.	$\frac{1}{s^2}$	t
3.	$\frac{1}{s^n}$ n = 1, 2, 3, . . .	$\frac{t^{n-1}}{\lfloor(n-1)}$, $\lfloor 0 = 1$
4.	$\frac{1}{s^n}$ n > 0	$\frac{t^{n-1}}{\lceil n}$
5.	$\frac{1}{s-a}$	e^{at}
6.	$\frac{1}{(s-a)^n}$ = 1, 2, 3 . . .	$\frac{t^{n-1}e^{at}}{\lfloor(n-1)}$, $\lfloor 0 = 1$
7.	$\frac{1}{(s-a)^n}$ n > 0	$\frac{t^{n-1}e^{at}}{\lceil(n)}$
8.	$\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$
9.	$\frac{s}{s^2+a^2}$	cosat
10.	$\frac{a}{s^2-a^2}$	sinh at
11.	$\frac{s}{s^2-a^2}$	cosh at

12.	$\frac{2as}{(s^2 + a^2)^2}$	t sin at
13.	$\frac{s^2 - a^2}{(s^2 + a^2)^2}$	t cos at
14.	$\frac{\lfloor n}{(s - a)^{n+1}}$	$t^n e^{at}$ $(n = 1, 2, 3 \ldots)$
15.	$\frac{b}{(s \div a)^2 + b^2}$	e^{at} sinbt
16.	$\frac{s - a}{(s - a)^2 + b^2}$	e^{at} cosbt
17.	$\frac{b}{(s - a)^2 - b^2}$	e^{at} sinhbt
18.	$\frac{s - a}{(s - a)^2 - b^2}$	e^{at} coshbt.

2.6 Definition of convolution

Let F(t) and G(t) be two functions of a class A, then the ***convolution*** of the two functions F(t) and G(t) denoted by $F * G$ is defined by the relation

$$F * G = \int_0^t F(u)\, G(t - u)\, du$$

This relation $F * G$ is also called the ***resultant*** or ***Faltung*** of F and G. Here we have to note the followings :

(1) $F * G$ is commutative i.e $F * G = G * F$

(ii) $F * G$ is associative i.e $(F * G) * H = F * (G * H)$

(iii) $F * G$ is distributive with respect to addition i.e $F * (G + H) = F * G + F * H$.

2.7 Convolution theorem (or convolution property)

Statement : If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ and $\mathcal{L}^{-1}\{g(s)\} = G(t)$, then

$$\mathcal{L}^{-1}\{f(s)\, g(s)\} = \int_0^t F(u)\, G(t - u)\, du = F * G.$$

Proof : Here the required result follows if we can prove that

$$\mathcal{L}\left\{\int_0^t F(u)\, G(t-u)\, du\right\} = f(s)\, g(s) \qquad (1)$$

where $f(s) = \mathcal{L}\{F(t)\}$ and $g(s) = \mathcal{L}\{G(t)\}$.

Now $\mathcal{L}\left\{\int_0^t F(u)\, G(t-u)\, du\right\} = \int_{t=0}^{\infty} e^{-st} \left\{\int_{u=0}^{t} F(u)\, G(t-u)\, du\right\} dt$

$$= \int_{t=0}^{\infty} \int_{u=0}^{t} e^{-st} F(u)\, G(t-u)\, du\, dt$$

$$= \lim_{T \to \infty} s_T \text{ where } s_T = \int_{t=0}^{T} \int_{u=0}^{t} e^{st} F(u)\, G(t-u)\, du\, dt \qquad (2)$$

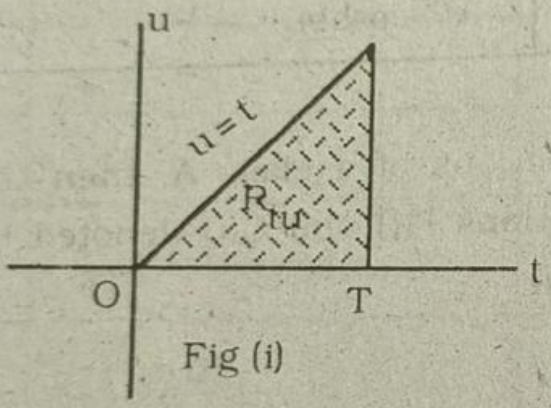


Fig (i)

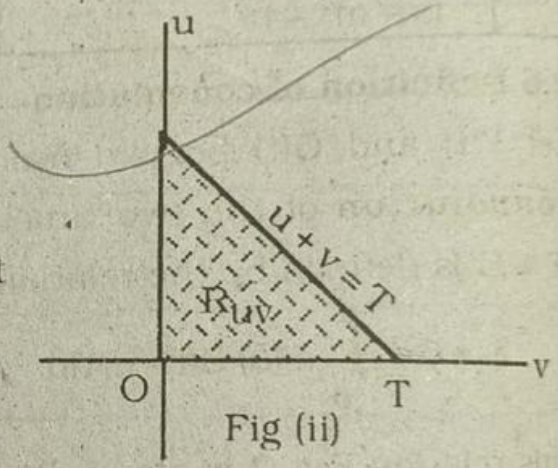


Fig (ii)

The region in the tu plane over which the integration (2) is defined is shown shaded in Fig (i). Letting $t - u = v$ or $t = u + v$, the shaded region R_{tu} of the tu plane is transformed into the shaded region R_{uv} of the uv plane shown in Fig (ii). Then by the theorem on transformation of multiple integrals, we have

$$s_T = \iint_{R_u} e^{-st} F(u)\, G(t-u)\, du\, dt$$

$$= \iint_{R_{uv}} e^{-s(u+v)} \cdot F(u)\, G(v) \frac{\partial(u,t)}{\partial(u,v)}\, du\, dv \qquad (3)$$

where the **Jacobian** of the transformation is

$$J = \frac{\partial(u,t)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\delta u} & \frac{\partial u}{\delta v} \\ \frac{\partial t}{\delta u} & \frac{\partial t}{\delta v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

Thus the right hand side of (3) is

$$s_T = \int_{v=0}^{T} \int_{u=0}^{T-v} e^{-s(u+v)} F(u)\, G(v)\, du dv \quad (4)$$

Let us define a new function

$$K(u, v) = \begin{cases} e^{-s(u+v)} F(u)\, G(v) & \text{if } u + v \le T \\ 0 & \text{if } u + v > T. \end{cases} \quad (5)$$

This function is defined over the square of Fig (iii). But, as indicated in (5) is zero over the unshaded portion of the square.

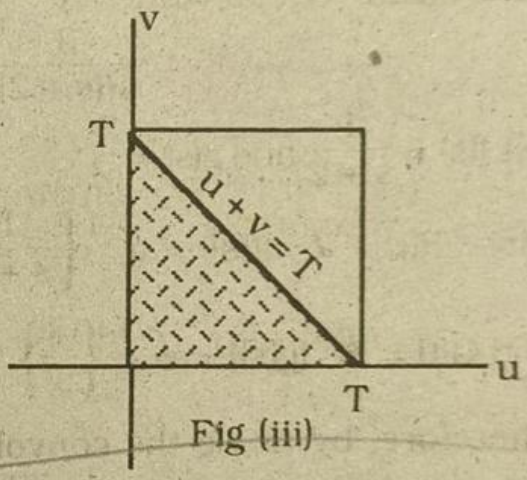


Fig (iii)

In terms of new function we can write (4) as

$$s_T = \int_{v=0}^{T} \int_{u=0}^{T} K(u, v)\, du dv$$

Then $\lim_{T \to \infty} s_T = \int_0^{\infty} \int_0^{\infty} K(u, v)\, du dv$

$$= \int_0^{\infty} \int_0^{\infty} e^{-s(u+v)} F(u)\, G(v)\, du dv.$$

$$= \left\{ \int_0^{\infty} e^{-su} F(u) du \right\} \left\{ \int_0^{\infty} e^{-sv} G(v) dv \right\} = f(s)\, g(s)$$

which establishes the theorem.

Note : $F * G = G * F$.

Proof : Letting $t - u = v$ or $u = t - v$, we have

$$F*G=\int_0^t F(u)\,G(t-u)\,du = \int_0^t F(t-v)\,G(v)\,dv$$

$$= \int_0^t G(v)\,F(t-v)\,dv = G * F.$$

$$\therefore \quad F * G = G * F.$$

WORKED OUT EXMAPLES

Example 17. Evaluate $\mathcal{L}^{-1}\left\{\frac{3}{s^2(s+2)}\right\}$ by use of the convolution theorem.

Solution : We can write

$$\frac{3}{s^2(s+2)} = \frac{1}{s+2}\cdot\frac{3}{s^2}$$

Let $f(s) = \frac{1}{s+2}$ and $g(s) = \frac{3}{s^2}$

Since $F(t) = \mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$

and $G(t) = \mathcal{L}^{-1}\{g(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{s^2}\right\} = 3\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = 3t$

Therefore, by using the convolution theorem,

we get $\mathcal{L}^{-1}\left\{\frac{3}{s^2(s+2)}\right\} = \int_0^t e^{-2u}\,3(t-u)\,du$

$= 3\int_0^t te^{-2u}du - 3\int_0^t ue^{-2u}du.$

$= -\frac{3t}{2}\left[e^{-2u}\right]_0^t + \frac{3}{2}\left[ue^{-2u}\right]_0^t - \frac{3}{2}\int_0^t e^{-2u}du$

$= -\frac{3t}{2}(e^{-2t}-1) + \frac{3}{2}te^{-2t} - 0 + \frac{3}{4}\left[e^{-2u}\right]_0^t$

$= -\frac{3t}{2}e^{-2t} + \frac{3t}{2} + \frac{3t}{2}e^{-2t} + \frac{3}{4}e^{-2t} - \frac{3}{4}.$

$= \frac{3t}{2} + \frac{3}{4}e^{-2t} - \frac{3}{4}.$

Hence $\mathcal{L}^{-1}\left\{\frac{3}{s^2(s+2)}\right\} = \frac{3t}{2} + \frac{3}{4}e^{-2t} - \frac{3}{4}.$

Example 18. Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\}$ by use of the convolution theorem.

Solution : We can write

$$\frac{1}{s^2(s^2+4)} = \frac{1}{s^2+4}\cdot\frac{1}{s^2}$$

Let $f(s) = \frac{1}{s^2+4}$ and $g(s) = \frac{1}{s^2}$

Then since $\mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t$

and $\mathcal{L}^{-1}\{g(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$

$\therefore$ By use of the convolution theorem

$\mathcal{L}^{-1}\{f(s)g(s)\} = \int_0^t F(u)\,G(t-u)\,du$, we have

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = \int_0^t \frac{1}{2}\sin 2u\,(t-u)\,du$$

$$= \frac{1}{2}t\int_0^t \sin 2u\,du - \frac{1}{2}\int_0^t u\sin 2u\,du$$

$$= \frac{1}{2}t\left[-\frac{1}{2}\cos 2u\right]_0^t + \frac{1}{4}\left[u\cos 2u\right]_0^t - \frac{1}{4}\int_0^t \cos 2u\,du$$

$$= -\frac{1}{4}t\,(\cos 2t - 1) + \frac{1}{4}t\cos 2t - 0 - \frac{1}{8}\left[\sin 2u\right]_0^t$$

$$= -\frac{1}{4}t\cos 2t + \frac{1}{4}t + \frac{1}{4}t\cos 2t - \frac{1}{8}\sin 2t + 0$$

$$= \frac{1}{4}t - \frac{1}{8}\sin 2t.$$

Hence $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = \frac{1}{4}t - \frac{1}{8}\sin 2t.$

Example 19. Evaluate $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$ by use of the convolution theorem. **[D. U. M. Sc(F) 1989]**

Solution : We can write $\frac{s}{(s^2+a^2)^2} = \frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}$

Then since $\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$ and $\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$

Therefore, by using the convolution theorem

$\mathcal{L}^{-1}\{f(s)g(s)\} = \int_0^t F(u)\,G(t-u)\,du$, we have

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \int_0^t \cos au.\frac{\sin a(t-u)}{a}\,du$$

$$= \frac{1}{a}\int_0^t \cos au \sin(at-au)\,du$$

$$= \frac{1}{a}\int_0^t \cos au(\sin at\cos au - \cos at\sin au)du$$

$$= \frac{1}{a}\sin at\int_0^t \cos^2 au\,du - \frac{1}{a}\cos at\int_0^t \sin au\cos au\,du$$

$$= \frac{1}{2a}\sin at\int_0^t (1+\cos 2au)du - \frac{1}{2a}\cos at\int_0^t \sin 2au\,du$$

$$= \frac{\sin at}{2a}\left[u+\frac{\sin 2au}{2a}\right]_0^t + \frac{\cos at}{2a}\left[\frac{\cos 2au}{2a}\right]_0^t$$

$$= \frac{\sin at}{2a}\left[t+\frac{\sin 2at}{2a}-0\right] + \frac{\cos at}{4a^2}(\cos 2at-1)$$

$$= \frac{t\sin at}{2a} + \frac{\sin at}{2a}\cdot\frac{2\sin at\cos at}{2a} - \frac{\cos at}{4a^2}\cdot 2\sin^2 at$$

$$= \frac{t\sin at}{2a} + \frac{\sin^2 at\cos at}{2a^2} - \frac{\sin^2 at\cos at}{2a^2} = \frac{t\sin at}{2a}.$$

Hence $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t\sin at}{2a}$.

Example 20. Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}$ by using the convolution theorem. **[D. U. M. Sc. (F) 1985]**

Solution : We can write $\frac{1}{s^2(s+1)^2} = \frac{1}{(s+1)^2} \cdot \frac{1}{s^2}$

Then since $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$ and $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t}$,

By using the convolution theorem

$\mathcal{L}^{-1}\{f(s)g(s)\} = \int_0^t F(u)\, G(t-u)du$, we have

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = \int_0^t ue^{-u}(t-u)du$$

$$= \int_0^t (ut - u^2)e^{-u}du$$

$$= [-(ut - u^2)e^{-u}]_0^t + \int_0^t (t-2u)\, e^{-u}\, du$$

$$= 0 + [-(t-2u)\, e^{-u}]_0^t + (0-2)\int_0^t e^{-u}du$$

$$= -t\,(e^{-t} - 1) + 2\,(te^{-t} - 0) + 2[e^{-u}]_0^t$$

$$= -te^{-t} + t + 2te^{-t} + 2e^{-t} - 2.$$

$$= te^{-t} + 2e^{-t} + t - 2$$

Therefore, $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = te^{-t} + 2e^{-t} + t - 2.$

2.8 Heaviside's expansion formula

Statement : Let P(s) and Q(s) be polynomials in s where P(s) has degree less than that of Q(s). If Q(s) has n distinct zeroes α_k, k = 1, 2, 3 ..... n, that is,

$Q(s) = (s-\alpha_1)(s-\alpha_2)(s-\alpha_3)\ldots\ldots\ldots(s-\alpha_n)$.

then $\mathcal{L}^{-1}\left\{\frac{p(s)}{Q(s)}\right\} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)} e^{\alpha_k t}$

which is often called **Heaviside's expansion theorem or formula.**

Proof : Since P(s) is a polynomial of degree less than that of Q(s) and Q(s) has distinct zeroes $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_n$, we can write according to the method of partial fractions

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-\alpha_1} + \frac{A_2}{s-\alpha_2} + \ldots\ldots\ldots + \frac{A_k}{s-\alpha_k} + \ldots\ldots\ldots + \frac{A_n}{s-\alpha_n} \quad (1)$$

Multiplying both sides of (1) by $s-\alpha_k$ and letting $s \to \alpha_k$, we find using L' Hospital's rule $A_k = \lim_{s \to \alpha_k} \frac{P(s)}{Q(s)} (s-\alpha_k)$

$$= \lim_{s \to \alpha_k} P(s) \left\{\frac{s-\alpha_k}{Q(s)}\right\}$$

$$= \lim_{s \to \alpha_k} P(s) \lim_{s \to \alpha_k} \left(\frac{s-\alpha_k}{Q(s)}\right) \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= P(\alpha_k) \lim_{s \to \alpha_k} \left\{\frac{1}{Q'(s)}\right\} = \frac{P(\alpha_k)}{Q'(\alpha_k)}$$

$$\therefore A_1 = \frac{P(\alpha_1)}{Q'(\alpha_1)}, A_2 = \frac{P(\alpha_2)}{Q'(\alpha_2)}, \ldots\ldots\ldots, A_n = \frac{P(\alpha_n)}{Q'(\alpha_n)}$$

$$\frac{P(s)}{Q(s)} = \frac{P(\alpha_1)}{Q'(\alpha_1)} \cdot \frac{1}{s-\alpha_1} + \frac{P(\alpha_2)}{Q'(\alpha_2)} \cdot \frac{1}{s-\alpha_2} + \ldots\ldots\ldots\ldots$$

$$+ \frac{p(\alpha_n)}{Q'(\alpha_n)} \cdot \frac{1}{s-\alpha_n} \quad (2)$$

Now taking the inverse Laplace transform of both sides of (2), we get

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \frac{P(\alpha_1)}{Q'(\alpha_1)} \mathcal{L}^{-1}\left\{\frac{1}{s-\alpha_1}\right\} + \frac{P(\alpha_2)}{Q'(\alpha_2)} \mathcal{L}^{-1}\left\{\frac{1}{s-\alpha_2}\right\}$$

$$+ \ldots\ldots\ldots + \frac{P(\alpha_n)}{Q'(a_n)} \mathcal{L}^{-1}\left\{\frac{1}{s-\alpha_n}\right\}$$

$$= \frac{P(\alpha_1)}{Q'(\alpha_1)} e^{\alpha_1 t} + \frac{P(\alpha_2)}{Q'(\alpha_2)} e^{\alpha_2 t} + \ldots\ldots\ldots + \frac{P(\alpha_n)}{Q'(\alpha_n)} e^{\alpha_n t}$$

$$= \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)} e^{\alpha_k t}$$

Hence $\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)} e^{\alpha_k t}$

Thus Heaviside's expansion theorem is proved.

Example 21. Using Heaviside's expansion formula evaluate $\mathcal{L}^{-1}\left\{\frac{3s+7}{(s+1)(s-3)}\right\}$

Solution : By Heaviside's expansion formula we have

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)} e^{\alpha_k t} \qquad (1)$$

In the given problem, we have $P(s) = 3s + 7$ and $Q(s) = (s+1)(s-3) = s^2 - 2s - 3$

$\therefore Q'(s) = 2s - 2$

$n = 2, \alpha_1 = -1, \alpha_2 = 3$

Thus $\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \frac{P(\alpha_1)}{Q'(\alpha_1)} e^{\alpha_1 t} + \frac{P(\alpha_2)}{Q'(\alpha_2)} e^{\alpha_2 t}$

$$= \frac{P(-1)}{Q'(-1)} e^{-t} + \frac{P(3)}{Q'(3)} e^{3t} \qquad (2)$$

$P(\alpha_1) = P(-1) = 3(-1) + 7 = -3 + 7 = 4$

$Q'(\alpha_1) = Q'(-1) = 2(-1) - 2 = -2 - 2 = -4$

$P(\alpha_2) = P(3) = 3.3 + 7 = 9 + 7 = 16$

$Q'(\alpha_2) = Q'(3) = 2.3 - 2 = 6 - 2 = 4$

Putting these values in (2), we get

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \frac{4}{-4} e^{-t} + \frac{16}{4} e^{3t} = -e^{-t} + 4e^{3t}$$

Hence $\mathcal{L}^{-1}\left\{\frac{3s+7}{(s+1)(s-3)}\right\} = 4e^{3t} - e^{-t}$.

Eample 22. Using Heaviside's expansion theorem evaluate

$$\mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$$

Solution : By Heaviside's expansion theorem we have

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)} e^{\alpha_k t} \quad (1)$$

In the given problem we have $P(s) = 2s^2 - 4$,

$$Q(s) = (s+1)(s-2)(s-3)$$
$$= s^3 - 4s^2 + s + 6$$
$$\therefore\ Q'(s) = 3s^2 - 8s + 1$$

$n = 3,\ \alpha_1 = -1,\ \alpha_2 = 2,\ \alpha_3 = 3$

$$\text{Thus } \mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \frac{P(\alpha_1)}{Q'(\alpha_1)} e^{\alpha_1 t} + \frac{P(\alpha_2)}{Q'(\alpha_2)} e^{\alpha_2 t} + \frac{P(\alpha_3)}{Q'(\alpha_3)} e^{\alpha_3 t}$$

$$= \frac{P(-1)}{Q'(-1)} e^{-t} + \frac{P(2)}{Q'(2)} e^{2t} + \frac{P(3)}{Q'(3)} e^{3t} \quad (2)$$

$P(\alpha_1) = P(-1) = 2(-1)^2 - 4 = 2 - 4 = -2$

$Q'(\alpha_1) = Q'(-1) = 3(-1)^2 - 8(-1) + 1 = 3 + 8 + 1 = 12$

$P(\alpha_2) = P(2) = 2.\ 2^2 - 4 = 8 - 4 = 4$

$Q'(\alpha_2) = Q'(2) = 3.2^2 - 8.2 + 1 = 12 - 16 + 1 = -3$

$P(\alpha_3) = P(3) = 2.3^2 - 4 = 18 - 4 = 14$

$Q'(\alpha_3) = Q'(3) = 3.3^2 - 8.3 + 1 = 27 - 24 + 1 = 4$

Putting these values in (2), we get

$$\mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\} = \frac{-2}{12} e^{-t} + \frac{4}{-3} e^{2t} + \frac{14}{4} e^{3t}$$

$$= -\frac{1}{6} e^{-t} - \frac{4}{3} e^{2t} + \frac{7}{2} e^{3t}.$$

Eample 23. Using Heaviside's expansion formula evaluate

$$\mathcal{L}^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\}$$

Solution : By Heaviside's expansion formula we have

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)} e^{\alpha_k t} \quad (1)$$

In the given problem, we have

$P(s) = 3s + 1$, $Q(s) = (s-1)(s^2+1) = s^3 - s^2 + s - 1$

$\therefore Q'(s) = 3s^2 - 2s + 1.$

$n = 3$, $\alpha_1 = 1$, $\alpha_2 = i$, $\alpha_3 = -i$,

since $s^2 + 1 = (s + i)(s - i)$

Thus $\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \frac{P(\alpha_1)}{Q'(\alpha_1)} e^{\alpha_1 t} + \frac{P(\alpha_2)}{Q'(\alpha_2)} e^{\alpha_2 t} + \frac{p(\alpha_3)}{Q'(\alpha_3)} e^{\alpha_3 t}$

$= \frac{P(1)}{Q'(1)} e^{t} + \frac{P(i)}{Q'(i)} e^{it} + \frac{P(-i)}{Q'(-i)} e^{-it}.$ (2)

$P(\alpha_1) = P(1) = 3 + 1 = 4$

$Q'(\alpha_1) = Q(1) = 3 - 2 + 1 = 2$

$P(\alpha_2) = P(i) = 3i + 1$

$Q'(\alpha_2) = Q'(i) = -3 - 2i + 1 = -2 - 2i$

$P(\alpha_3) = P(-i) = -3i + 1$

$Q'(\alpha_3) = Q'(-i) = -3 + 2i + 1 = -2 + 2i$

Putting these values in (2), we get

$$\mathcal{L}^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\} = \frac{4}{2} e^{t} + \frac{3i+1}{-2-2i} e^{it} + \frac{-3i+1}{-2+2i} e^{-it}$$

$$= 2e^{t} - \frac{1}{2}\left\{\frac{3i+1}{1+i}\right\} e^{it} + \frac{1}{2}\left\{\frac{3i-1}{1-i}\right\} e^{-it}$$

$$= 2e^{t} - \frac{1}{2}\left\{\frac{(3i+1)(1-i)}{1-i^2}\right\} e^{it} + \frac{1}{2}\left\{\frac{(3i-1)(1+i)}{1-i^2}\right\} e^{-it}$$

$$= 2e^{t} - \frac{1}{4}(3i+1)(1-i)\, e^{it} + \frac{1}{4}(3i-1)(1+i)\, e^{-it}$$

$$= 2e^{t} - \frac{1}{4}(3i - 3i^2 + 1 - i)\, e^{it} + \frac{1}{4}(3i + 3i^2 - 1 - i)\, e^{-it}$$

$$= 2e^{t} - \frac{1}{4}(3i + 3 + 1 - i)\, e^{it} + \frac{1}{4}(3i - 3 - 1 - i)\, e^{-it}$$

$$= 2e^{t} - \frac{1}{4}(2i + 4)\, e^{it} + \frac{1}{4}(2i - 4)\, e^{-it}$$

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$$= 2e^t - \frac{1}{2}(i+2)\,e^{it} + \frac{1}{2}(i-2)\,e^{-it}$$

$$= 2e^t - \frac{1}{2}(i+2)(\cos t + i\sin t) + \frac{1}{2}(i-2)(\cos t - i\sin t)$$

$$= 2e^t - \frac{1}{2}i\cos t - \frac{1}{2}i^2\sin t - \cos t - i\sin t + \frac{1}{2}i\cos t - \frac{1}{2}i^2\sin t - \cos t + i\sin t.$$

$$= 2e^t - 2\cos t + \frac{1}{2}\sin t + \frac{1}{2}\sin t$$

$$= 2e^t - 2\cos t + \sin t.$$

Hence $\mathcal{L}^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\} = 2e^t - 2\cos t + \sin t.$

2.9 The beta function

Example 24. Using Laplace transform prove that $\beta(m, n) = \frac{\lceil m \lceil n}{\lceil m+n}$

Proof : By definition of beta function, we have

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx,\ m > 0,\ n > 0$$

Let $G(t) = \int_0^1 x^{m-1}(t-x)^{n-1}\,dx$

Again let $F_1(t) = t^{m-1}$ and $F_2(t) = t^{n-1}$ then

$$G(t) = \int_0^1 F_1(x)\,F_2(t-x)\,dx = F_1 * F_2$$

$$\therefore \mathcal{L}\{G(t)\} = \mathcal{L}\{F_1 * F_2\}$$

$$= \mathcal{L}\{F_1(t)\}.\ \mathcal{L}\{F_2(t)\}$$

$$= \mathcal{L}\{t^{m-1}\}.\ \mathcal{L}\{t^{n-1}\}$$

$$= \frac{\lceil m}{s^m} \cdot \frac{\lceil n}{s^n} = \frac{\lceil m \lceil n}{s^{m+n}}$$

$$\therefore G(t) = \int_0^1 x^{m-1}(t-x)^{n-1}\,dx = \mathcal{L}^{-1}\left\{\frac{\lceil m \lceil n}{s^{m+n}}\right\}$$

$$= \lceil m \lceil n \; \mathcal{L}^{-1}\left\{\frac{1}{s^{m+n}}\right\}$$

$$= \lceil m \lceil n \cdot \frac{t^{m+n-1}}{\lceil (m+n)}$$

since $\mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{\lceil n+1}$

Therefore, $G(t) = \frac{\lceil m \lceil n}{\lceil (m+n)} t^{m+n-1}$

or, $\int_0^1 x^{m-1} (t-x)^{n-1} dx = \frac{\lceil m \lceil n}{\lceil (m+n)} t^{m+n-1}$ (1)

Putting $t = 1$ in (1), we get

$$\int_0^1 x^{m-1} (1-x)^{n-1} dx = \frac{\lceil m \lceil n}{\lceil (m+n)}$$

Hence $\beta(m, n) = \frac{\lceil m \lceil n}{\lceil (m+n)}$.

2.10 Evaluation of integrals

Example 25. Prove that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

by use of Laplace transform. **[D. U. M. SC (F) 1992]**

Proof : Let $(G(t) = \int_0^\infty e^{-tx^2} dx$

From definition of Laplace transform, we have

$$\mathcal{L}\{G(t)\} = \int_0^\infty e^{-st} G(t) \, dt$$

$$= \int_0^\infty e^{-st} \left[\int_0^\infty e^{-tx^2} dx\right] dt$$

$$= \int_0^\infty \left[\int_0^\infty e^{-st} e^{-tx^2} dt\right] dx$$

$$= \int_0^\infty \mathcal{L}\{e^{-tx^2}\} \, dx$$

$$= \int_0^\infty \frac{1}{s + x^2}\, dx$$

since $\mathcal{L}\{e^{-tx^2}\} = \frac{1}{s + x^2}$

$$= \int_0^\infty \frac{dx}{x^2 + (\sqrt{s})^2}$$

$$= \left[\frac{1}{\sqrt{s}} \tan^{-1} \frac{x}{\sqrt{s}} \right]_0^\infty$$

$$= \frac{1}{\sqrt{s}} \tan^{-1} \infty - \frac{1}{\sqrt{s}} \tan^{-1} 0$$

$$= \frac{1}{\sqrt{s}} \cdot \frac{\pi}{2} - \frac{1}{\sqrt{s}} 0 = \frac{\pi}{2\sqrt{s}}.$$

Thus $\mathcal{L}\{G(t)\} = \frac{\pi}{2\sqrt{s}}$.

Now taking the inverse Laplace transform we have

$$G(t) = \mathcal{L}^{-1}\left\{\frac{\pi}{2\sqrt{s}}\right\} = \frac{\pi}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^{\frac{1}{2}}}\right\}$$

$$= \frac{\pi}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^{-\frac{1}{2}+1}}\right\}$$

$$= \frac{\pi}{2} \cdot \frac{t^{-\frac{1}{2}}}{\left|\left(-\frac{1}{2}+1\right)\right|}$$

$$= \frac{\pi}{2} \cdot \frac{t^{-\frac{1}{2}}}{\left|\frac{1}{2}\right|}$$

$$= \frac{\pi}{2} \cdot \frac{1}{\sqrt{t}\sqrt{\pi}} = \frac{1}{2}\sqrt{\frac{\pi}{t}}.$$

Hence $\int_0^\infty e^{-tx^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{t}}$.

Putting t = 1, we have $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

Example 26. Prove that $\int_0^\infty \cos x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$

by use of Laplace transform. **[D. U. M. SC. (F) 1992]**

Proof : Let $G(t) = \int_0^\infty \cos t\, x^2 dx$

Then taking the Laplace transform, we have

$$\mathcal{L}\{G(t)\} = \int_0^\infty e^{-st}\left[\int_0^\infty \cos t\, x^2 dx\right] dt$$

$$= \int_0^\infty \left[\int_0^\infty e^{-st} \cos t\, x^2 dt\right] dx$$

$$= \int_0^\infty \mathcal{L}\{\cos t\, x^2\} dx$$

$$= \int_0^\infty \frac{s}{s^2 + x^4}\, dx \text{ Since } \mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2},\ s > 0$$

Let $x^2 = s\tan\theta$ or $x = \sqrt{s}\sqrt{\tan\theta}$

$s^2 + x^4 = s^2 + s^2\tan^2\theta = s^2\sec^2\theta$

$dx = \sqrt{s}.\frac{1}{2\sqrt{\tan\theta}}\sec^2\theta d\theta$

Limits $\left.\begin{matrix} x = 0 \\ \theta = 0 \end{matrix}\right\} \left.\begin{matrix} x = \infty \\ \theta = \frac{\pi}{2} \end{matrix}\right\}$

$$\therefore \int_0^\infty \frac{s}{s^2 + x^4} dx = \int_0^{\frac{\pi}{2}} \frac{s}{s^2\sec^2\theta}\sqrt{s}.\frac{1}{2\sqrt{\tan\theta}}\sec^2\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{d\theta}{2\sqrt{s}\sqrt{\tan\theta}}$$

$$= \frac{1}{2\sqrt{s}} \int_0^{\frac{\pi}{2}} (\tan\theta)^{-\frac{1}{2}}\, d\theta$$

$$= \frac{1}{2\sqrt{s}} \int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{2}}\theta \cos^{\frac{1}{2}}\theta d\theta$$

$$= \frac{1}{2\sqrt{s}} \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}-1}\theta \cos^{\frac{3}{2}-1}\theta d\theta$$

$$= \frac{1}{2\sqrt{s}} \int_0^{\frac{\pi}{2}} \sin^{2.\frac{1}{4}-1}\theta \cos^{2.\frac{3}{4}-1}\theta d\theta$$

$$= \frac{1}{2\sqrt{s}} \cdot \frac{1}{2} \frac{\lceil\frac{1}{4} \lceil\frac{3}{4}}{\lceil\frac{1}{4}+\frac{3}{4}}$$

[Using the formula $\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$

$= \frac{1}{2}\beta(m, n) = \frac{\lceil m \lceil n}{2\lceil m+n}$]

$$= \frac{1}{4\sqrt{s}} \frac{\lceil\frac{1}{4} \lceil 1-\frac{1}{4}}{\lceil 1}$$

$$= \frac{1}{4\sqrt{s}} \cdot \frac{\pi}{\sin\frac{\pi}{4}} \quad \text{since } \lceil p \lceil 1-p = \frac{\pi}{\sin p\pi}$$

$$= \frac{1}{4\sqrt{s}} \cdot \frac{\pi}{\frac{1}{\sqrt{2}}}$$

$$= \frac{\pi\sqrt{2}}{4\sqrt{s}}$$

$$\therefore \mathcal{L}\{G(t)\} = \frac{\pi\sqrt{2}}{4\sqrt{s}}$$

Now taking the inverse Laplace transform, we have

$$G(t) = \int_0^\infty \cos tx^2\, dx = \frac{\pi\sqrt{2}}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^{\frac{1}{2}}}\right\}$$

$$= \frac{\pi\sqrt{2}}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^{-\frac{1}{2}+1}}\right\}$$

$$= \frac{\pi\sqrt{2}}{4}\cdot\frac{t^{-\frac{1}{2}}}{\left|-\frac{1}{2}+1\right.}$$

$$= \frac{\pi\sqrt{2}}{4}\cdot\frac{t^{-\frac{1}{2}}}{\left|\frac{1}{2}\right.}$$

$$= \frac{\pi\sqrt{2}}{4}\cdot\frac{1}{\sqrt{\pi}\sqrt{t}} = \frac{\sqrt{2\pi}}{4}\cdot\frac{1}{\sqrt{t}}$$

$$\therefore \int_0^\infty \cos tx^2 dx = \frac{\sqrt{2\pi}}{4}\cdot\frac{1}{\sqrt{t}}$$

Putting $t = 1$, we have

$$\int_0^\infty \cos x^2 dx = \frac{\sqrt{2\pi}}{4} = \frac{\sqrt{\pi}}{2\sqrt{2}} = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

Hence $\int_0^\infty \cos x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$

Example 27. Prove that $\int_0^\infty \sin x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$ by use of Laplace transform.

Proof : Let $F(t) = \int_0^\infty \sin tx^2 dx$

Then taking the Laplace transform, we have

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st}\left[\int_0^\infty \sin tx^2 dx\right] dt$$

$$= \int_0^\infty \left[\int_0^\infty e^{-st} \sin tx^2 dt\right] dx$$

$$= \int_0^\infty \mathcal{L}\{\sin tx^2\}\, dx$$

$$= \int_0^\infty \frac{x^2}{s^2 + x^4} dx \text{ since } \mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}$$

Let $x^2 = s\tan\theta$ or $x = \sqrt{s}\sqrt{\tan\theta}$

$s^2 + x^4 = s^2 + s^2\tan^2\theta = s^2\sec^2\theta$

$dx = \sqrt{s}.\frac{1}{2\sqrt{\tan\theta}}.\sec^2\theta d\theta$

Limits $\left.\begin{matrix} x = 0 \\ \theta = 0 \end{matrix}\right\} \left.\begin{matrix} x = \infty \\ \theta = \frac{\pi}{2} \end{matrix}\right\}$

$$\therefore \int_0^\infty \frac{x^2}{s^2 + x^4} = \int_0^{\frac{\pi}{2}} \frac{s\tan\theta}{s^2\sec^2\theta} \cdot \sqrt{s}.\frac{1}{2\sqrt{\tan\theta}} \sec^2\theta d\theta$$

$$= \frac{1}{2\sqrt{s}} \int_0^{\frac{\pi}{2}} (\tan\theta)^{\frac{1}{2}} d\theta$$

$$= \frac{1}{2\sqrt{s}} \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta \cos^{-\frac{1}{2}}\theta d\theta.$$

$$= \frac{1}{2\sqrt{s}} \int_0^{\frac{\pi}{2}} \sin^{\frac{3}{2}-1}\theta \cos^{\frac{1}{2}-1}\theta d\theta$$

$$= \frac{1}{2\sqrt{s}} \int_0^{\frac{\pi}{2}} \sin^{2.\frac{3}{4}-1}\theta \cos^{2.\frac{1}{4}-1}\theta d\theta$$

$$= \frac{1}{2\sqrt{s}} \frac{\sqrt{\frac{3}{4}}\sqrt{\frac{1}{4}}}{2\sqrt{\frac{3}{4}+\frac{1}{4}}}$$

$$= \frac{1}{2\sqrt{s}} \frac{2\sqrt{1-\frac{1}{4}}\sqrt{\frac{1}{4}}}{2\sqrt{1}}$$

$$= \frac{1}{2\sqrt{s}} \cdot \sqrt{\frac{1}{4}}\sqrt{1-\frac{1}{4}}$$

$$= \frac{1}{4\sqrt{s}} \cdot \frac{\pi}{\sin\frac{\pi}{4}}$$

$$= \frac{1}{4\sqrt{s}} \cdot \frac{\pi}{\frac{1}{\sqrt{2}}}$$

$$= \frac{\pi\sqrt{2}}{4\sqrt{s}}$$

$$\therefore \mathcal{L}\{F(t)\} = \frac{\pi\sqrt{2}}{4\sqrt{s}}$$

Now taking the inverse Laplace transform, we have

$$F(t) = \frac{\pi\sqrt{2}}{4} \mathcal{L}^{-1}\left\{\frac{1}{s^{\frac{1}{2}}}\right\}$$

$$= \frac{\pi\sqrt{2}}{4} \mathcal{L}^{-1}\left\{\frac{1}{s^{-\frac{1}{2}+1}}\right\}$$

$$= \frac{\pi\sqrt{2}}{4} \cdot \frac{t^{-\frac{1}{2}}}{\sqrt{\frac{1}{2}}}$$

$$= \frac{\pi\sqrt{2}}{4} \cdot \frac{1}{\sqrt{\pi}\sqrt{t}} = \frac{\sqrt{2\pi}}{4} \cdot \frac{1}{\sqrt{t}}$$

$$\therefore \int_0^\infty \sin t\, x^2 dx = \frac{\sqrt{2\pi}}{4} \cdot \frac{1}{\sqrt{t}}$$

Putting t = 1, we have

$$\int_0^\infty \sin x^2 dx = \frac{\sqrt{2\pi}}{4} = \frac{\sqrt{\pi}}{2\sqrt{2}} = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

Hence $\int_0^\infty \sin x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$

Example 28. Using Laplace transform prove that $J_o(t) = \frac{1}{\pi}\int_0^\pi \cos(t\cos\theta)d\theta$

Proof : Let $F(t) = \frac{1}{\pi}\int_0^\pi \cos(t\cos\theta)d\theta$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(t\cos\theta)d\theta \quad (1)$$

Taking Laplace transform, we have

$$\mathcal{L}\{F(t)\} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \mathcal{L}\{\cos(t\cos\theta)\}d\theta$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{s}{s^2 + \cos^2\theta}d\theta$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{s\sec^2\theta d\theta}{s^2\sec^2\theta + 1}$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{s\sec^2\theta d\theta}{s^2\tan^2\theta + s^2 + 1} \qquad (2)$$

Putting $x = s\tan\theta$ so that $dx = s\sec^2\theta d\theta$.

Limits $\left.\begin{matrix}\theta = 0 \\ x = 0\end{matrix}\right\} \left.\begin{matrix}\theta = \frac{\pi}{2} \\ x = \infty\end{matrix}\right\}$

$$\therefore \mathcal{L}\{F(t)\} = \frac{2}{\pi}\int_0^{\infty} \frac{dx}{x^2 + s^2 + 1}$$

$$= \frac{2}{\pi}\int_0^{\infty} \frac{dx}{x^2 + (\sqrt{s^2+1})^2}$$

$$= \frac{2}{\pi}\frac{1}{\sqrt{s^2+1}}\left[\tan^{-1}\frac{x}{\sqrt{s^2+1}}\right]_0^{\infty}$$

$$= \frac{2}{\pi}\cdot\frac{1}{\sqrt{s^2+1}}\left(\frac{\pi}{2} - 0\right) = \frac{1}{\sqrt{s^2+1}}$$

Taking the inverse Laplace transform, we have

$$F(t) = \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s^2+1}}\right\} = J_o(t).$$

Hence $J_o(t) = \frac{1}{\pi}\int_0^{\pi} \cos(t\cos\theta)d\theta.$

EXERCISES 2

1. Evaluate $\mathcal{L}^{-1}\left\{\frac{s}{s^2+9} + \frac{6s}{s^2-4} + \frac{3}{s-2}\right\}$

Answer : $\cos 3t + 6\cosh 2t + 3e^{2t}$.

2. Evaluate $\mathcal{L}^{-1}\left\{\frac{2}{s^2} + \frac{5}{s-a} - \frac{3s}{s^2-4} + \frac{10}{s^2-25}\right\}$

Answer : $2t + 5e^{at} - 3\cos 2t + 2\sinh 5t$.

3. Evaluate $\mathcal{L}^{-1}\left\{\frac{12}{s^4}+\frac{3}{s+4}-\frac{15}{s^2-9}+\frac{2s}{s^2-4}\right\}$

Answer : $2t^3 + 3e^{-4t} - 5\sinh 3t + 2\cosh 2t$.

4. Evaluate $\mathcal{L}^{-1}\left\{\frac{3s+2}{s^3}-\frac{s-9}{s^2+9}+\frac{6-30\sqrt{s}}{s^4}\right\}$

Answer : $3t + t^2 - \cos 3t + 3\sin 3t + t^3 - 16\frac{t^{5/2}}{\sqrt{\pi}}$

5. Evaluate $\mathcal{L}^{-1}\left\{\frac{6}{2s-3}-\frac{4s+3}{9s^2-16}+\frac{8-6s}{16s^2+9}\right\}$

Answer : $3e^{\frac{3t}{2}} - \frac{4}{9}\cosh\frac{4t}{3} - \frac{1}{4}\sinh\frac{4t}{3} + \frac{2}{3}\sin\frac{3t}{4} - \frac{3}{8}\cos\frac{3t}{4}$.

6. Evaluate $\mathcal{L}^{-1}\left\{\frac{3s-14}{s^2-4s+8}\right\}$

Answer : $e^{2t}(3\cos 2t - 4\sin 2t)$

7. Prove that $\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \frac{t^2}{2} + \cos t - 1$.

[D. U. M. SC. (F) 1986]

8. Prove that $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+s+1}\right\}$

Answer : $\frac{e^{-\frac{1}{2}t}}{\sqrt{3}}\left(\sqrt{3}\cos\frac{\sqrt{3}t}{2} + \sin\frac{\sqrt{3}t}{2}\right)$.

9. Evaluate $\mathcal{L}^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\}$

Answer : $4e^{3t} - e^{-t}$.

10. Evaluate $\mathcal{L}^{-1}\left\{\frac{4s+12}{s^2+8s+16}\right\}$.

Answer : $4e^{-4t}(1-t)$.

11. Evaluate $\mathcal{L}^{-1}\left\{\frac{3s+2}{4s^2+12s+9}\right\}$

Answer : $\frac{1}{8}e^{-\frac{3t}{2}}(6-5t)$.

12. Evaluate (i) $\mathcal{L}^{-1}\left\{\frac{1-3s}{2s^2-7}\right\}$ (ii) $\mathcal{L}^{-1}\left\{\frac{2s+7}{3s^2+5}\right\}$

Answers : (i) $\frac{1}{\sqrt{14}}\sinh\sqrt{\frac{7}{2}}\,t-\frac{3}{2}\cosh\sqrt{\frac{7}{2}}\,t$.

(ii) $\frac{2}{3}\cos\sqrt{\frac{5}{3}}\,t+\frac{7}{\sqrt{15}}\sin\sqrt{\frac{5}{3}}\,t$

13. Find $\mathcal{L}^{-1}\left\{\frac{7s+4}{4s^2+4s+9}\right\}$ as a function of t.

Answer : $e^{-\frac{t}{2}}\left(\frac{7}{4}\cos\sqrt{2}t+\frac{1}{8\sqrt{2}}\sin\sqrt{2}t\right)$.

14. Evaluate $\mathcal{L}^{-1}\left\{\frac{s+3}{4s^2+4s+1}\right\}$

Answer : $\frac{1}{4}e^{-\frac{t}{2}}+\frac{5}{8}te^{-\frac{t}{2}}$.

15. Find $\mathcal{L}^{-1}\left\{\frac{e^{4-3s}}{(s+4)^{\frac{5}{2}}}\right\}$ as a function of t.

Answer : $\frac{4(t-3)^{\frac{3}{2}}e^{-4(t-4)}}{3\sqrt{\pi}}u(t-3)$.

PARTIAL FRACTION DECOMPOSITIONS

16. Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s+2)(s^2+1)}\right\}$

Answer : $\frac{1}{6}e^{t}-\frac{1}{15}e^{-2t}-\frac{1}{10}\cos t-\frac{3}{10}\sin t$.

17. Evaluate $\mathcal{L}^{-1}\left\{\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right\}$

Answer : $-\frac{I}{3}e^{-t} - \frac{7}{2}t^2e^{2t} + 4te^{2t} + \frac{1}{3}e^{2t}$.

18. Evaluate $\mathcal{L}^{-1}\left\{\frac{s^2 + 2s - 4}{s^4 + 2s^3 + s^2}\right\}$

Answer : $10 - 4t - 10e^{-t} - 5te^{-t}$.

19. Evaluate $\mathcal{L}^{-1}\left\{\frac{8}{s^3(s^2 - s - 2)}\right\}$

Answer : $-3 + 2t - 2t^2 + \frac{1}{3}e^{2t} + \frac{8}{3}e^{-t}$.

20 . Evaluate $\mathcal{L}^{-1}\left\{\frac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)}\right\}$

Answer : $\frac{1}{3}e^{-t}(\sin t + \sin 2t)$.

21. Evaluate $\mathcal{L}^{-1}\left\{\frac{s^2 - 2s + 3}{(s - 1)^2(s + 1)}\right\}$

Answer : $\frac{1}{2}(2t - 1)e^t + \frac{3}{2}e^{-t}$.

22. Evaluate $\mathcal{L}^{-1}\left\{\frac{6s^2 + 22s + 18}{s^3 + 6s^2 + 11s + 6}\right\}$

Answer : $e^{-t} + 2e^{-2t} + 3e^{-3t}$.

23. Evaluate $\mathcal{L}^{-1}\left\{\frac{4s + 5}{(s + 3)(s - 4)^2}\right\}$

Answer : $-\frac{1}{7}e^{-3t} + \frac{1}{7}e^{4t} + 3te^{4t}$.

24. Evaluate $\mathcal{L}^{-1}\left\{\frac{s}{s^4 + s^2 + 1}\right\}$

Answer : $\frac{2}{\sqrt{3}}\sinh\frac{t}{2}\sin(\sqrt{3}\frac{t}{2})$.

25. Prove that

$$\mathcal{L}^{-1}\left\{\frac{2s^3 + 2s^2 + 4s + 1}{(s^2 + 1)(s^2 + s + 1)}\right\} = \cos t + 2\sin t + e^{-\frac{t}{2}}\left[\cos\left(\frac{\sqrt{3}}{2}t\right) - \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}t\right)\right].$$

26. Prove that

$$\mathcal{L}^{-1}\left\{\frac{s^2}{s^4+4a^4}\right\}=\frac{1}{2a}(\text{coshat sinat + sinhat cosat}).$$

27. Prove that

$$\mathcal{L}^{-1}\left\{\frac{11s^2-2s+5}{(s+1)(s-2)(2s-1)}\right\}=5e^{2t}-\frac{3}{2}e^{\frac{t}{2}}+2e^{-t}.$$

28. Prove that

$$\mathcal{L}^{-1}\left\{\frac{3s^3-3s^2-40s+36}{(s^2-4)^2}\right\}=(5t+3)e^{-2t}-2te^{2t}.$$

29. Prove that

$$\mathcal{L}^{-1}\left\{\frac{2s^3-s^2-1}{(s+1)^2(s^2+1)^2}\right\}=\frac{1}{2}\sin t+\frac{1}{2}t\cos t-te^{-t}.$$

Evaluate the followings using the **convolution theorem :**

30. $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\}$

Answer : $\frac{1}{2}(\sin t-\cos t+e^{-t})$

31. $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\}$

Answer : $\frac{1}{2}t\cos 2t+\frac{1}{4}\sin 2t.$

32. $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+3)}\right\}$

Answer : $\frac{1}{3}t-\frac{1}{3\sqrt{3}}\sin\sqrt{3}t.$

33. $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s-2)}\right\}$

Answer : $e^{2t}-e^{t}.$

34. $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$

Answer: $\frac{1}{2}\sin t - \frac{t}{2}\cos t$

35. $\mathcal{L}^{-1}\left\{\frac{8}{(s^2+1)^3}\right\}$

Answer : $(3-t^2)\sin t - 3t\cos t$.

36. $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$

Answer : $\frac{1}{a^2-b^2}(a\sin at - b\sin bt)$.

37. $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^3}\right\}$

Answer : $\frac{1}{64}\, t(\sin 2t - 2t\cos 2t)$.

38. $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-a^2)}\right\}$

Answer : $\frac{1}{a^3}(\sinh at - at)$

Apply covolution theorem to prove the following s:

39. $\int_0^t J_0(t)\, J_0(t-u)\, du = \sin t$

40. $\int_0^t \sin u \cos(t-u)\, du = \frac{1}{2}\, t \sin t.$

41. $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)\, s^{\frac{1}{2}}}\right\} = e^t \operatorname{erf}(\sqrt{t})$

42. $J_0(t) = \frac{1}{\pi}\int_{-1}^{1} e^{itw}\,(1-w^2)^{-\frac{1}{2}}\, dw.$

Using Heaviside's expansion formula prove the followings:

43. $\mathcal{L}^{-1}\left\{\frac{s+5}{(s+1)(s^2+1)}\right\} = 2e^{-t} + 3\sin t - 2\cos t$

44. $\mathcal{L}^{-1}\left\{\frac{19s+37}{(s+1)(s-2)(s+3)}\right\} = 5e^{2t} - 3e^{-t} - 2e^{-3t}.$

45. $\mathcal{L}^{-1}\left\{\frac{s^2-6}{s^3+4s^2+3s}\right\} = \frac{1}{2}e^{-3t} + \frac{5}{2}e^{-t} - 2.$

46. $\mathcal{L}^{-1}\left\{\frac{2s^2+5s-4}{s^3+s^2-2s}\right\} = 2 + e^{t} - e^{-2t}$

47. $\mathcal{L}^{-1}\left\{\frac{1}{s^3+1}\right\}$

$= \frac{1}{3}\left[e^{-t} - e^{\frac{t}{2}}\left(\cos\frac{\sqrt{3}t}{2} - \sqrt{3}\sin\frac{\sqrt{3}t}{2}\right)\right].$

48. $\mathcal{L}^{-1}\left\{\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\} = \frac{1}{2}e^{t} - e^{2t} + \frac{5}{2}e^{3t}.$

49. $\mathcal{L}^{-1}\left\{\frac{s^2}{(s+1)(s+2)(s+3)}\right\} = \frac{1}{2}e^{-t} - 4e^{-2t} - \frac{9}{2}e^{-3t}.$

50. Prove that $\int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2}$ by use of Laplace transform.

51. Using Laplace transform prove the followings :

(i) $\int_0^{\frac{\pi}{2}} \sin(t\sin^2\theta)d\theta = \frac{1}{2}\sin\left(\frac{t}{2}\right)J_0\left(\frac{t}{2}\right)$

(ii) $\int_0^{\frac{\pi}{2}} \cos(t\cos^2\theta)d\theta = \frac{1}{2}\cos\left(\frac{t}{2}\right)J_0\left(\frac{t}{2}\right)$

52. Prove that if $0 < p < 1$

(i) $\int_0^\infty \frac{\sin x}{x^p}dx = \frac{\pi}{2\sqrt{p}\sin\left(\frac{p\pi}{2}\right)}$

(ii) $\int_0^\infty \frac{\cos x}{x^p}dx = \frac{\pi}{2\sqrt{p}\cos\left(\frac{p\pi}{2}\right)}.$

Method–7

CHAPTER THREE

APPLICATIONS OF LAPLACE TRANSFORMS TO DIFFERENTIAL EQUATIONS

3.1. Introduction

Lapalce transform can be applied for solving the ordinary linear differential equations with constant coefficients (or with variable coefficients), the partial differential equations and the boundary value problems. It can also be applied with advantage to the solution of the simultaneous linear differential equations. The Laplace transform methods of solving different types of differential equations are explained through worked out examples.

3.2 Application of Laplace transform in solving ordinary linear differential equations with constant coefficients.

WORKED OUT EXAMPLES

Example 1. Solve the following differential equation by using Laplace transform :

$\frac{dy}{dt} - 3y = o$; $y(o) = 1$.

Solution : The given differential equation can be written as $y' - 3y = o$ (1)

Taking the Laplace transform of both sides of (1) we get

$\mathcal{L}\{y'\} - 3\,\mathcal{L}\{y\} = \mathcal{L}\{o\}$

or, $sY(s) - y(o) - 3Y(s) = 0$

Since $\mathcal{L}(F'(t)\} = sf(s) - F(o)$

or, $sY(s) - 1 - 3Y(s) = o$ since $y(o) = 1$

or, $(s - 3)\,Y(s) = 1$

or, $Y(s) = \frac{1}{s - 3}$ (2)

Now taking the inverse Laplace transform of both sides of (2), we get

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

or, $y(t) = e^{3t}$. since $\mathcal{L}(e^{at}) = \frac{1}{s-a}$, $s > a$.

Example 2. Solve the differential equation

$y' + 2y = e^t$; $y(o) = 1$

by using Laplace transform.

Solution : The given differential equation is

$$y' + 2y = e^t \qquad (1)$$

Taking the Laplace transform of both sides of (1), we get

$$\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^t\}$$

$$sY(s) - y(o) + 2Y(s) = \frac{1}{s-1}$$

$$\text{or, } sY(s) - 1 + 2Y(s) = \frac{1}{s-1}$$

$$\text{or, } (s+2)Y(s) = 1 + \frac{1}{s-1} = \frac{s}{s-1}$$

$$\text{or, } Y(s) = \frac{s}{(s-1)(s+2)} = \frac{1}{3}\cdot\frac{1}{s-1} + \frac{2}{3}\cdot\frac{1}{s+2} \qquad (2)$$

Now taking inverse Laplace transform of both sides of (2), we get

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$\text{or, } y(t) = \frac{1}{3}e^t + \frac{2}{3}e^{-2t}.$$

Example 3. Solve the differential equation

$x' - 5x = e^{5t}$; $x(o) = o$

by using Laplace transform.

Solution : The given differential equation is

$x' - 5x = e^{5t}$ (1)

Taking the Laplace transform of both sides of (1), we get

$\mathcal{L}\{x'\} - 5\mathcal{L}\{x\} = \mathcal{L}\{e^{5t}\}$

Or, $s\,X(s) - x(0) - 5\,X(s) = \frac{1}{s-5}$

or, $s\,X(s) - o - 5\,X(s) = \frac{1}{s-5}$

or, $(s-5)\,X(s) = \frac{1}{s-5}$

or, $X(s) = \frac{1}{(s-5)^2}$ (2)

Now taking the inverse Laplace transform of both sides of (2), we get

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-5)^2}\right\}$$

Or, $x(t) = te^{5t}$

Since $\mathcal{L}^{-1}\left\{\frac{\lfloor n-1}{(s-a)^n}\right\} = t^{n-1}e^{at}$ $(n = 1, 2, ...)$ $(s > a)$.

Example 4. Solve the differential equation
$y' + y = \sin t$; $y(o) = 1$
by using the Laplace transform.

Solution : The given differential equation is

$y' + y = \sin t$ (1)

Taking the Laplace transform of both sides of (1), we get

$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{\sin t\}$

or, $sY(s) - y(o) + Y(s) = \frac{1}{s^2+1}$

since $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$

or, $sY(s) - 1 + Y(s) = \frac{1}{s^2+1}$

$$\text{or, } (s+1)Y(s) = 1 + \frac{1}{s^2+1}$$

$$\text{or, } Y(s) = \frac{1}{s+1} + \frac{1}{(s+1)(s^2+1)} \quad (2)$$

$$\frac{1}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

$$\text{or, } 1 = A(s^2+1) + (Bs+C)(s+1) \quad (3)$$

Taking $s = -1$ in (3), we get

$$1 = 2A + 0, \therefore \boxed{A = \frac{1}{2}}.$$

Again taking the coefficients of s^2 from both sides of (3), we get $0 = A + B$.

$$\therefore B = -A = -\frac{1}{2}. \text{ Or, } B = -\frac{1}{2}$$

Lastly, taking the constant terms from both sides of (3), we get

$$1 = A + C \quad \therefore \boxed{C = \frac{1}{2}}$$

Therefore, $$\frac{1}{(s+1)(s^2+1)} = \frac{1}{2}\cdot\frac{1}{s+1} + \frac{1}{2}\cdot\frac{(-s+1)}{s^2+1}$$

$$= \frac{1}{2}\cdot\frac{1}{s+1} - \frac{1}{2}\cdot\frac{s}{s^2+1} + \frac{1}{2}\cdot\frac{1}{s^2+1}$$

Thus from (2), we get

$$Y(s) = \frac{1}{s+1} + \frac{1}{2}\cdot\frac{1}{s+1} - \frac{1}{2}\cdot\frac{s}{s^2+1} + \frac{1}{2}\cdot\frac{1}{s^2+1}$$

$$= \frac{3}{2}\cdot\frac{1}{s+1} - \frac{1}{2}\cdot\frac{s}{s^2+1} + \frac{1}{2}\cdot\frac{1}{s^2+1} \quad (4)$$

Taking the inverse Laplace transform of both sides of (4), we get

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{2}\cdot\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\text{or, } y(t) = \frac{3}{2}e^{-t} - \frac{1}{2}\cos t + \frac{1}{2}\sin t.$$

Example 5. Solve the differential equation $x' + x = te^{-t}$ by using the Laplace transform.

Solution : The given differential equation is

$x' + x = te^{-t}$ (1)

Taking the Laplace transform of both sides of (1), we get

$\mathcal{L}\{x'\} + \mathcal{L}\{x\} = \mathcal{L}\{te^{-t}\}$

or, $s\,X(s) - x(0) + X(s) = \frac{1}{(s+1)^2}$ (2)

Since no initial condition is given, Let $x(0) = c$, where c denotes an arbitrary constant.

Thus from (2), we have

$s\,X(s) - c + X(s) = \frac{1}{(s+1)^2}$

or, $(s+1)\,X(s) = c + \frac{1}{(s+1)^2}$

Or, $X(s) = \frac{c}{s+1} + \frac{1}{(s+1)^3}$ (3)

Taking the inverse Laplace transform of both sides of (3), we get

$$\mathcal{L}^{-1}\{X(s)\} = c\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^3}\right\}$$

or, $x(t) = ce^{-t} + \frac{1}{2}t^2e^{-t}$ since $\mathcal{L}^{-1}\left\{\frac{\lfloor n-1}{(s+a)^n}\right\} = = t^{n-1}e^{-at}$.

The constant c can be determined (n = 1, 2 , ...) only if an initial condition is given.

Example 6. Solve following differential equation by using Laplace transform : $Y''(t) + Y(t) = t$; $Y(0) = 1$, $Y'(0) = -2$

[D. U. H. (T) 1991]

Solution : The given differential equation is

$Y''(t) + Y(t) = t$ (1)

Taking the Laplace transfrom of both sides of (1) and using the given conditions we get $\mathcal{L}\{Y''(t)\} + \mathcal{L}\{Y(t)\} = \mathcal{L}(t)$

$$\text{or, } s^2y - sY(o) - Y'(o) + y = \frac{1}{s^2}$$

$$\text{or, } s^2y - s + 2 + y = \frac{1}{s^2}$$

$$\text{or, } (s^2 + 1)y = s - 2 + \frac{1}{s^2} = \frac{s^3 - 2s^2 + 1}{s^2}$$

$$\text{or, } y = \frac{s^3 - 2s^2 + 1}{s^2(s^2 + 1)}$$

$$\text{Now } \frac{s^3 - 2s^2 + 1}{s^2(s^2 + 1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 1}$$

$$\text{or, } s^3 - 2s^2 + 1 = As(s^2 + 1) + B(s^2 + 1) + Cs^3 + Ds^2 \quad (2)$$

Equating the coefficients of s^3 from both sides of (2), we get $C = 1$. Putting $s = o$, in (2), we get $B = 1$.

Equating the coefficients of s^2 from both sides of (2), we get $-2 = B + D$, or, $-2 = 1 + D \therefore D = -3$.

Equating the coefficients of s from both sides of (2), wet ge $+0 = A + 0 \therefore A = 0$

$$\therefore y = 0 + \frac{1}{s^2} + \frac{s - 3}{s^2 + 1} = \frac{1}{s^2} + \frac{s}{s^2 + 1} - \frac{3}{s^2 + 1} \quad (3)$$

Taking the inverse Laplace transform of both sides of (3), we get

$$\mathcal{L}^{-1}\{y\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left(\frac{s}{s^2 + 1}\right) - 3\,\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\}$$

or, $Y(t) = t + \cos t - 3\sin t$ which is the required solution.

Example 7. Solve the following differential equation by using the Laplace transform :

$y'' - 3y' + 2y = e^{2t}$; $y(o) = -3$, $y'(o) = 5$.

Solution : The given differential equation is

$$y'' - 3y' + 2y = e^{2t} \quad (1)$$

Taking the Laplace transform of both sides of (1), we get

$$\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\,\mathcal{L}\{y\} = \mathcal{L}\{e^{2t}\}$$

$$\text{or, } s^2Y(s) - sy(o) - y'(o) - 3\{sY(s) - y(o)\} + 2Y(s) = \frac{1}{s - 2}$$

$$\text{or, } s^2Y(s) + 3s - 5 - 3sY(s) - 9 + 2Y(s) = \frac{1}{s-2}$$

$$\text{or, } (s^2 - 3s + 2)\, Y(s) = -3s + 14 + \frac{1}{s-2}$$

$$\text{or, } (s-1)(s-2)\, Y(s) = -3s + 14 + \frac{1}{s-2}$$

$$\text{or, } Y(s) = -\frac{3s}{(s-1)(s-2)} + \frac{14}{(s-1)(s-2)} + \frac{1}{(s-2)^2(s-1)}$$

$$= \frac{-3s^2 + 6s + 14s - 28 + 1}{(s-1\ \ 's-2)^2}$$

$$= \frac{-3s^2 + 20s - 27}{(s-1)(s-2)^2}$$

$$\text{Now } \frac{-3s^2 + 20s - 27}{(s-1)(s-2)^2} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$\text{or, } -3s^2 + 20s - 27 = A(s-2)^2 + B(s-1)(s-2) + C(s-1) \quad (2)$$

Putting $s = 1$ in (2), we get

$-3 + 20 - 27 = A, \therefore A = -10$

Putting $s = 2$ in (2), we get

$-12 + 40 - 27 = C, \therefore C = 1$

Equating the coefficients of s^2 from both sides of (2), we get

$-3 = A + B$

$$\text{or, } -3 = -10 + B$$

$$\text{or, } B = 7$$

$$\therefore Y(s) = \frac{-10}{s-1} + \frac{7}{s-2} + \frac{1}{(s-2)^2} \quad (3)$$

Taking the inverse Laplace transform of both sides of (3), we get

$$\mathcal{L}^{-1}\{Y(s)\} = -10\,\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 7\,\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

or, $y = y(t) = -10e^t + 7e^{2t} + te^{2t}$

which is the required solution.

Example 8. Solve the following differential equation using Laplace transform :

$$Y'' + 9Y = \cos 2t;\ Y(0) = 1,\ Y\left(\frac{\pi}{2}\right) = -1$$

[D. U. H. 1988, 1993]

Solution : The given differential equation is

$$Y'' + 9Y = \cos 2t \quad (1)$$

Taking Laplace transform of both sides of (1), we get

$$\mathcal{L}\{Y''\} + 9\,\mathcal{L}\{Y\} = \mathcal{L}\{\cos 2t\}$$

$$\text{or, } s^2y(s) - sY(0) - Y'(0) + 9y(s) = \frac{s}{s^2+4}$$

$$\text{or, } s^2y(s) - s - c + 9\,y(s) = \frac{s}{s^2+4}$$

where $Y'(0) = c$ (say)

$$\text{or, } (s^2+9)y(s) = s + c + \frac{s}{s^2+4}$$

$$\text{or, } y(s) = \frac{s+c}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}$$

$$\text{or, } y(s) = \frac{s}{s^2+3^2} + \frac{c}{s^2+3^2} + \frac{s}{5(s^2+4)} - \frac{s}{5(s^2+9)} \quad (2)$$

Taking the inverse Laplace transform of both sides of (2), we get

$$\mathcal{L}^{-1}\{y(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \mathcal{L}^{-1}\left\{\frac{c}{s^2+3^2}\right\}$$

$$+\frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\}$$

$$\text{or, } Y(t) = \cos 3t + \frac{c}{3}\sin 3t + \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t$$

$$= \frac{4}{5}\cos 3t + \frac{c}{3}\sin 3t + \frac{1}{5}\cos 2t \quad (3)$$

$$\therefore Y\left(\frac{\pi}{2}\right) = \frac{4}{5}\cos\frac{3\pi}{2} + \frac{c}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\cos\pi$$

$$= -\frac{4}{5}\cos\frac{\pi}{2} - \frac{c}{3}\sin\frac{\pi}{2} + \frac{1}{5}\cos\pi$$

$$\text{or, } -1 = 0 - \frac{c}{3} - \frac{1}{5}, \text{ since } Y\left(\frac{\pi}{2}\right) = -1$$

$$\text{or, } \frac{c}{3} = 1 - \frac{1}{5} = \frac{4}{5}$$

$$\therefore c = \frac{12}{5}$$

Thus from (3), we get

$$Y(t) = \frac{4}{5}\cos 3t + \frac{1}{3}\cdot\frac{12}{5}\sin 3t + \frac{1}{5}\cos 2t$$

$$= \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t + \frac{1}{5}\cos 2t.$$

which is the required solution.

Example 9. Solve the differential equation

$x'' - 4x' + 4x = 4\cos 2t$; $x(0) = 2$, $x'(0) = 5$ using Laplace transform. **[D. U. H.1986]**

Solution : The given differential equation is

$$x'' - 4x' + 4x = 4\cos 2t \quad (1)$$

Taking Laplace transform of both sides of (1), we get

$$\mathcal{L}\{x''\} - 4\mathcal{L}\{x'\} + 4\mathcal{L}\{x\} = 4\mathcal{L}\{\cos 2t\}$$

$$\text{or, } s^2 X(s) - sx(0) - x'(0) - 4\{sX(s) - x(0)\} + 4X(s) = 4.\frac{s}{s^2+4}$$

$$\text{or, } s^2 X(s) - 2s - 5 - 4s X(s) + 8 + 4X(s) = 4.\frac{s}{s^2+4}$$

$$\text{or, } (s^2 - 4s + 4)X(s) = 2s - 3 + \frac{4s}{s^2+4}$$

$$\text{or, } (s-2)^2 X(s) = 2s - 3 + \frac{4s}{s^2+4}$$

$$\text{or, } X(s) = \frac{2s}{(s-2)^2} - \frac{3}{(s-2)^2} + \frac{4s}{(s-2)^2(s^2+4)}$$

$$\text{or, } X(s) = \frac{2(s-2)}{(s-2)^2} + \frac{1}{(s-2)^2} + \frac{4s}{(s-2)^2(s^2+4)} \quad (2)$$

$$\text{Now } \frac{4s}{(s-2)^2(s^2+4)} = \frac{A}{s-2} + \frac{B}{(s-2)^2} + \frac{Cs+D}{s^2+4}$$

$$\text{or, } 4s = A(s-2)(s^2+4) + B(s^2+4) + (Cs+D)(s-2)^2 \quad (3)$$

Putting $s = 2$ in (3), we get

$8 = o + 8B + o \quad \therefore B = 1$

Equating the coefficients of s^3 from both sides of (3), we get

$o = A + C.$

Equating the coefficients of s^2 from both sides of (3), we get

$o = -2A + B + D - 4C$

or, $o = -2(A + c) + 1 + D - 2c$

or, $1 + D - 2c = o$ since $A + C = o$

Equating the constant terms from both sides of (3), we get

$o = -8A + 4B + 4D$

or, $o = -2A + B + D$

or, $2C + 1 + D = o \quad$ since $A + C = 0$

$$\left.\begin{array}{l} \text{Now } 2c + 1 + D = 0 \\ -2c + 1 + D = 0 \end{array}\right\} \text{ Adding}$$

$$2 + 2D = O$$

$\therefore D = -1, C = o, A = o$

$$\therefore \frac{4s}{(s-2)^2(s^2+4)} = \frac{1}{(s-2)^2} - \frac{1}{s^2+2^2}$$

Thus from (2), we have

$$X(s) = \frac{2}{s-2} + \frac{1}{(s-2)^2} + \frac{1}{(s-2)^2} - \frac{1}{s^2+2^2}$$

$$\text{or, } X(s) = \frac{2}{s-2} + \frac{2}{(s-2)^2} - \frac{1}{s^2+2^2} \quad (4)$$

Taking the inverse Laplace transform of both sides of (4), we get

$$\mathcal{L}^{-1}\{X(s)\} = 2.\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\}$$

or, $x = x(t) = 2e^{2t} + 2te^{2t} - \frac{1}{2}\sin 2t.$

which is the required solution.

Example 10. Solve the differential equation

$y'' - y' - 2y = t^2;\ y(0) = 1, y'(0) = 3$

by using Laplace transform. **[D. U. H. 1989]**

Also verify the result.

Solution : The given differential equation is

$$y'' - y' - 2y = t^2 \quad (1)$$

Taking the Laplace transform of both sides of (1), we get

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = \mathcal{L}\{t^2\}$$

or, $s^2Y(s) - sy(0) - y'(0) - \{sY(s) - y(0)\} - 2Y(s) = \frac{2}{s^3}$

or, $s^2Y(s) - s - 3 - sY(s) + 1 - 2Y(s) = \frac{2}{s^3}$

or, $(s^2 - s - 2)\,Y(s) = s + 2 + \frac{2}{s^3}$

or, $Y(s) = \frac{s+2}{s^2 - s - 2} + \frac{2}{s^3(s^2 - s - 2)}$

or, $Y(s) = \frac{s+2}{(s-2)(s+1)} + \frac{2}{s^3(s-2)(s+1)}$

$$\frac{s+2}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$$

or, $s + 2 = A(s+1) + B(s-2) \quad (2)$

Putting $s = 2$ in (2), we get

$4 = 3A \ \therefore A = \frac{4}{3}.$

Again putting $s = -1$ in (2), we get

$1 = -3B \ \therefore B = -\frac{1}{3}.$

$$\therefore \frac{s+2}{(s-2)(s+1)} = \frac{4}{3}\cdot\frac{1}{s-2} - \frac{1}{3}\cdot\frac{1}{s+1} \quad (3)$$

$$\frac{1}{s^3(s-2)(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s-2} + \frac{E}{s+1}$$

or, $1 = As^2(s-2)(s+1) + Bs(s-2)(s+1)$

$+ C(s-2)(s+1) + Ds^3(s+1) + Es^3(s-2)$ (4)

Putting $s = o$ in (4), we get

$1 = -2C \therefore C = -\frac{1}{2}$

Putting $s = 2$ in (4), we get

$1 = 24D, \therefore D = \frac{1}{24}$

Putting $s = -1$, in (4), we get

$1 = 3E, \therefore E = \frac{1}{3}$

Equating the coefficients of s^4 from both sides of (4), we get

$0 = A + o + o + D + E$

or, $A = -\frac{1}{24} - \frac{1}{3} = -\frac{9}{24} = -\frac{3}{8}$

Equating the coefficients of s^3 from both sides of (4) we get

$0 = -A + B + D - 2E$

or, $o = \frac{3}{8} + B + \frac{1}{24} - \frac{2}{3}$

or, $B = -\frac{3}{8} - \frac{1}{24} + \frac{2}{3} = \frac{6}{24} = \frac{1}{4}$

$$\therefore \frac{1}{s^3(s-2)(s+1)} = -\frac{3}{8}\cdot\frac{1}{s} + \frac{1}{4}\cdot\frac{1}{s^2} - \frac{1}{2}\cdot\frac{1}{s^3} + \frac{1}{24}\cdot\frac{1}{s-2} + \frac{1}{3}\cdot\frac{1}{s+1} \quad (5)$$

Using (3) and (5) in the equation

$Y(s) = \frac{s+2}{(s-2)(s+1)} + \frac{2}{s^3(s-2)(s+1)}$, we get

$$Y(s) = \frac{4}{3}\cdot\frac{1}{s-2} - \frac{1}{3}\cdot\frac{1}{s+1} - \frac{3}{4}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{1}{s^2} - \frac{1}{s^3} + \frac{1}{12}\cdot\frac{1}{s-2} + \frac{2}{3}\cdot\frac{1}{s+1} \quad (6)$$

Taking the inverse Laplace transform of both sides of (6), we get

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} + \frac{1}{12}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$\text{or, } y = y(t) = \frac{4}{3}e^{2t} - \frac{1}{3}e^{-t} - \frac{3}{4}\cdot 1 + \frac{1}{2}t - \frac{t^2}{2} + \frac{1}{12}e^{2t} + \frac{2}{3}e^{-t}$$

$$\text{or, } y = \frac{17}{12}e^{2t} + \frac{1}{3}e^{-t} - \frac{3}{4} + \frac{1}{2}t - \frac{1}{2}t^2.$$

which is the required solution.

Verification of the result.

$$y = \frac{17}{12}e^{2t} + \frac{1}{3}e^{-t} - \frac{3}{4} + \frac{1}{2}t - \frac{1}{2}t^2$$

$$y' = \frac{17}{6}e^{2t} - \frac{1}{3}e^{-t} - o + \frac{1}{2} - t$$

$$y'' = \frac{17e^{2t}}{3} + \frac{1}{3}e^{-t} + o - 1$$

$$\therefore y'' - y' - 2y = \frac{17}{3}e^{2t} + \frac{1}{3}e^{-t} - 1 - \frac{17}{6}e^{2t} + \frac{1}{3}e^{-t} - \frac{1}{2} + t - \frac{17}{6}e^{2t} - \frac{2}{3}e^{-t} + \frac{3}{2} - t + t^2$$

$$= \frac{17}{3}e^{2t} - \frac{17e^{2t}}{3} + \frac{2}{3}e^{-t} - \frac{2}{3}e^{-t} - \frac{3}{2} + \frac{3}{2} + t - t + t^2 = t^2$$

$$\therefore y'' - y' - 2y = t^2.$$

$$y = \frac{17}{12}e^{2t} + \frac{1}{3}e^{-t} - \frac{3}{4} + \frac{1}{2}t - \frac{1}{2}t^2$$

$$\therefore y(0) = \frac{17}{12} + \frac{1}{3} - \frac{3}{4} + 0 = \frac{17 + 4 - 9}{12} = \frac{12}{12} = 1$$

$$y' = \frac{17}{6}e^{2t} - \frac{1}{3}e^{-t} + \frac{1}{2} - t$$

$$\therefore y'(0) = \frac{17}{6} - \frac{1}{3} + \frac{1}{2} = \frac{17 - 2 + 3}{6} = \frac{18}{6} = 3$$

Hence the result is verified.

Example 11. Solve the differential equation

$y''(x) + y(x) = x$; $y(0) = 0$, $y'(0) = \pi$ **[D. U. H. 1990]**

Solution : The given differential equation is

$$y''(x) + y(x) = x \quad (1)$$

Taking the Laplace transform of both sides of (1), we get

$$\mathcal{L}\{y''(x)\} + \mathcal{L}\{y(x)\} = \mathcal{L}\{x\}$$

$$\text{or, } s^2Y(s) - s\,y(0) - y'(0) + Y(s) = \frac{1}{s^2}$$

$$\text{or, } s^2Y(s) - 0 - \pi + Y(s) = \frac{1}{s^2}$$

$$\text{or, } (s^2 + 1)\,Y(s) = \pi + \frac{1}{s^2}$$

$$\text{or, } Y(s) = \pi.\frac{1}{s^2 + 1} + \frac{1}{s^2(s^2 + 1)}$$

$$\text{or, } Y(s) = \pi.\frac{1}{s^2 + 1} + \frac{s^2 + 1 - s^2}{s^2(s^2 + 1)}$$

$$\text{or, } Y(s) = \pi.\frac{1}{s^2 + 1} + \frac{1}{s^2} - \frac{1}{s^2 + 1} \quad (2)$$

Taking the inverse Laplace transform of both sides of (2) we get

$$\mathcal{L}^{-1}\{Y(s)\} = \pi\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\}$$

$$\text{or, } y = y(x) = \pi \sin x + x - \sin x$$

$$\text{or, } y = (\pi - 1)\sin x + x$$

which is the required solution.

Example 12. Find using Laplace transform the solution of the following differential equation which satisfy the given conditions :

$y'' - 9y = e^t$; $y(0) = 1, y'(0) = 0$

Also verify the result. **D.U. M. SC. (T) 1991**

Solution : The given differential equation is

$$y'' - 9y = e^t \quad (1)$$

Taking the Laplace transform of both sides of (1), we get

$$\mathcal{L}\{y''\} - 9\,\mathcal{L}\{y\} = \mathcal{L}\{e^t\}$$

$$\text{Or, } s^2Y(s) - sy(0) - y'(0) - 9Y(s) = \frac{1}{s-1}$$

$$\text{Or, } s^2Y(s) - s - 0 - 9Y(s) = \frac{1}{s-1}$$

$$\text{Or, } (s^2 - 9)Y(s) = s + \frac{1}{s-1}$$

$$\text{Or, } Y(s) = \frac{s}{s^2 - 3^2} + \frac{1}{(s^2 - 3^2)(s-1)}$$

$$\text{Or, } Y(s) = \frac{s}{s^2 - 3^2} + \frac{1}{(s-1)(s+3)(s-3)} \quad (2)$$

Taking inverse Laplace transform of both sides of (1), we get

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2 - 3^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s+3)(s-3)}\right\}$$

$$\text{Or, } y = y(t) = \cosh 3t + \mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s+3)(s-3)}\right\} \quad (3)$$

$$\frac{1}{(s-1)(s+3)(s-3)} = -\frac{1}{8}\cdot\frac{1}{s-1} + \frac{1}{24}\cdot\frac{1}{s+3} + \frac{1}{12}\cdot\frac{1}{s-3} \quad (4)$$

Taking the inverse Laplace transform of both sides of (4) we have

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s+3)(s-3)}\right\} = -\frac{1}{8}\,\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{24}\,\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$+ \frac{1}{12}\,\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= -\frac{1}{8}e^t + \frac{1}{24}e^{-3t} + \frac{1}{12}e^{3t}.$$

Thus from (3), we have

$$y = \cosh 3t - \frac{1}{8}e^{t} + \frac{1}{24}e^{-3t} + \frac{1}{12}e^{3t}.$$

which is the required solution.

Verification of the result : Since $\cosh t = \frac{1}{2}(e^{3t} + e^{-3t})$

$$\therefore \quad y = \frac{1}{2}e^{3t} + \frac{1}{2}e^{-3t} - \frac{1}{8}e^{t} + \frac{1}{24}e^{-3t} + \frac{1}{12}e^{3t}$$

$$= \frac{7}{12}e^{3t} + \frac{13}{24}e^{-3t} - \frac{1}{8}e^{t}$$

$$y' = \frac{dy}{dt} = \frac{7}{4}e^{3t} - \frac{13}{8}e^{-3t} - \frac{1}{8}e^{t}$$

$$y'' = \frac{d^2y}{dt^2} = \frac{21}{4}e^{3t} + \frac{39}{8}e^{-3t} - \frac{1}{8}e^{t}$$

$$\text{Or, } y'' = \frac{9.7}{12}e^{3t} + \frac{9.13}{24}e^{-3t} - \frac{9}{8}e^{t} + e^{t}$$

$$= 9\left(\frac{7}{12}e^{3t} + \frac{13}{24}e^{-3t} - \frac{1}{8}e^{t}\right) + e^{t}$$

$$= 9y + e^{t}$$

$$\text{Or, } y'' - 9y = e^{t}.$$

$$y = \frac{7}{12}e^{3t} + \frac{13}{24}e^{-3t} - \frac{1}{8}e^{t}$$

$$\therefore \quad y(o) = \frac{7}{12} + \frac{13}{24} - \frac{1}{8} = \frac{14 + 13 - 3}{24} = \frac{24}{24} = 1.$$

$$y' = \frac{7}{4}e^{3t} - \frac{13}{8}e^{-3t} - \frac{1}{8}e^{t}$$

$$\therefore \quad y'(o) = \frac{7}{4} - \frac{13}{8} - \frac{1}{8} = \frac{14 - 14}{8} = \frac{0}{8} = 0.$$

Hence the result is verified.

Example 13. Solve the following initial value problem using Laplace transform :

$Y'' - Y' = t$; $Y(o) = 2$, $Y'(o) = -3$. **D.U.H. 1990**

Solution : The given differential equation is $Y'' - Y' = t$. (1)

Taking the Laplace transform of both sides of (1), we get

$$\mathcal{L}\{Y''\} - \mathcal{L}\{Y'\} = \mathcal{L}\{t\}$$

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Or, $s^2y(s) - sY(o) - Y'(o) - sy(s) + Y(o) = \frac{1}{s^2}$

Or, $s^2y(s) - 2s + 3 - sy(s) + 2 = \frac{1}{s^2}$

Or, $(s^2 - s)\, y(s) = 2s - 5 + \frac{1}{s^2}$

Or, $y(s) = \frac{2s}{s(s-1)} - \frac{5}{s(s-1)} + \frac{1}{s^3(s-1)}$

Or, $y(s) = \frac{2}{s-1} - \frac{5}{s-1} + \frac{5}{s} + \frac{1}{s^3(s-1)}$

Or, $y(s) = -\frac{3}{s-1} + \frac{5}{s} + \frac{1}{s^3(s-1)}$ (2)

$$\frac{1}{s^3(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s-1}$$

Or, $1 = As^2(s-1) + Bs(s-1) + C(s-1) + Ds^3$ (3)

putting $s = o$ in (3), we get $1 = -C$, $\boxed{\therefore C = -1.}$

Again putting $s = 1$, in (3), we get $D = 1$.

Equating the coefficients of s^3 from both sides of (3), we get

$o = A + D$

or, $o = A + 1$, $\therefore A = -1$.

Equating the coefficients of s^2 from both sides of (3), we get

$o = -A + B$

or, $o = 1 + B$, $\therefore B = -1$.

$$\therefore \frac{1}{s^3(s-1)} = \frac{-1}{s} - \frac{1}{s^2} - \frac{1}{s^3} + \frac{1}{s-1}$$

Thus from (3), we have

$$y(s) = -\frac{3}{s-1} + \frac{5}{s} - \frac{1}{s} - \frac{1}{s^2} - \frac{1}{s^3} + \frac{1}{s-1}$$

or, $y(s) = -\frac{2}{s-1} + \frac{4}{s} - \frac{1}{s^2} - \frac{1}{s^3}$ (4)

Taking the inverse Laplace transform of both sides of (4), we get

$$\mathcal{L}^{-1}\{y(s)\} = -2\,\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 4\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$$

or, $Y = Y(t) = -2e^t + 4 - t - \frac{t^2}{2}$

which is the required solution.

Verification of the result :

$$Y = -2e^t + 4 - t - \frac{t^2}{2} \qquad \begin{cases} Y(o) = -2 + 4 = 2 \\ Y'(o) = +2 - 1 = -3 \end{cases}$$

$$Y' = -2e^t - 1 - t$$

$$Y'' = -2e^t - 1$$

$$\therefore\ Y'' - Y' = -2e^t - 1 + 2e^t + 1 + t = t$$

Therefore, $Y'' - Y' = t$

Exmple 14. Solve the differential equation

$Y'' + 2Y' + 5Y = e^{-t}\sin t;\ Y(o) = o,\ Y'(o) = 1$ using Laplace transform. **D.U.M. SC (F) 1986**

Solution : The given differential equation is

$$Y'' + 2Y' + 5Y = e^{-t}\sin t \quad (1)$$

Taking the Laplace transform of both sides of (1) and using the given conditions, we get

$$\mathcal{L}\{Y''\} + 2\mathcal{L}\{Y'\} + 5\mathcal{L}\{Y\} = \mathcal{L}\{e^{-t}\sin t\}$$

or, $s^2y(s) - sY(o) - Y'(o) + 2\{sy(s) - Y(o)\} + 5y(s) = \frac{1}{(s+1)^2+1}$

or, $s^2y(s) - 1 + 2sy(s) + 5y(s) = \frac{1}{(s+1)^2+1}$

or, $(s^2 + 2s + 5)\,y(s) = 1 + \frac{1}{(s+1)^2+1}$

$$\text{or, } y(s) = \frac{1}{s^2+2s+5} + \frac{1}{(s^2+2s+5)(s^2+2s+2)}$$

$$= \frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+3)} \quad (2)$$

$$\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5}$$

$$\text{or, } s^2+2s+3=(As+B)(s^2+2s+5)+(Cs+D)(s^2+2s+2)$$

$$\text{or, } s^2+2s+3 = (A+C)s^3+(2A+B+2C+D)s^2$$

$$+(5A+2B+2C+2D)s+5B+2D \quad (3)$$

Equating the coefficients of s^3, s^2, s and the constant terms from both sides of (3), we get

$$A+C=0,\ 2A+B+2C+D=1$$

$$5A+2B+2C+2D=2$$

$$5B+2D=3$$

Solving these equations for A, B, C and D, we get

$$A=0,\ B=\frac{1}{3},\ C=0,\ D=\frac{2}{3}.$$

Thus from (3), we get

$$y(s) = \frac{1/3}{s^2+2s+2} + \frac{2/3}{s^2+2s+5}$$

$$\text{or, } y(s) = \frac{1}{3}\cdot\frac{1}{(s+1)^2+1} + \frac{2}{3}\cdot\frac{1}{(s+1)^2+2^2} \quad (4)$$

Taking the inverse Laplace transform of both sides of (4), we get

$$\mathcal{L}^{-1}\{y(s)\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}$$

$$\text{or, } Y = Y(t) = \frac{1}{3}e^{-t}\sin t + \frac{2}{3}e^{-t}\cdot\frac{\sin 2t}{2}$$

$$\text{or, } Y = \frac{1}{3}e^{-t}(\sin t + \sin 2t).$$

which is the required solution.

Example 15. Solve the following initial value problem using Laplace transform :

$$\frac{d^2y}{dt^2} + y = t\cos 2t,\ y = 0,\ \frac{dy}{dt} = 0 \text{ when } t = 0.$$

Solution : The given differential equation is

$$\frac{d^2y}{dt^2} + y = t\cos 2t \quad (1)$$

Taking the Laplace transform of both sides of (1), we get

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} + \mathcal{L}\{y\} = \mathcal{L}\{t\cos 2t\}$$

$$s^2Y(s) - sy(o) - y'(o) + Y(s) = \frac{s^2 - 2^2}{(s^2 + 2^2)^2}$$

Since $\mathcal{L}\{t\cos at\} = \dfrac{s^2 - a^2}{(s^2 + a^2)^2}$

or, $s^2Y(s) - o - o + Y(s) = \dfrac{s^2 - 2^2}{(s^2 + 2^2)^2}$

or, $(s^2 + 1)\,Y(s) = \dfrac{s^2 - 2^2}{(s^2 + 2^2)^2}$

or, $Y(s) = \dfrac{s^2 - 2^2}{(s^2 + 1)(s^2 + 2^2)^2}$ (2)

$$\frac{s^2 - 4}{(s^2 + 1)(s^2 + 4)^2} = \frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 4} + \frac{Es + F}{(s^2 + 4)^2}$$

or, $s^2 - 4 = (As + B)(s^2 + 4)^2 + (Cs + D)(s^2 + 4)(s^2 + 1) + (Es + F)(s^2 + 1)$ (3)

Putting $s^2 = -1$ in (3), we get

$-5 = (As + B)\,9 + 0$ or, $As + B = -\dfrac{5}{9}$

$\therefore\ A = 0,\ B = -\dfrac{5}{9}$

Putting $s^2 = -4$ in (3), we get

$-8 = 0 + (Es + F)(-3)$

or, $8 = 3\,(Es + F)$ or, $Es + F = \dfrac{8}{3}$

$\therefore\ E = o,\ F = \dfrac{8}{3}$.

Equating the coefficients of s^5 from both sides of (3), we get

$0 = A + C \therefore A + C = o$

Equating the coefficients of s from both sides of (3), we get

$$0 = 16A + 4C + E$$

or, $16A + 4C = 0$ Since $E = o$

or, $4A + C = o$

Solving two equations $A + C = o$, $4A + C = o$ we have $A = C = o$

Lastly, equating the constant terms from both sides of (3), we get

$$-4 = 16B + 4D + F$$

$$\text{or, } -4 = \frac{-80}{9} + 4D + \frac{8}{3}$$

$$\text{or, } -1 = -\frac{20}{9} + D + \frac{2}{3}$$

$$\text{or, } D = -1 + \frac{20}{9} - \frac{2}{3} = \frac{-9 + 20 - 6}{9} = \frac{5}{9}.$$

Therefore, from (2), we have

$$Y(s) = \frac{s^2 - 2^2}{(s+1)^2 (s^2+2^2)^2} = -\frac{5}{9}\cdot\frac{1}{s^2+1} + \frac{5}{9}\cdot\frac{1}{s^2+4} + \frac{8}{3}\cdot\frac{1}{(s^2+4)^2} \quad (4)$$

Taking inverse Laplace transform of both sides of (4), we get

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{-5}{9}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \frac{5}{9}\mathcal{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\} + \frac{8}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)^2}\right\}$$

$$\text{or, } y = y(t) = -\frac{5}{9}\sin t + \frac{5}{9}\cdot\frac{1}{2}\sin 2t + \frac{8}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)^2}\right\}$$

$$\text{or, } y = -\frac{5}{9}\sin t + \frac{5}{18}\sin 2t + \frac{8}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)^2}\right\} \quad (5)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\cdot\frac{1}{s^2+4}\right\} \text{ (using convolution theorem)}$$

$$= \int_0^t \frac{1}{2}\sin 2v \cdot \frac{1}{2}\sin 2(t-v)\,dv$$

$$= \frac{1}{4}\int_0^t \sin 2v \sin(2t-2v)dv.$$

$$= \frac{1}{8}\int_0^t [\cos(2v-2t+2v) - \cos 2t]dv$$

$$= \frac{1}{8}\int_0^t [\cos(4v-2t) - \cos 2t]dv$$

$$= \frac{1}{8}\left[\frac{1}{4}\sin(4v-2t) - v\cos 2t\right]_0^t$$

$$= \frac{1}{8}\left[\frac{1}{4}\sin(4t-2t) - \frac{1}{4}\sin(-2t) - t\cos 2t\right]$$

$$= \frac{1}{8}\left[\frac{1}{4}\sin 2t + \frac{1}{4}\sin 2t - t\cos 2t\right]$$

$$= \frac{1}{8}\left[\frac{1}{2}\sin 2t - t\cos 2t\right]$$

$$\therefore \quad \mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)^2}\right\} = \frac{1}{8}\left[\frac{1}{2}\sin 2t - t\cos 2t\right]$$

Putting this value in (5), we get

$$y = -\frac{5}{9}\sin t + \frac{5}{18}\sin 2t + \frac{1}{3}\left[\frac{1}{2}\sin 2t - t\cos 2t\right]$$

$$\text{or, } y = -\frac{5}{9}\sin t + \frac{5}{18}\sin 2t + \frac{1}{6}\sin 2t - \frac{1}{3}t\cos 2t$$

$$\text{or, } y = -\frac{5}{9}\sin t + \frac{4}{9}\sin 2t - \frac{1}{3}t\cos 2t.$$

which is the required solution.

Example 16. Solve the following initial value problem using Laplace transform :

$y''(t) + a^2y(t) = A\sin \omega t;$

$y(0) = 1, y'(0) = 0, \omega \neq a.$

Solution : The given differential equation is

$$y''(t) + a^2y(t) = A\sin \omega t \quad (1)$$

Taking the Laplace transform of both sides of (1), we get

$$\mathcal{L}\{y''(t)\} + a^2\ \mathcal{L}\{y(t)\} = A\,\mathcal{L}\{\sin \omega t\}$$

$$\text{or, } s^2Y(s) - sy(o) - y'(o) + a^2Y(s) = \frac{A\omega}{s^2+\omega^2}.$$

$$\text{or, } s^2Y(s) - s - o + a^2\,Y(s) = \frac{A\omega}{s^2+\omega^2}$$

$$\text{or, } (s^2+a^2)Y(s) = s + \frac{A\omega}{s^2+\omega^2}$$

$$\text{or, } Y(s) = \frac{s}{s^2+a^2} + \frac{A\omega}{(s^2+a^2)(s^2+\omega^2)}.$$

$$\text{or, } Y(s) = \frac{s}{s^2+a^2} + \frac{A\omega}{a^2-\omega^2}\left\{\frac{1}{s^2+\omega^2} - \frac{1}{s^2+a^2}\right\} \quad (2)$$

Taking the inverse Laplace transform of both sides of (2), we get

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} + \frac{A\omega}{a^2-\omega^2}$$

$$\left[\mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\}\right]$$

$$\text{or, } y(t) = \cos at + \frac{A\omega}{a^2-\omega^2}\left[\frac{1}{\omega}\sin\omega t - \frac{1}{a}\sin at\right]$$

$$\text{or, } y(t) = \cos at. + \frac{A\omega}{a^2-\omega^2}\left[\frac{a\sin\omega t - \omega\sin at}{\omega a}\right]$$

$$\text{or, } y(t) = \cos at + \frac{A}{a(a^2-\omega^2)}(a\sin\omega t - \omega\sin at)$$

which is the required solution.

Example 17. Using Laplace transform solve the differential equation $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = t + \sin t$ subject to $x(o) = 1$ and $x'(o) = -2$. Also verify the result.

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Solution : The given differential equation is

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = t + \sin t \quad (1)$$

Taking the Laplace transform of both sides of (1), we get

$$\mathcal{L}\left\{\frac{d^2x}{dt^2}\right\} + 3\,\mathcal{L}\left\{\frac{dx}{dt}\right\} + 2\;\mathcal{L}\{x\} = \mathcal{L}\{t + \sin t\}$$

or, $s^2\,X(s) - s\,x(0) - x'(0) + 3\{s\,X(s) - x(0)\} + 2\,X(s)$

$$= \frac{1}{s^2} + \frac{1}{s^2+1}$$

or, $s^2\,X(s) - s + 2 + 3s\,X(s) - 3 + 2\,X(s) = \dfrac{2s^2+1}{s^2(s^2+1)}$

or, $(s^2 + 3s + 2)\;X(s) = s + 1 + \dfrac{2s^2+1}{s^2(s^2+1)}$

or, $(s+1)(s+2)\,X(s) = \dfrac{s^5 + s^4 + s^3 + 3s^2 + 1}{s^2(s^2+1)}$

or, $X(s) = \dfrac{s^5 + s^4 + s^3 + 3s^2 + 1}{s^2(s^2+1)(s+1)(s+2)}$

Now $\dfrac{s^5+s^4+s^3+3s^2+1}{s^2(s^2+1)(s+1)(s+2)} = \dfrac{A}{s^2} + \dfrac{B}{s} + \dfrac{Cs+D}{s^2+1} + \dfrac{E}{s+1} + \dfrac{F}{s+2}$ (2)

$\therefore\quad s^5 + s^4 + s^3 + 3s^2 + 1 = A(s^2+1)(s+1)(s+2)$

$+ Bs\,(s^2+1)(s+1)(s+2) + (Cs + D)\,s^2(s+1)(s+2)$

$+ Es^2(s^2+1)(s+2) + Fs^2(s^2+1)(s+1)$ (3)

Equating the costant terms from both sides of (3), we get

$2A = 1 \therefore \boxed{A = \frac{1}{2}}$

Equating the coefficients of s^5 from both sides of (3), we get

$1 = o + B + C + E + F$

or, $B + C + E + F = 1$ (4)

Putting $s = -1$ in (3), we get

$-1 + 1 - 1 + 3 + 1 = 0 + 0 + 0 + 2E + 0$

$\therefore \quad 2E = 3$ or, $\boxed{E = \frac{3}{2}}$

Putting $s = -2$ in (3), we get

$$-32 + 16 - 8 + 12 + 1 = 0 - 20F$$

or, $-11 = -20F$, $\therefore \boxed{F = \frac{11}{20}}$

Equating the coefficients of s from both sides of (3), we get

$0 = 3A + 2B$ or, $\frac{3}{2} + 2B = 0 \therefore \boxed{B = -\frac{3}{4}}$

Putting the values of B, E & F in (4), we get

$$-\frac{3}{4} + C + \frac{3}{2} + \frac{11}{20} = 1$$

or, $-\frac{3}{4} + C + \frac{1}{2} + \frac{11}{20} = 0$

or, $-\frac{1}{4} + C + \frac{11}{20} = 0$

or, $-\frac{5}{20} + C + \frac{11}{20} = 0 \therefore \boxed{C = -\frac{6}{20}}$

Equating the coefficients of s^2 from both sides of (3) we get

$$3 = 3A + 3B + 2D + 2E + F$$

putting the values of A, B, E and F, we get

$$3 = \frac{3}{2} - \frac{9}{4} + 2D + 3 + \frac{11}{20}$$

or, $\frac{3}{2} = 2D + \frac{3}{4} + \frac{11}{20}$

or, $\frac{3}{4} = 2D + \frac{11}{20}$

or, $\frac{15}{20} = 2D + \frac{11}{20}$

or, $\frac{4}{20} = 2D \therefore \boxed{D = \frac{1}{10}}$

Putting the values of A, B, C, D, E and F in (2), we get

$$X(s) = \frac{s^5 + s^4 + s^3 + 3s^2 + 1}{s^2(s^2+1)(s+1)(s+2)} = \frac{1}{2}\cdot\frac{1}{s^2} - \frac{3}{4}\cdot\frac{1}{s} + \frac{\frac{-6}{20}s + \frac{1}{10}}{s^2+1}$$

$$+ \frac{3}{2}\cdot\frac{1}{s+1} + \frac{11}{20}\cdot\frac{1}{s+2} \quad (5)$$

Taking the inverse Laplace transform of both sides of (5), we get

$$\mathcal{L}^{-1}\{X(s)\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{6}{20}\cdot\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$+ \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{11}{20}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$\text{or, } x = \frac{1}{2}t - \frac{3}{4} - \frac{6}{20}\cos t + \frac{1}{10}\sin t + \frac{3}{2}e^{-t} + \frac{11}{20}e^{-2t}$$

$$\text{or, } x = \frac{1}{2}t - \frac{3}{4} - \frac{3}{10}\cos t + \frac{1}{10}\sin t + \frac{3}{2}e^{-t} + \frac{11}{20}e^{-2t}$$

which is the required solution.

Verification of the result :

$$x = \frac{1}{2}t - \frac{3}{4} - \frac{6}{20}\cos t + \frac{1}{10}\sin t + \frac{3}{2}e^{-t} + \frac{11}{20}e^{-2t}$$

$$\frac{dx}{dt} = \frac{1}{2} - 0 + \frac{6}{20}\sin t + \frac{1}{10}\cos t - \frac{3}{2}e^{-t} - \frac{11}{10}e^{-2t}$$

$$\frac{d^2x}{dt^2} = 0 + \frac{6}{20}\cos t - \frac{1}{10}\sin t + \frac{3}{2}e^{-t} + \frac{11}{5}e^{-2t}$$

Therefore

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = \frac{3}{10}\cos t - \frac{1}{10}\sin t + \frac{3}{2}e^{-t}$$

$$+\frac{11}{5}e^{-2t}+\frac{3}{2}+\frac{9}{10}\sin t+\frac{3}{10}\cos t-\frac{9}{2}e^{-t}-\frac{33}{10}e^{-2t}$$

$$+t-\frac{3}{2}-\frac{3}{5}\cos t+\frac{1}{5}\sin t+3e^{-t}+\frac{11}{10}e^{-2t}$$

$$=t+\sin t.$$

Hence $\frac{d^2x}{dt^2}+3\frac{dx}{dt}+2x=t+\sin t.$

$$x(o)=o-\frac{3}{4}-\frac{6}{20}+o+\frac{3}{2}+\frac{11}{20}$$

$$=-\frac{6}{20}+\frac{3}{4}+\frac{11}{20}=\frac{20}{20}=1$$

$$=\frac{-6+15+11}{20}=\frac{20}{20}=1$$

$$x'(t)=\frac{1}{2}-o+\frac{6}{20}\sin t+\frac{1}{10}\cos t-\frac{3}{2}e^{-t}-\frac{11}{10}e^{-2t}$$

$$x'(o)=\frac{1}{2}+o+\frac{1}{10}-\frac{3}{2}-\frac{11}{10}$$

$$=\frac{1}{2}-\frac{5}{5}-\frac{3}{2}$$

$$=\frac{5-10-15}{10}=\frac{-20}{10}=-2.$$

Thus the result is verified.

Example 18. Solve the following problem by the Laplace transformation method :

$$y''(t)+6y'(t)+9y(t)=6t^2e^{-3t};\ y(o)=y'(o)=0$$

Also verify that your solution satisfies the differential equation and the initial conditions.

Solution : The given differential equation is

$$y''(t) + 6y'(t) + 9y(t) = 6t^2e^{-3t} \quad (1)$$

Taking the Laplace transform of both sides of (1), we get

$$\mathcal{L}\{y''(t)\} + 6\mathcal{L}\{y'(t)\} + 9\mathcal{L}\{y(t)\} = 6\mathcal{L}\{t^2e^{-3t}\}$$

or, $s^2Y(s) - s\,y(o) - y'(o) + 6\{sY(s)\} - y(o)\} + 9Y(s) = 6.\frac{2}{(s+3)^3}$

since $\mathcal{L}\{t^k e^{at}\} = \frac{\lfloor k}{(s-a)^{k+1}}$

or, $s^2Y(s) - o + 6sY(s) - o + 9Y(s) = \frac{12}{(s+3)^3}$

or, $(s^2 + 6s + 9)Y(s) = \frac{12}{(s+3)^3}$.

or, $(s+3)^2 Y(s) = \frac{12}{(s+3)^3}$

or, $Y(s) = \frac{12}{(s+3)^5} = \frac{1}{2}.\frac{\lfloor 4}{(s+3)^{4+1}}$ (2)

Taking the inverse Laplace transform of both sides of (2), we get

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{\lfloor 4}{(s+3)^{4+1}}\right\}$$

or, $y(t) = \frac{1}{2}t^4e^{-3t}$ which is the required solution.

Verification of the result :

$$y(t) = \frac{1}{2}t^4e^{-3t}$$

$$y'(t) = 2t^3e^{-3t} - \frac{3}{2}t^4e^{-3t}$$

$$y''(t) = 6t^2e^{-3t} - 6t^3e^{-3t} - 6t^3e^{-3t} + \frac{9}{2}t^4e^{-3t}.$$

$$\therefore\ y''(t) + 6y'(t) + 9\,y(t) = 6t^2e^{-3t}$$

$$-6t^3 e^{-3t} - 6t^3 e^{-3t} + \frac{9}{2}t^4 e^{-3t}$$

$$+ 12t^3 e^{-3t} - 9t^4 e^{-3t} + \frac{9}{2}t^4 e^{-3t}$$

$$= 6t^2e^{-3t} - 12t^3e^{-3t} + 12t^3e^{-3t} + 9t^4 e^{-3t} - 9t^4 e^{-3t}$$

$$= 6t^2 e^{-3t} + o + o = 6t^2 e^{-3t}.$$

Thus $y''(t) + 6y'(t) + 9\,y(t) = 6t^2e^{-3t}$.

Also $y(o) = o,\ y'(o) = 0$

Hence the result is verified.

Example 19. Using the Laplace transform find the general solution of the differential equation

$y'' + 2y' + 5y = e^{-t} \sin t$ **D. U. M. SC (F) 1990**

Solution : The given differential equation is

$$y'' + 2y' + 5y = e^{-t} \sin t \quad (1)$$

Let $y(o) = a$ and $y'(o) = b$ where a and b are arbitrary constants.

Taking the Laplace transform of both sides of (1), we get

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 5\,\mathcal{L}\{y\} = \mathcal{L}\left\{e^{-t} \sin t\right\}$$

$$\text{Or, } s^2 Y(s) - sy(o) - y'(o) + 2\{s\,Y(s) - y(o)\} + 5Y(s) = \frac{1}{(s+1)^2 + 1^2}$$

$$\text{Or, } s^2 Y(s) - as - b + 2sY(s) - 2a + 5Y(s) = \frac{1}{s^2 + 2s + 2}$$

$$\text{Or, } (s^2 + 2s + 5)\,Y(s) = as + 2a + b + \frac{1}{s^2 + 2s + 2}$$

$$\text{Or, } Y(s) = \frac{a(s+1)}{(s+1)^2 + 2^2} + (a+b)\,\frac{1}{(s+1)^2 + 2^2} + \frac{1}{(s^2 + 2s + 5)(s^2 + 2s + 2)} \quad (2)$$

Taking the inverse Laplace transform of both sides of (2),

we get $\mathcal{L}^{-1}\{Y(s)\} = a\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\} +$

$(a+b)\,\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s^2+2s+5)(s^2+2s+2)}\right\}$

Or, $y = ae^{-t}\cos 2t + \frac{1}{2}(a+b)\,e^{-t}\sin 2t$

$$+\mathcal{L}^{-1}\left\{\frac{1}{(s^2+2s+5)\,(s^2+2s+2)}\right\} \quad (3)$$

Now $\frac{1}{(s^2+2s+5)\,(s^2+2s+2)} = \frac{\frac{1}{3}\{(s^2+2s+5)-(s^2+2s+2)\}}{(s^2+2s+5)\,(s^2+2s+2)}$

$$= \frac{1}{3}\cdot\frac{1}{s^2+2s+2} - \frac{1}{3}\cdot\frac{1}{s^2+2s+5}$$

$$= \frac{1}{3}\cdot\frac{1}{(s+1)^2+1^2} - \frac{1}{3}\cdot\frac{1}{(s+1)^2+2^2}.$$

$$\therefore \quad \mathcal{L}^{-1}\left\{\frac{1}{(s^2+2s+5)(s^2+2s+2)}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1^2}\right\}$$

$$-\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}$$

$$= \frac{1}{3}e^{-t}\sin t - \frac{1}{3}\cdot\frac{1}{2}e^{-t}\sin 2t$$

Putting this value in (3), we get

$$y = ae^{-t}\cos 2t + \frac{1}{2}(a+b)\,e^{-t}\sin 2t + \frac{1}{3}e^{-t}\sin t - \frac{1}{6}e^{-t}\sin 2t.$$

$$= ae^{-t}\cos 2t + \frac{1}{2}\left(a+b-\frac{1}{3}\right)e^{-t}\sin 2t + \frac{1}{3}e^{-t}\sin t \quad (4)$$

Let $\frac{1}{2}\left(a+b-\frac{1}{3}\right) = c_1$ and $a = c_2$.

Then from (4), we have

$$y = e^{-t}(c_1\sin 2t + c_2\cos 2t) + \frac{1}{3}e^{-t}\sin t.$$

which is the required general solution.

Example 20. Using Laplace transform solve the differential equation $y''' + 2y'' - y' - 2y = 0$

Subject to $y(o) = 0, y'(o) = 0, y''(o) = 6$.

Solution : The given differential equation is

$$y''' + 2y'' - y' - 2y = 0 \quad (1)$$

Taking the Laplace transform of both sides of (1) and using the given conditions,

we get $\mathcal{L}\{y'''\} + 2\mathcal{L}\{(y'')\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0$

Or, $\{s^3 Y(s) - s^2 y(o) - sy'(o) - y''(o)\}$

$+ 2\{s^2 Y(s) - s\,y(o) - y'(o)\} - \{sY(s) - y(o)\} - 2Y(s) = 0$

Or, $\{s^3 Y(s) - o - o - 6\} + 2\{s^2 Y(s) - o - o\} - \{s\,Y(s) - o\} - 2Y(s) = 0$

Or, $s^3 Y(s) - 6 + 2s^2 Y(s) - sY(s) - 2Y(s) = 0$

Or, $(s^3 + 2s^2 - s - 2)Y(s) = 6$

Or, $(s - 1)(s + 1)(s + 2)\,Y(s) = 6$

Or, $Y(s) = \dfrac{6}{(s - 1)(s + 1)(s + 2)} \quad (2)$

Now $\dfrac{6}{(s - 1)(s + 1)(s + 2)} = \dfrac{A}{s - 1} + \dfrac{B}{s + 1} + \dfrac{C}{s + 2}$

Or, $6 = A(s + 1)(s + 2) + B(s - 1)(s + 2) + C(s - 1)(s + 1) \quad (3)$

Putting $s = 1$ in (3), we get $6 = 6A \therefore \boxed{A = 1}$

Putting $s = -1$ in (3), we get

$$6 = -2B \quad \therefore \boxed{B = -3}$$

Putting $s = -2$ in (3), we get $6 = 3C$

$$\therefore \boxed{C = 2}$$

Thus from (3), we get

$$Y(s) = \frac{6}{(s-1)(s+1)(s+2)} = \frac{1}{s-1} - 3.\frac{1}{s+1} + 2.\frac{1}{s+2} \quad (4)$$

Taking the inverse Laplace transform of both sides of (4), we get

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - 3\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

Or, $y = y(t) = e^t - 3e^{-t} + 2e^{-2t}$

which is the required solution.

Example 21. Solve the following differential equation using the Laplace transform :

$(D^3 - D^2 - D + 1)y = 8te^{-t}$; $y = D^2y = 0$, $Dy = 1$ when $t = 0$.

Solution : The given differential equation can be written as

$$y''' - y'' - y' + y = 8te^{-t} \quad (1)$$

$$y(0) = 0,\ y'(0) = 1,\ y''(0) = 0.$$

Taking the Laplace transform of both sides of (1)

we get $\mathcal{L}\{y'''\} - \mathcal{L}\{(y'')\} - \mathcal{L}\{y'\} + \mathcal{L}\{y\} = 8\,\mathcal{L}\{te^{-t}\}$

Or, $s^3 Y(s) - s^2y(0) - sy'(0) - y''(0) - \{s^2Y(s) - sy(0) - y'(0)\}$

$$- \{sY(s) - y(0)\} + Y(s) = 8.\frac{1}{(s+1)^2}$$

Or, $s^3Y(s) - 0 - s - 0 - s^2Y(s) + 0 + 1 - sY(s) - 0 + Y(s) = \frac{8}{(s+1)^2}$

Or, $(s^3 - s^2 - s + 1)\,Y(s) = s - 1 + \frac{8}{(s+1)^2}$

Or, $(s-1)^2\,(s+1)\,Y(s) = s - 1 + \frac{8}{(s+1)^2}$

Or, $Y(s) = \frac{1}{(s+1)(s-1)} + \frac{8}{(s+1)^3(s-1)^2}$

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Or, $Y(s) = \frac{1}{2}\left\{\frac{1}{s-1} - \frac{1}{s+1}\right\} + \frac{8}{(s+1)^3(s-1)^2}$ (2)

$$\frac{8}{(s+1)^3(s-1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3} + \frac{D}{(s-1)} + \frac{E}{(s-1)^2}$$

Or, $8 = A(s+1)^2(s-1)^2 + B(s+1)(s-1)^2 + C(s-1)^2 +$

$$D(s-1)(s+1)^3 + E(s+1)^3 \quad (3)$$

Putting $s = 1$ in (3), we get

$8 = 0 + 0 + 0 + 0 + 8E \quad \therefore E = 1$

Putting $s = -1$ in (3), we get

$8 = 0 + 0 + 4c + 0 + 0 \quad \therefore C = 2$

Equating the coefficients of s^4 from both sides of (3), we get $0 = A + D$

Equating the coefficients of s^3 from both sides of (3), we get $B + 2D + E = 0$

$\therefore \quad B + 2D = -1.$

Equating the constant terms from both sides of (3), we get $8 = A + B + C - D + E$

Or, $8 = A + B + 2 - D + 1$

Or, $B - 2D = 5$ since $A + D = 0$

Now $\left.\begin{matrix} B + 2D = -1 \\ B - 2D = 5 \end{matrix}\right\}$ Adding

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$2B = 4 \quad \therefore B = 2, D = \frac{-3}{2}$

$$A = \frac{3}{2}$$

$$\therefore \frac{8}{(s+1)^3(s-1)^2} = \frac{3}{2}\cdot\frac{1}{s+1} + \frac{2}{(s+1)^2} + \frac{2}{(s+1)^3} - \frac{3}{2}\cdot\frac{1}{s-1} + \frac{1}{(s-1)^2}$$

Putting this value in (2), we get

$$Y(s) = \frac{1}{2}\cdot\frac{1}{s-1} - \frac{1}{2}\cdot\frac{1}{s+1} + \frac{3}{2}\cdot\frac{1}{s+1}$$

$$+ \frac{2}{(s+1)^2} + \frac{2}{(s+1)^3} - \frac{3}{2}\cdot\frac{1}{s-1} + \frac{1}{(s-1)^2}$$

$$\text{Or, } Y(s) = -\frac{1}{s-1} + \frac{1}{s+1} + \frac{1}{(s-1)^2} + \frac{2}{(s+1)^2} + \frac{2}{(s+1)^3} \quad (4)$$

Taking the inverse Laplace transform of both sides of (4), we get

$$\mathcal{L}^{-1}\{Y(s)\} = -\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{(s+1)^3}\right\}$$

$$\text{Or, } y = y(t) = -e^t + e^{-t} + te^t + 2te^{-t} + t^2e^{-t}$$

$$= (1 + 2t + t^2)e^{-t} - e^t + te^t$$

$$= (1+t)^2e^{-t} - (1-t)e^t$$

which is the required solution.

Example 22. Using Laplace transform solve the differential equation $Y''' - 9Y'' + 27Y' - 27Y = 5t^4e^{3t}$ subject to $Y(0) = Y'(0) = Y''(0) = 0$.

Also verify the result.

Solution : The given differential equation is

$$Y''' - 9Y'' + 27Y' - 27Y = 5t^4e^{3t} \quad (1)$$

Taking the Laplace transform of both sides of (1), we get

$$\mathcal{L}\{Y'''\} - 9\mathcal{L}\{Y''\} + 27\mathcal{L}\{Y'\} - 27\mathcal{L}\{Y\} = 5\mathcal{L}\{t^4e^{3t}\}$$

$$s^3y(s) - s^2Y(0) - sY'(0) - Y''(0) - 9\{s^2y(s) - sY(0) - Y'(0)\} + 27\{sy(s) - Y(0)\} - 27y(s) = \frac{5\lfloor 4}{(s-3)^5}$$

$$\text{or, } s^3y(s) - 0 - 9s^2y(s) + 0 + 27sy(s) - 0 - 27y(s) = \frac{5.24}{(s-3)^5}$$

$$\text{or, } (s^3 - 9s^2 + 27s - 27)\,y(s) = \frac{120}{(s-3)^5}$$

or, $(s-3)^3\, y(s) = \dfrac{120}{(s-3)^5}$

or, $y(s) = \dfrac{120}{(s-3)^8}$ (2)

Taking the inverse Laplace transform of both sides of (2), we get

$$\mathcal{L}^{-1}\{y(s)\} = 120\,\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^8}\right\} = \frac{1}{42}\mathcal{L}^{-1}\left\{\frac{\lfloor 7}{(s-3)^{7+1}}\right\}$$

or, $Y = Y(t) = \dfrac{1}{42}t^7e^{3t}$ which is the required solution.

Verification of the result :

$$Y = Y(t) = \frac{1}{42}t^7e^{3t}$$

$$Y' = \frac{1}{6}t^6e^{3t} + \frac{1}{14}t^7e^{3t}$$

$$Y'' = t^5e^{3t} + \frac{1}{2}t^6e^{3t} + \frac{1}{2}t^6e^{3t} + \frac{3}{14}t^7e^{3t}$$

or, $$Y'' = t^5e^{3t} + t^6e^{3t} + \frac{3}{14}t^7e^{3t}$$

$$Y''' = 5t^4e^{3t} + 3t^5e^{3t} + 6t^5e^{3t} + 3t^6e^{3t} + \frac{3}{2}t^6e^{3t} + \frac{9}{14}t^7e^{3t}$$

$$Y'''-9Y'' + 27Y' - 27Y = 5t^4e^{3t} + 9t^5e^{3t} + \frac{9}{2}t^6e^{3t}$$

$$+\frac{9}{14}t^7e^{3t} - 9t^5e^{3t} - 9t^6e^{3t} - \frac{27}{14}t^7e^{3t}$$

$$+\frac{9}{2}t^6e^{3t} + \frac{27}{14}t^7e^{3t} - \frac{9}{14}t^7e^{3t} = 5t^4e^{3t}$$

$\therefore$ $Y'''-9Y'' + 27Y' - 27Y = 5t^4e^{3t}$

$Y(o) = o,\ Y'(o) = o, Y''(o) = o.$

Hence the result is verified.

Example 23. Solve the following differential equation using Laplace transform:

$y^{iv}(t) - 16\, y(t) = 30\sin t;$

$y(o) = o,\ y'(o) = 2,\ y''(\pi) = o,\ y'''(\pi) = -18.$

Solution: The given differential equation is

$$y^{iv}(t)-16\,y(t)=30\sin t \qquad (1)$$

Taking the Laplace transform of both sides of (1), we get

$$\mathcal{L}\{y^{iv}(t)\}-16\,\mathcal{L}\{y(t)\}=30\,\mathcal{L}\{\sin t\}$$

$$s^4Y(s)-s^3y(o)-s^2y'(o)-sy''(o)-y'''(o)-16Y(s)=30.\frac{1}{s^2+1}$$

or, $$s^4Y(s)-o-2s^2-as-b-16Y(s)=30.\frac{1}{s^2+1}$$

where $y''(o)=a$ and $y'''(o)=b$ (say).

or, $$(s^4-16)\,Y(s)=2s^2+as+b+\frac{30}{s^2+1}$$

or, $$Y(s)=\frac{2s^2}{s^4-16}+\frac{as+b}{s^4-16}+\frac{30}{(s^2+1)(s^4-16)}$$

or, $$Y(s)=\frac{2s^2}{(s^2+4)(s^2-4)}+\frac{as+b}{(s^2+4)(s^2-4)}+\frac{30}{(s^2+1)(s^2+4)(s^2-4)}$$

$$=\frac{s^2+4+s^2-4}{(s^2+4)(s^2-4)}+\frac{as+b}{(s^2+4)(s+2)(s-2)}+\frac{30}{(s^2+1)(s^2+4)(s+2)(s-2)}$$

$$=\frac{1}{s^2-4}+\frac{1}{s^2+4}+\frac{as+b}{(s^2+4)(s+2)(s-2)}$$

$$+\frac{30}{(s^2+1)(s^2+4)(s+2)(s-2)} \qquad (2)$$

$$\frac{as+b}{(s^2+4)(s+2)(s-2)}=\frac{As+B}{s^2+4}+\frac{C}{s+2}+\frac{D}{s-2}$$

or, $$as+b=(As+B)(s+2)(s-2)+C(s-2)(s^2+4)$$

$$+D(s+2)(s^2+4) \qquad (3)$$

Putting $s=2$, in (3), we get

$$2a+b=o+o+32D$$

$$\therefore \boxed{D=\frac{1}{32}(2a+b)}$$

Putting s= –2 in (3), we get

$$-2a+b = o -32C+o \qquad \therefore \boxed{C = \frac{2a-b}{32}}$$

Taking the coefficients of s^3 from both sides of (3), we get

$$o=A+C+D$$

$$\therefore A=\frac{-2a+b-2a-b}{32} = -\frac{1}{8}a \quad \therefore \boxed{A= -\frac{1}{8}a}$$

Taking the constant terms from both sides of (3),

we get $b= -4B - 8C+8D$

or, $$b= -4B- 8\left(\frac{2a-b}{32}\right) + 8\left(\frac{2a+b}{32}\right)$$

or, $$b= -4B- \left(\frac{2a-b}{4}\right) + \frac{2a+b}{4}$$

or, $$4b = -16B - 2a+b + 2a+b$$

or, $$2b = -16B$$

or, $$\boxed{B= -\frac{b}{8}}$$

$$\therefore \frac{as+b}{(s^2+4)\,(s+2)\,(s-2)} = \frac{1}{8}\cdot\frac{-as-b}{s^2+4} + \frac{2a-b}{32}\cdot\frac{1}{s+2} + \frac{2a+b}{32}\cdot\frac{1}{s-2}$$

Again, $$\frac{30}{(s^2+1)\,(s^2+4)\,(s+2)\,(s-2)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} + \frac{E}{s+2} + \frac{F}{s-2}$$

or, $$30 = (As+B)\,(s^2+4)\,(s+2)\,(s-2)$$

$$+ (Cs+D)\,(s^2+1)\,(s+2)\,(s-2)$$

$$+ E(s^2+1)(s^2+4)(s-2) + F\,(s^2+1)\,(s^2+4)\,(s+2) \quad (4)$$

Putting s=2 in (4), we get

$$30 = 160F \qquad \therefore \boxed{F=\frac{3}{16}}$$

Putting $s = -2$ in (4), we get

$$30 = -160E \qquad \therefore \quad \boxed{E = -\frac{3}{16}}$$

Taking the coefficients of s^5 from both sides of (4) we get

$$0 = A + C + E + F$$

or, $$\boxed{A + C = 0}$$

Taking the constant terms from both sides of (4), we get

$$30 = -16B - 4D - 8E + 8F$$

or, $$30 = -16B - 4D + \frac{3}{2} + \frac{3}{2}$$

or, $$\boxed{27 = -16B - 4D}$$

Taking the coefficients of s^4 from both sides of (4) we get

$$B + D - 2E + 2F = 0$$

or, $$B + D + \frac{3}{8} + \frac{3}{8} = 0$$

or, $$B + D + \frac{3}{4} = 0$$

or, $$4B + 4D = -3$$

$$\left.\begin{aligned} 4B + 4D &= -3 \\ -16B - 4D &= 27 \end{aligned}\right\}$$

$$-12B = 24 \qquad \therefore \quad \boxed{\begin{aligned} B &= -2 \\ D &= \frac{5}{4} \end{aligned}}$$

Taking the coefficients of s from both sides of (4) we get

$$0 = -8A - 4C + 4E + 4F$$

or, $$-8A - 4C = 0$$

or, $$\boxed{2A + C = 0}$$

Also $A + C = 0$

Solving we have $\boxed{A = 0,\ C = 0.}$

$$\therefore \frac{30}{(s^2+1)(s^2+4)(s+2)(s-2)} = \frac{-2}{s^2+1} + \frac{5}{4}\cdot\frac{1}{s^2+4} - \frac{3}{16}\cdot\frac{1}{s+2} + \frac{3}{16}\cdot\frac{1}{s-2}$$

Therefore, $Y(s) = \frac{1}{s^2-4} + \frac{1}{s^2+4} - \frac{a}{8}\cdot\frac{s}{s^2+4} - \frac{b}{8}\cdot\frac{1}{s^2+4}$

$$+\frac{2a-b}{32}\cdot\frac{1}{s+2} + \frac{2a+b}{32}\cdot\frac{1}{s-2} - \frac{2}{s^2+1}$$

$$+\frac{5}{4}\cdot\frac{1}{s^2+4} - \frac{3}{16}\cdot\frac{1}{s+2} + \frac{3}{16}\cdot\frac{1}{s-2} \quad (5)$$

Taking the inverse Laplace transform of both sides of (5), we get

$$y = y(t) = \frac{1}{2}\sinh 2t + \frac{1}{2}\sin 2t - \frac{a}{8}\cdot\cos 2t - \frac{b}{16}\sin 2t$$

$$+\frac{2a-b}{32}e^{-2t} + \frac{2a+b}{32}e^{2t} - 2\sin t + \frac{5}{8}\sin 2t - \frac{3}{16}e^{-2t} + \frac{3}{16}e^{2t}$$

$$= \frac{1}{2}\sinh 2t + \frac{1}{2}\sin 2t - \frac{a}{8}\cos 2t - \frac{b}{16}\sin 2t$$

$$+\frac{a}{8}\cdot\frac{1}{2}(e^{2t}+e^{-2t}) + \frac{b}{16}\cdot\frac{1}{2}(e^{2t}-e^{-2t}) - 2\sin t$$

$$+\frac{5}{8}\sin 2t + \frac{3}{8}\cdot\frac{1}{2}(e^{2t}-e^{-2t})$$

or, $$y(t) = \frac{1}{2}\sinh 2t + \frac{9}{8}\sin 2t - \frac{a}{8}\cos 2t - \frac{b}{16}\sin 2t$$

$$+\frac{a}{8}\cosh 2t + \frac{b}{16}\sinh 2t - 2\sin t + \frac{3}{8}\sinh 2t \quad (6)$$

$$\therefore y'(t) = \cosh 2t + \frac{9}{4}\cos 2t + \frac{a}{4}\sin 2t - \frac{b}{8}\cos 2t$$

$$+\frac{a}{4}\sinh 2t + \frac{b}{8}\cosh 2t - 2\cos t + \frac{3}{4}\cosh 2t$$

$$y''(t) = 2\sinh 2t - \frac{9}{2}\sin 2t + \frac{a}{2}\cos 2t + \frac{b}{4}\sin 2t$$

$$+\frac{a}{2}\cosh 2t + \frac{b}{4}\sinh 2t + 2\sin t + \frac{3}{2}\sinh 2t$$

$$y''(\pi) = o - o + \frac{a}{2} + o + \frac{a}{2} + o + o + o$$

or, $o = \frac{a}{2} + \frac{a}{2}$ since $y''(\pi) = o$

$\therefore \boxed{a = o}$

$$y'''(t) = 4\cosh 2t - 9\cos 2t - a\sin 2t + \frac{b}{2}\cos 2t + a\sinh 2t + \frac{b}{2}\cosh 2t + 2\cos t + 3\cosh 2t$$

$$y'''(\pi) = 4 - 9 - o + \frac{b}{2} + o + \frac{b}{2} - 2 + 3$$

or, $-18 = -4 + b$, $\therefore \boxed{b = -14}$

Putting $a = o$ and $b = -14$ in (6), we get

$$y = y(t) = \frac{1}{2}\sinh 2t + \frac{9}{8}\sin 2t - o + \frac{14}{16}\sin 2t + o - \frac{14}{16}\sinh 2t - 2\sin t + \frac{3}{8}\sinh 2t$$

or, $y = 2\sin 2t - 2\sin t = 2(\sin 2t - \sin t)$

which is the required solution.

3.3 APPLICATION OF CONVOLUTION THEOREM IN DIFFERENTIAL EQUATIONS.

Example 24. Solve the following differential equation by using Laplace transform :

$Y'' - 4Y' + 3Y = F(t)$ if $Y(o) = 1$, $Y'(o) = o$.

Solution: The given differential equation is

$$Y'' - 4Y' + 3Y = F(t) \quad (1)$$

Taking the Laplace transform of both sides of (1), we get

$$\mathcal{L}\{Y''\} - 4\mathcal{L}\{Y'\} + 3\mathcal{L}\{Y\} = \mathcal{L}\{F(t)\}$$

or, $s^2 y(s) - Y(o) - Y'(o) - 4\{sy(s) - Y(o)\} + 3y(s) = f(s)$ (say)

or, $s^2 y(s) - s - o - 4s\,y(s) + 4 + 3y(s) = f(s)$

or, $(s^2 - 4s + 3)\,y(s) = s - 4 + f(s)$

or, $(s-3)(s-1)\,y(s) = (s-4) + f(s)$

or, $y(s) = \dfrac{s-4}{(s-3)(s-1)} + \dfrac{f(s)}{(s-3)(s-1)}$.

or, $y(s) = \dfrac{1}{s-1} - \dfrac{1}{(s-3)(s-1)} + \dfrac{f(s)}{(s-3)(s-1)}$

or, $y(s) = \dfrac{1}{s-1} + \dfrac{1}{2}\cdot\dfrac{1}{s-1} - \dfrac{1}{2}\cdot\dfrac{1}{s-3} + \dfrac{f(s)}{(s-3)(s-1)}$ (2)

Taking the inverse Laplace transform of both sides of (2), we get

$$\mathcal{L}^{-1}\{y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \mathcal{L}^{-1}\left\{\frac{f(s)}{(s-3)(s-1)}\right\}$$

or, $Y = Y(t) = e^t + \dfrac{1}{2}e^t - \dfrac{1}{2}e^{3t} + \mathcal{L}^{-1}\left\{\dfrac{f(s)}{(s-3)(s-1)}\right\}$

or, $Y = \dfrac{3}{2}e^t - \dfrac{1}{2}e^{3t} + \mathcal{L}^{-1}\left\{\dfrac{f(s)}{(s-3)(s-1)}\right\}$ (3)

Now $\mathcal{L}^{-1}\left\{\dfrac{f(s)}{(s-3)(s-1)}\right\} = \mathcal{L}^{-1}\left[\dfrac{1}{2}\left\{\dfrac{1}{s-3} - \dfrac{1}{s-1}\right\}f(s)\right]$

(Using convolution theorem)

$$= \frac{1}{2}\int_0^t (e^{3u} - e^u)\,F(t-u)\,du,$$

$$\left\{\begin{array}{l} \text{since } \mathcal{L}^{-1}\left\{\dfrac{1}{s-3} - \dfrac{1}{s-1}\right\} = e^{3t} - e^t \\ \text{and } \mathcal{L}^{-1}\{f(s)\} = F(t). \end{array}\right.$$

Putting this value in (3), we get

$$Y = \frac{3}{2}e^t - \frac{1}{2}e^{3t} + \frac{1}{2}\int_0^t (e^{3u} - e^u)\,F(t-u)\,du$$

which is the required solution.

Example 25. Solve the following differential equation using the Laplace transform:

$$Y'' + a^2Y = F(t)\ ;\ Y(0) = 1,\ Y'(0) = -2.$$

Solution : The given differential equation is

$$Y'' + a^2 Y = F(t) \quad (1)$$

Taking the Laplace transform of both sides of (1), we get

$$\mathcal{L}\{Y''\} + a^2 \mathcal{L}\{Y\} = \mathcal{L}\{F(t)\}$$

Or, $s^2 y(s) - s\,Y(o) - Y'(o) + a^2 y(s) = f(s)$ (say)

Or, $s^2 y(s) - s + 2 + a^2 y(s) = f(s)$

Or, $(s^2 + a^2)\, y(s) = s - 2 + f(s)$

Or, $$y(s) = \frac{s}{s^2+a^2} - \frac{2}{s^2+a^2} + \frac{f(s)}{s^2+a^2} \quad (2)$$

Taking the inverse Laplace transform of both sides of (2), we get

$$\mathcal{L}^{-1}\{y(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} + \mathcal{L}^{-1}\left\{\frac{f(s)}{s^2+a^2}\right\}$$

Or, $$Y = Y(t) = \cos at - \frac{2}{a}\sin at + \mathcal{L}^{-1}\left\{f(s).\frac{1}{s^2+a^2}\right\} \quad (3)$$

Now $\mathcal{L}^{-1}\{f(s)\} = F(t)$ and

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$$

$\therefore$ Using **convolution theorem**, we have

$$\mathcal{L}^{-1}\left\{f(s).\frac{1}{s^2+a^2}\right\} = \int_0^t F(u).\frac{1}{a}\sin a(t-u)\,du$$

$$= \frac{1}{a}\int_0^t F(u)\sin a(t-u)\,du.$$

Thus from (3), we have

$$Y = \cos at - \frac{2}{a}\sin at + \frac{1}{a}\int_0^t F(u)\sin a(t-u)\,du$$

which is the required solution.

Example 26. Solve the following differential equation by the Laplace transform:

$y'' + 4y' + 4y = f(t)$; $y(o) = 1$, $y'(o) = 2$.

Solution : The given differential equation is

$$y'' + 4y' + 4y = f(t) \quad (1)$$

Taking the Laplace transform of both sides of (1), we get

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$

Or, $s^2 Y(s) - s\,y(0) - y'(0) + 4\{s\,Y(s) - y(0)\} + 4Y(s) = F(s)$ (say)

Or, $s^2 Y(s) - s - 2 + 4s\,Y(s) - 4 + 4Y(s) = F(s)$

Or, $(s^2 + 4s + 4)\,Y(s) = s + 6 + F(s)$

Or, $(s+2)^2 Y(s) = (s+2) + 4 + F(s)$

Or, $$Y(s) = \frac{1}{s+2} + \frac{4}{(s+2)^2} + \frac{F(s)}{(s+2)^2} \quad (2)$$

Taking the inverse Laplace transform of both sides of (1), we get

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{F(s)}{(s+2)^2}\right\}$$

Or, $$y = y(t) = e^{-2t} + 4te^{-2t} + \mathcal{L}^{-1}\left\{\frac{F(s)}{(s+2)^2}\right\} \quad (3)$$

Using convolution theorem, we have

$$\mathcal{L}^{-1}\left\{F(s).\frac{1}{(s+2)^2}\right\} = \int_0^t f(u)\,(t-u)e^{-2(t-u)}\,du$$

since $\mathcal{L}^{-1}\{F(s)\} = f(u)$ and

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} = te^{-2t}.$$

Thus from (3), we have

$$y = e^{-2t} + 4te^{-2t} + \int_0^t f(u)\,(t-u)\,e^{-2(t-u)}\,du.$$

which is the required solution.

Example 27. Find the general solution of the differential equation $Y'' - a^2 Y = F(t)$

Solution: The given differential equation is

$$Y'' - a^2 Y = F(t) \quad (1)$$

Let $Y(o) = c_1$ and $Y'(o) = c_2$. Then

taking the Laplace transform of both sides of (1), we get

$$\mathcal{L}\{Y''\} - a^2 \mathcal{L}\{Y\} = \mathcal{L}\{F(t)\}$$

Or, $s^2 y(s) - s\,Y(o) - Y'(o) - a^2 y(s) = f(s)$ (say)

Or, $s^2 y(s) - c_1 s - c_2 - a^2 y(s) = f(s)$

Or, $(s^2 - a^2)\, y(s) = c_1 s + c_2 + f(s)$

Or, $$y(s) = c_1 \frac{s}{s^2-a^2} + c_2 \frac{1}{s^2-a^2} + \frac{f(s)}{s^2-a^2} \quad (2)$$

Taking the inverse Laplace transform of both sides of (2), we get

$$\mathcal{L}^{-1}\{y(s)\} = c_1 \mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} + c_2 \mathcal{L}^{-1}\left\{\frac{1}{s^2-a^2}\right\} + \mathcal{L}^{-1}\left\{\frac{f(s)}{s^2-a^2}\right\}$$

Or, $$Y = Y(t) = c_1 \cosh at + \frac{c_2}{a} \sinh at + \mathcal{L}^{-1}\left\{\frac{f(s)}{s^2-a^2}\right\} \quad (3)$$

Now using the **convolution theorem,** we have

$$\mathcal{L}^{-1}\left\{f(s).\frac{1}{s^2-a^2}\right\} = \int_o^t F(u).\frac{1}{a}\sinh a(t-u)\,du$$

since $\mathcal{L}^{-1}\{f(s)\} = F(u)$ and $\mathcal{L}^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at$.

Thus from (3), we have

$$Y = c_1 \cosh at + \frac{c_2}{a}\sinh at + \frac{1}{a}\int_o^t F(u)\sinh a(t-u)\,du$$

Let $c_1 = A$ and $\frac{c_2}{a} = B$. Then we have

$$Y = A \cosh at + B \sinh at + \frac{1}{a}\int_o^t F(u)\sinh a(t-u)\,du.$$

which is the required solution.

3.4 APPLICATION OF LAPLACE TRANSFORM IN SOLVING ORDINARY LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS.

Example 28. Solve by Laplace transform the differential equation $ty'' + (t-1)y' - y = 0$,

$y(o) = 5$, $y(\infty) = 0$. **D. U.M. SC (F)1985**

Solution: The given differential equation is

$$ty'' + (t-1)y' - y = 0 \quad (1)$$

Taking the Laplace transform of both sides of (1) we have

$$\mathcal{L}\{ty''\} + \mathcal{L}\{ty'\} - \mathcal{L}\{y'\} - \mathcal{L}\{y\} = 0$$

or, $$-\frac{d}{ds}\{s^2Y - s\,y(0) - y'(0)\} - \frac{d}{ds}\{sY - y(0)\}$$

$$- \{sY - y(0)\} - Y = 0$$

or, $$-\frac{d}{ds}\{s^2Y - 5s - c\} - \frac{d}{ds}(sY - 5) - sY + 5 - Y = 0$$

where $y'(0) = c$ (say).

or, $$-s^2Y' - 2sY + 5 + 0 - sY' - Y + 0 - sY + 5 - Y = 0$$

or, $$(s^2+s)Y' + 3sY + 2Y - 10 = 0$$

or, $$(s^2+s)Y' + (3s+2)Y = 10$$

or, $$Y' + \frac{3s+2}{s^2+s}Y = \frac{10}{s^2+s}$$

or, $$Y' + \left\{\frac{(2s+1)+(s+1)}{s^2+s}\right\}Y = \frac{10}{s(s+1)}.$$

or, $$Y' + \left\{\frac{2s+1}{s^2+s} + \frac{1}{s}\right\}Y = \frac{10}{s(s+1)} \quad (2)$$

$$\text{I.F} = e^{\int\left\{\frac{2s+1}{s^2+s} + \frac{1}{s}\right\}ds} = e^{\log(s^2+s) + \log s}$$

$$= e^{\log s^2(s+1)} = s^2(s+1).$$

$$\therefore \quad Y s^2(s+1) = 10\int \frac{s^2(s+1)}{s(s+1)}\,ds = 10\int s\,ds = 5s^2 + c$$

or, $$Y = \frac{5}{s+1} + \frac{c}{s^2(s+1)} \quad (3)$$

Taking inverse Laplace transform of both sides of (3), we get

$$\mathcal{L}^{-1}\{Y\} = 5\,\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + c\,\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)}\right\}$$

or, $$y = 5e^{-t} + c\,\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)}\right\} \quad (4)$$

Now $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$ and $\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$

Therefore, using **convolution theorem** we have

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)}\right\} = \int_0^t e^{-u}\,(t-u)\,du$$

$$= t\int_0^t e^{-u}\,du - \int_0^t ue^{-u}\,du$$

$$= t[-e^{-u}]_0^t + [ue^{-u}]_0^t - \int_0^t e^{-u}\,du$$

$$= t\left[-e^{-t}+1\right] + te^{-t} - o + [e^{-u}]_0^t$$

$$= -te^{-t} + t + te^{-t} + e^{-t} - 1$$

$$= t + e^{-t} - 1.$$

Thus from (4), we have

$$y = 5e^{-t} + c(t+e^{-t}-1) \quad (5)$$

$$y(\infty) = 5e^{-\infty} + c\,(\infty + e^{-\infty} - 1)$$

or, $o = o + c(\infty + o - 1)$

or, $o = c\infty \therefore c = \frac{o}{\infty} = o$

or, $c = o$.

Putting $c = o$ in (5), we get

$y = 5e^{-t}$ which is the required solution.

Example 29. Solve the Bessel's differential equation of order zero $ty''(t) + y'(t) + ty(t) = o$ under the conditions that $y(o) = 1$ and $y(t)$ and its derivatives have transforms.

D. U. M. SC. (F) 1991

Solution : The given equation is

$$ty''(t) + y'(t) + ty(t) = o \quad (1)$$

Taking the Laplace transform of both sides of (1), we have

$$\mathcal{L}\{t\,y''(t)\} + \mathcal{L}\{y'(t)\} + \mathcal{L}\{ty(t)\} = o$$

$$\text{or, } -\frac{d}{ds}\{s^2Y - sy(o) - y'(o)\} + sY - y(o) - \frac{d}{ds}Y = o$$

$$\text{or, } -\frac{d}{ds}\{s^2Y - s - c\} + sY - 1 - \frac{d}{ds}Y = o$$

where $y(o) = 1$ and $y'(o) = c$(say).

$$\text{or, } -s^2Y' - 2sY + 1 - 0 + sY - 1 - Y' = 0$$

$$\text{or, } (s^2 + 1)Y' + sY = o$$

$$\text{or, } Y' + \frac{s}{s^2 + 1}Y = o$$

$$\text{or, } \frac{dY}{ds} + \frac{1}{2}\cdot\frac{2s}{s^2 + 1}Y = o$$

$$\text{or, } \frac{dY}{Y} + \frac{1}{2}\frac{2sds}{s^2 + 1} = 0$$

Integrating, we have

$$\log Y + \frac{1}{2}\log(s^2 + 1) = \log A.$$

$$\text{or, } \log Y = \log A - \log\sqrt{s^2 + 1} = \log\frac{A}{\sqrt{s^2 + 1}}$$

$$\text{or, } Y = \frac{A}{\sqrt{s^2 + 1}} \quad (2)$$

Now taking the inverse Laplace transform of both sides of (2) we have $\mathcal{L}^{-1}\{Y\} = A\,\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s^2+1}}\right\}$.

or, $y = A\,J_o(t)$ (3) since $\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s^2+1}}\right\} = J_o(t)$

$y(o) = A\,J_o(o)$ when $t = o$

or, $1 = A$ since $J_o(o) = 1$ & $y(o) = 1$

Putting $A = 1$ in (3), we get

$y = Jo(t)$. which is the required solution.

Example 30. Solve the differential equation

$tY'' + Y' + 4tY = o$; $Y(o) = 3$, $Y'(o) = o$

by using the Laplace transform. **D. U. H. 1991**

Solution : The given differential equation is

$$tY'' + Y' + 4tY = 0 \qquad (1)$$

Taking the Laplace transform of both sides of (1), we have

$$\mathcal{L}\{tY''\} + \mathcal{L}\{Y'\} + 4\,\mathcal{L}\{tY\} = o$$

or, $$-\frac{d}{ds}\{s^2y - sY(o) - Y'(o)\} + sy - Y(o) - 4\frac{dy}{ds} = o$$

or, $$-\frac{d}{ds}\{s^2y - 3s - 0\} + sy - 3 - 4\frac{dy}{ds} = o$$

or, $$-s^2\frac{dy}{ds} - 2sy + 3 + sy - 3 - 4\frac{dy}{ds} = o$$

or, $$s^2\frac{dy}{ds} + 2sy + o - sy + 4\frac{dy}{ds} = o$$

or, $$(s^2 + 4)\frac{dy}{ds} + sy = o$$

or, $$\frac{dy}{y} + \frac{s}{s^2+4}\,ds = o$$

Integrating, we have

$$\log y + \frac{1}{2}\log(s^2 + 4) = \log C$$

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or, $\log y\sqrt{s^2+4} = \log C$

or, $y\sqrt{s^2+4} = C$

or, $y = \dfrac{C}{\sqrt{s^2+4}}$ (2)

Taking the inverse Laplace transform of both sides of (2), we have

$$\mathcal{L}^{-1}\{y\} = C\,\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s^2+2^2}}\right\}$$

or, $Y = CJ_0(2t)$ (3) Since $\mathcal{L}^{-1}\left\{\dfrac{1}{\sqrt{s^2+2^2}}\right\} = J_0(2t)$

$\therefore$ $Y(o) = C\,J_0(o)$

or, $3 = C.1$ Since $Y(o) = 3$, $J_0(o) = 1$

$\therefore$ $C = 3$.

Putting $C = 3$ in (3), we have

$Y = 3J_0(2t)$ which is the required solution.

Example 31. Solve the following differential equation by using the Laplace transform :

$Y'' - tY' + Y = 1$; $Y(o) = 1$, $Y'(o) = 2$. **D.U.M.SC.(F)1989**

Solution : The given differential equation is

$$Y'' - tY' + Y = 1 \quad (1)$$

Taking Laplace transform of both sides of (1) we get

$$\mathcal{L}\{Y''\} - \mathcal{L}\{tY\} + \mathcal{L}\{Y\} = \mathcal{L}\{1\}$$

or, $s^2y - sY(o) - Y'(o) + \dfrac{d}{ds}\{sy - Y(o)\} + y = \dfrac{1}{s}$

or, $s^2y - s - 2 + sy' + y - o + y = \dfrac{1}{s}$

or, $sy' + (s^2+2)y = s + 2 + \dfrac{1}{s}$

or, $y' + \left(\frac{s^2+2}{s}\right)y = 1 + \frac{2}{s} + \frac{1}{s^2}$

or, $\frac{dy}{ds} + \left(s + \frac{2}{s}\right)y = 1 + \frac{2}{s} + \frac{1}{s^2}$ (2)

which is a first order differential equation.

An integrating factor is

$$I.F = e^{\int\left(s+\frac{2}{s}\right)ds}$$

$$= e^{\frac{s^2}{2} + 2\log s}$$

$$= e^{\frac{s^2}{2} + \log s^2}$$

$$= e^{\frac{s^2}{2}} \cdot e^{\log s^2} = s^2 e^{\frac{s^2}{2}}.$$

$$\therefore \frac{d}{ds}\left\{s^2 e^{\frac{s^2}{2}} \cdot y\right\} = \left(1 + \frac{2}{s} + \frac{1}{s^2}\right) s^2 e^{\frac{s^2}{2}}$$

Integrating both sides, we have

$$s^2 e^{\frac{s^2}{2}} \cdot y = \int\left(1 + \frac{2}{s} + \frac{1}{s^2}\right) s^2 e^{\frac{s^2}{2}}\, ds$$

or, $y = \frac{1}{s^2} \cdot e^{-\frac{s^2}{2}} \int (s^2 + 2s + 1)\, e^{\frac{s^2}{2}}\, ds$ (3)

Now $\int (s^2 + 2s + 1)e^{\frac{s^2}{2}}\, ds$

$$= \int s^2 e^{\frac{s^2}{2}}\, ds + 2\int s e^{\frac{s^2}{2}}\, ds + \int e^{\frac{s^2}{2}} .1.\, ds$$

$$= \int s^2 e^{\frac{s^2}{2}}\, ds + 2e^{\frac{s^2}{2}} + se^{\frac{s^2}{2}} - \int e^{\frac{s^2}{2}} \frac{1}{2}.\, 2s.\, sds$$

$$= \int s^2 e^{\frac{s^2}{2}}\, ds + 2e^{\frac{s^2}{2}} + se^{\frac{s^2}{2}} - \int s^2 e^{\frac{s^2}{2}}\, ds$$

$$= se^{\frac{s^2}{2}} + 2e^{\frac{s^2}{2}} + c.$$

Thus from (3), we have

$$y = \frac{1}{s^2} e^{-\frac{s^2}{2}} \left(se^{\frac{s^2}{2}} + 2e^{\frac{s^2}{2}} + c \right)$$

$$= \frac{1}{s} + \frac{2}{s^2} + \frac{c}{s^2}\, e^{-\frac{s^2}{2}}$$

[Using $e^{-x} = 1 - \frac{x}{\underline{1}} + \frac{x^2}{\underline{2}} - \frac{x^3}{\underline{3}} + \ldots\ldots]$

$$= \frac{1}{s} + \frac{2}{s^2} + \frac{c}{s^2} \left(1 - \frac{1}{2}s^2 + \frac{1}{8}s^4 - \ldots\ldots \right)$$

$$= \frac{1}{s} + \frac{c+2}{s^2} - c \left(\frac{1}{2} - \frac{1}{8}s^2 + \ldots\ldots \right) \quad (4)$$

Then Since $\mathcal{L}^{-1}\{s^k\} = o$ for $k = 1, 2, 3, \ldots\ldots$ taking inverse Laplace transform of both side of (4), we have

$$\mathcal{L}^{-1}\{y\} = \mathcal{L}^{-1}\left\{\frac{1}{2}\right\}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + (c+2)\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - c\mathcal{L}^{-1}\left\{\frac{1}{2} - \frac{1}{8}s^2 + \ldots\ldots\right\}$$

or, $Y = 1 + (c+2)\,t - c.o$

or, $Y = 1 + (c+2)t$ (5)

or, $Y' = o + c + 2$ but given $Y'(o) = 2$

or, $Y'(o) = c + 2$ or, $2 = c + 2 \therefore c = o$

Hence from (5), we have $Y = 1 + 2t$ which is the required solution.

Example 32. Solve the following differential equation using Laplace transform :

$$ty'' + (1 - 2t)y' - 2y = o;\ y(o) = 1,\ y'(o) = 2.$$

D. U. H. 1993

Solution : The given differential equation can be written as

$$ty'' + y' - 2ty' - 2y = o \quad (1)$$

Taking the Laplace transform of both sides of (1) we get

$$\mathcal{L}\{ty''\} + \mathcal{L}\{y'\} - 2\,\mathcal{L}\{t\,y'\} - 2\,\mathcal{L}\{y\} = o$$

or, $-\frac{d}{ds}\{s^2Y - sy(o) - y'(o)\} + sY - y(o) + 2\frac{d}{ds}\{sY - y(o)\} - 2Y = o.$

or, $-\frac{d}{ds}(s^2Y - s - 2) + sY - 1 + 2\frac{d}{ds}(sY - 1) - 2Y = o$

or, $-s^2\frac{dY}{ds} - 2sY + 1 + o + sY - 1 + 2s\frac{dY}{ds} + 2Y - o - 2Y = o$

or, $-s^2\frac{dY}{ds} - sY + 2s\frac{dY}{ds} = o$

$$\text{or, } s^2\frac{dY}{ds} - 2s\frac{dY}{ds} + sY = o$$

$$\text{or, } s\frac{dY}{ds} - 2\frac{dY}{ds} + Y = o$$

$$\text{or, } (s-2)\frac{dY}{ds} + Y = o$$

$$\text{or, } \frac{dY}{Y} + \frac{ds}{s-2} = o$$

Integrating, we get

$$\log Y + \log(s-2) = \log C$$

$$\text{or, } \text{Log}Y = \log C - \log(s-2) = \log\frac{C}{s-2}$$

$$\therefore \quad Y = \frac{C}{s-2} \quad (2)$$

Taking the inverse Laplace transform of both sides of (2), we get

$$\mathcal{L}^{-1}\{Y\} = C\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$\text{or, } y = y(t) = Ce^{2t} \quad (3)$$

$$\therefore \quad y(o) = Ce^0 = C.1 = C$$

$$\text{or, } 1 = C \text{ Since } y(o) = 1$$

$$\therefore \quad \boxed{C = 1}$$

Putting $C = 1$ in (3), we get

$y = e^{2t}$ which is the required solution.

Example 33. Using the Laplace transform find the general solution of the differential equation

$ty'' + (2t + 3)y' + (t + 3)\,y = ae^{-t}$ if $y = y(t)$ and its derivatives have transforms.

Solution : The given differential equation is

$$ty'' + (2t + 3)y' + (t + 3)y = ae^{-t} \quad (1)$$

Let $y(0) = A$ and $y'(0) = B$ where A and B are arbitrary constants.

The given equation (1) can be written as

$ty'' + 2ty' + 3y' + ty + 3y = ae^{-t}$ (2)

Taking the Laplace transform of both sides of (2) we get

$$\mathcal{L}\{ty''\} + 2\mathcal{L}\{ty'\} + 3\mathcal{L}\{y'\} + \mathcal{L}\{ty\} + 3\mathcal{L}\{y\} = a\mathcal{L}\{e^{-t}\}$$

or, $$-\frac{d}{ds}\{s^2Y - s\,y(0) - y'(0)\} - 2\frac{d}{ds}\{sY - y(0)\} + 2\{sY - y(0)\}$$

$$-\frac{dY}{ds} + 3y = \frac{a}{s+1}$$

or, $$-\frac{d}{ds}\{s^2Y - sA - B\} - 2\frac{d}{ds}\{sY - A\} + 3(sY - A) - \frac{dY}{ds} + 3y$$

$$= \frac{a}{s+1}$$

or, $$-s^2\frac{dY}{ds} - 2sY + A + 0 - 2s\frac{dY}{ds} - 2Y + o + 3sY - 3A$$

$$-\frac{dY}{ds} + 3Y = \frac{a}{s+1}$$

or, $$-(s^2 + 2s + 1)\frac{dY}{ds} + sY + Y = 2A + \frac{a}{s+1}$$

or, $$(s^2 + 2s + 1)\frac{dY}{ds} - (s+1)Y = -2A - \frac{a}{s+1}$$

or, $$(s+1)^2\frac{dY}{ds} - (s+1)Y = -2A - \frac{a}{s+1}$$

or, $$\frac{dY}{ds} - \frac{1}{s+1}Y = -\frac{2A}{(s+1)^2} - \frac{a}{(s+1)^3}. \quad (3)$$

which is a linear differential equation of first order.

To solve the equation (3) we will use the integrating factor method.

$$\text{I.F} = e^{-\int \frac{1}{s+1}\,ds} = e^{-\log(s+1)} = e^{\log\frac{1}{s+1}} = \frac{1}{s+1}.$$

$$\therefore \quad Y\cdot\frac{1}{s+1} = -2A\int\frac{ds}{(s+1)^3} - a\int\frac{ds}{(s+1)^4}$$

$$= 2A\cdot\frac{1}{2(s+1)^2} + a\cdot\frac{1}{3(s+1)^3} + B$$

$$\text{Or, } \frac{Y}{s+1} = \frac{A}{(s+1)^2} + \frac{a}{3(s+1)^3} + B \quad \ldots\ldots\ldots (*)$$

$$\text{Or, } Y = \frac{A}{s+1} + \frac{a}{3(s+1)^2} + B(s+1) \qquad (4)$$

where B is a constant of integration. Now the integrating constant B must vanish [Making $s \to \infty$ in (*) we obtain $B = 0$].

Thus from (4), we have $Y = \frac{A}{s+1} + \frac{a}{3(s+1)^2}$ (5)

Taking the inverse Laplace transform of both sides of (5), we have

$$\mathcal{L}^{-1}\{Y\} = A.\,\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{a}{3}\,\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$

or, $y = y(t) = Ae^{-t} + \frac{a}{3}te^{-t}$

or, $y = \left(A + \frac{at}{3}\right)e^{-t}$ which is the required solution.

3.5 APPLICATION OF LAPLACE TRANSFORM IN SOLVING SIMULTANEOUS ORDINARY DIFFERENTIAL EQUATIONS

Example 34. Solve the following simultaneous differential equation by the Laplace Transform :

$$\left.\begin{aligned} (D-2)x + 3y &= 0 \\ 2x + (D-1)y &= 0 \end{aligned}\right\}$$

subject to $x(0) = 8$, $y(0) = 3$.

Solution : The given simultaneous differential equations can be written in the form

$$\left.\begin{aligned} \frac{dx}{dt} &= 2x - 3y \\ \frac{dy}{dt} &= y - 2x \end{aligned}\right\} \text{ subject to } x(0) = 8,\ y(0) = 3.$$

Let $\mathcal{L}\{x\} = X$, $\mathcal{L}\{y\} = Y$.

Then taking .Laplace transform, we get

$$\mathcal{L}\left\{\frac{dx}{dt}\right\} = \mathcal{L}\{2x - 3y\}$$

$$\mathcal{L}\left\{\frac{dx}{dt}\right\} = \mathcal{L}\{y - 2x\}$$

$$\left.\begin{aligned} sX - x(0) &= 2X - 3Y \\ sY - y(0) &= Y - 2X \end{aligned}\right\}$$

or,

$$\left.\begin{aligned} sX - 8 &= 2X - 3Y \\ sY - 3 &= Y - 2X \end{aligned}\right\}$$

or,

$$(s-2)X + 3Y - 8 = 0 \quad (1)$$

$$2X + (s-1)Y - 3 = 0 \quad (2)$$

Solving (1) and (2), for X and Y, we get

$$\frac{X}{8s-17} = \frac{Y}{3s-22} = \frac{1}{(s-1)(s-2)-6}$$

or,

$$\frac{X}{8s-17} = \frac{Y}{3s-22} = \frac{1}{s^2-3s-4}$$

$$\therefore \quad X = \frac{8s-17}{s^2-3s-4} = \frac{8s-17}{(s+1)(s-4)} = \frac{5}{s+1} + \frac{3}{s-4} \quad (3)$$

$$Y = \frac{3s-22}{s^2-3s-4} = \frac{3s-22}{(s+1)(s-4)} = \frac{5}{s+1} - \frac{2}{s-4} \quad (4)$$

Now taking the inverse Laplace transform of both sides of (3) and (4) respectively, we get .

$$\mathcal{L}^{-1}\{X\} = 5\,\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + 3\,\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}$$

or,

$$x = 5e^{-t} + 3e^{4t}.$$

$$\mathcal{L}^{-1}\{Y\} = 5\,\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - 2\,\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}$$

or,

$$y = 5e^{-t} - 2e^{4t}.$$

Hence the required solution is

$$\left.\begin{aligned} x &= 5e^{-t} + 3e^{4t} \\ y &= 5e^{-t} - 2e^{4t}. \end{aligned}\right\}$$

Example 35. Solve the following simultaneous differential equations by the Laplace transform:

$$\frac{dx}{dt} + y = \sin t, \quad \frac{dy}{dt} + x = \cos t$$

given that $x(0) = 2$, $y(0) = 0$.

Solution : The given simultaneous differential equations are

$$\frac{dx}{dt} + y = \sin t \quad (1)$$

$$\frac{dy}{dt} + x = \cos t \quad (2)$$

Let $\mathcal{L}\{x\} = X$, $\mathcal{L}\{y\} = Y$

Then taking the Laplace transform of both sides of (1) and (2) respectively, we get

$$\mathcal{L}\left\{\frac{dx}{dt}\right\} + \mathcal{L}\{y\} = \mathcal{L}\{\sin t\}$$

or, $$sX - x(0) + Y = \frac{1}{s^2 + 1}$$

or, $$sX - 2 + Y - \frac{1}{s^2 + 1} = 0$$

or, $$sX + Y - \frac{(2s^2 + 3)}{s^2 + 1} = 0 \quad (3)$$

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} + \mathcal{L}\{x\} = \mathcal{L}\{\cos t\}$$

or, $$sY - y(0) + X = \frac{s}{s^2 + 1}$$

or, $$sY - 0 + X - \frac{s}{s^2 + 1} = 0$$

or, $$sY + X - \frac{s}{s^2 + 1} = 0$$

or, $$X + sY - \frac{s}{s^2 + 1} = 0 \quad (4)$$

Solving (3) and (4) for X and Y, we get

$$\frac{X}{-\frac{s}{s^2+1}+\frac{2s^3+3s}{s^2+1}}=\frac{Y}{-\frac{-2s^2-3}{s^2+1}+\frac{s^2}{s^2+1}}=\frac{1}{s^2-1}$$

or, $$\frac{X}{\frac{2s^3+2s}{s^2+1}}=\frac{Y}{\frac{-s^2-3}{s^2+1}}=\frac{1}{s^2-1}$$

or, $$\frac{X}{2s}=\frac{Y}{\frac{-s^2-3}{s^2+1}}=\frac{1}{s^2-1}$$

$$\therefore \quad X=\frac{2s}{s^2-1}=\frac{1}{s+1}+\frac{1}{s-1} \quad (5)$$

Again $$Y=\frac{-s^2-3}{(s^2+1)(s^2-1)}$$

$$=\frac{-(s^2+1)-2}{(s^2+1)(s^2-1)}$$

$$=-\frac{1}{s^2-1}-\frac{2}{(s^2+1)(s^2-1)}$$

$$=-\frac{1}{s^2-1}-\left[\frac{1}{s^2-1}-\frac{1}{s^2+1}\right]$$

$$=-\frac{1}{s^2-1}-\frac{1}{s^2-1}+\frac{1}{s^2+1}$$

$$=-\frac{2}{s^2-1}+\frac{1}{s^2+1}$$

$$=\frac{-2}{(s+1)(s-1)}+\frac{1}{s^2+1}$$

$$=-\frac{1}{s-1}+\frac{1}{s+1}+\frac{1}{s^2+1} \quad (6)$$

Now taking the inverse Laplace transform of both sides of (5) and (6) respectively, we have

$$\mathcal{L}^{-1}\{X\}=\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

Or, $x=e^{-t}+e^{t}$

Also $\mathcal{L}^{-1}\{Y\} = -\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$

$$+ \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

Or, $y = -e^t + e^{-t} + \sin t$

Hence the required solution is

$$\left.\begin{aligned} x &= e^t + e^{-t} \\ y &= e^{-t} - e^t + \sin t \end{aligned}\right\}.$$

Example 36. Solve the following simultaneous differential equations by Laplace transform:

$$\left.\begin{aligned} \frac{dx}{dt} + \frac{dy}{dt} &= t \\ \frac{d^2x}{dt^2} - y &= e^{-t} \end{aligned}\right\}$$

subject to the conditions $x(0) = 3$,

$x'(0) = -2$, $y(0) = 0$.

Solution : The given simultaneous differential equations are

$$\frac{dx}{dt} + \frac{dy}{dt} = t \quad (i)$$

$$\frac{d^2x}{dt^2} - y = e^{-t} \quad (2)$$

Let $\mathcal{L}\{x\} = X$ and $\mathcal{L}\{y\} = Y$.

Then taking the Laplace transform of both sides of (1) and (2), we get

$$\mathcal{L}\left\{\frac{dx}{dt}\right\} + \mathcal{L}\left\{\frac{dy}{dt}\right\} = \mathcal{L}\{t\}$$

or, $sX - x(0) + sY - y(0) = \frac{1}{s^2}$

or, $sX - 3 + sY - 0 = \frac{1}{s^2}$

or, $sX + sY = 3 + \frac{1}{s^2}$ (3)

$$\mathcal{L}\left\{\frac{d^2x}{dt^2}\right\} - \mathcal{L}\{y\} = \mathcal{L}\{e^{-t}\}$$

or, $s^2X - sx(0) - x'(0) - Y = \frac{1}{s+1}$

or, $s^2X - 3s + 2 - Y = \frac{1}{s+1}$

or, $s^2X - Y = 3s - 2 + \frac{1}{s+1}$ (4)

Now we will solve (3) and (4) for X and Y.

$$\left.\begin{aligned} sX + sY &= 3 + \frac{1}{s^2} \\ s^3X - sY &= 3s^2 - 2s + \frac{s}{s+1} \end{aligned}\right\} \text{ Adding}$$

$$(s^3+s)X = 3s^2 - 2s + \frac{s}{s+1} + 3 + \frac{1}{s^2}$$

or, $X = \frac{3s^2}{s(s^2+1)} - \frac{2s}{s(s^2+1)} + \frac{s}{(s+1)s(s^2+1)}$

$$+ \frac{3}{s(s^2+1)} + \frac{1}{s^3(s^2+1)}$$

or, $X = \frac{3s}{s^2+1} - \frac{2}{s^2+1} + \frac{1}{(s+1)(s^2+1)}$

$$+ \frac{3}{s(s^2+1)} + \frac{1}{s^3(s^2+1)}$$

or, $X = \frac{3s}{s^2+1} - \frac{2}{s^2+1} + \frac{1}{(s+1)(s^2+1)}$

$$+ \frac{3s^2+1}{s^3(s^2+1)} \quad (5)$$

$$\frac{1}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

or, $1 = A(s^2+1) + (Bs+C)(s+1)$ (6)

Putting $s = -1$ in (6), we get $1 = 2A$

$$\therefore \boxed{A = \frac{1}{2}}$$

Taking the coefficients of s^2 from both sides of (6),

we get $o = A + B$ $\therefore$ $\boxed{B = -\frac{1}{2}}$

Equating the constant terms from both sides of (6), we get

$1 = A + C$ $\therefore$ $\boxed{C = \frac{1}{2}}$

Thus $$\frac{1}{(s+1)(s^2+1)} = \frac{1}{2}\cdot\frac{1}{s+1} - \frac{1}{2}\cdot\frac{(s-1)}{s^2+1}$$

$$= \frac{1}{2}\cdot\frac{1}{s+1} - \frac{1}{2}\cdot\frac{s}{s^2+1} + \frac{1}{2}\cdot\frac{1}{s^2+1}$$

$$\frac{3s^2+1}{s^3(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds+E}{s^2+1}$$

Or, $3s^2 + 1 = A s^2(s^2+1) + Bs(s^2+1) + C(s^2+1) + (Ds+E)s^3$ (7)

Putting $s = O$ in (7), we get

$1 = o + o + C + o$ $\therefore$ $\boxed{C = 1}$

Equating the coefficients of s^4 from both sides of (7), we get

$o = A + D$

Equating the coefficients of s^3 from both sides of (7),

we get $o = O + B + E$

$\therefore$ $B + E = O.$

Equating the coefficients of s^2 from both sides of (7), we get

$3 = A + O + C$

or, $3 = A + 1,$ $\therefore$ $\boxed{A = 2}$

$\boxed{D = -2}$

Equating the coefficients of s from both sides of (7), we get

$O + O + B = O$ $\therefore$ $\boxed{B = o, E = O}$

Therefore, $$\frac{3s^2+1}{s^3(s^2+1)} = \frac{2}{s} + \frac{1}{s^3} - 2\cdot\frac{s}{s^2+1}$$

Thus from (5), we have

$$X = 3.\frac{s}{s^2+1} - 2.\frac{1}{s^2+1} + \frac{1}{2}.\frac{1}{s+1} - \frac{1}{2}\frac{s}{s^2+1} + \frac{1}{2}.\frac{1}{s^2+1} + \frac{2}{s} + \frac{1}{s^3} - 2.\frac{s}{s^2+1}$$

$$= \frac{1}{2}.\frac{s}{s^2+1} - \frac{3}{2}.\frac{1}{s^2+1} + \frac{1}{2}.\frac{1}{s+1} + \frac{2}{s} + \frac{1}{s^3} \quad (8)$$

Taking the inverse Laplace transform of both sides of (8), we get

$$\mathcal{L}^{-1}(X) = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+1}\right\}$$

or, $x = \frac{1}{2}\cos t - \frac{3}{2}\sin t + \frac{1}{2}e^{-t} + 2 + \frac{1}{2}t^2$.

Again
$$\left.\begin{aligned} s^2X + s^2Y &= 3s + \frac{1}{s} \\ s^2X - Y &= 3s - 2 + \frac{1}{s+1} \end{aligned}\right\}$$ Subtracting

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$$(s^2+1)Y = \frac{1}{s} + 2 - \frac{1}{s+1}$$

or, $Y = \frac{1}{s(s^2+1)} + \frac{2}{s^2+1} - \frac{1}{(s+1)(s^2+1)}$

or, $Y = \frac{1}{s(s^2+1)} + \frac{2}{s^2+1} - \frac{1}{2}.\frac{1}{s+1} + \frac{1}{2}.\frac{s}{s^2+1} - \frac{1}{2}.\frac{1}{s^2+1}$

Now $\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$

or, $1 = A(s^2+1) + (Bs+C)s$ (9)

Putting $s = o$ in (9), we get $\boxed{A = 1}$

Equating the coefficients of s^2 from both sides of (9), we get $o = A + B$, $\therefore$ $\boxed{B = -1}$

Equating the coefficients of s from both sides of (9), we get

$$\boxed{C = o}$$

$$\therefore \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

Therefore, $Y = \frac{1}{s} - \frac{s}{s^2+1} + 2.\frac{1}{s^2+1} - \frac{1}{2}.\frac{1}{s+1} + \frac{1}{2}.\frac{s}{s^2+1}$

$$-\frac{1}{2}.\frac{1}{s^2+1}$$

or, $Y = \frac{1}{s} - \frac{1}{2}.\frac{s}{s^2+1} + \frac{3}{2}.\frac{1}{s^2+1} - \frac{1}{2}.\frac{1}{s+1}$ (10)

Taking the inverse Laplace transform of both sides of (10), we get

$$\mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$+\frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

or, $y = 1 - \frac{1}{2}\cos t + \frac{3}{2}\sin t - \frac{1}{2}e^{-t}$

Hence the required solution is

$$x = \frac{1}{2}\cos t - \frac{3}{2}\sin t + \frac{1}{2}e^{-t} + 2 + \frac{1}{2}t^2.$$

$$y = 1 - \frac{1}{2}\cos t + \frac{3}{2}\sin t - \frac{1}{2}e^{-t}.$$

3.6 APPLICATION OF LAPLACE TRANSFORM IN SOLVING PARTIAL DIFFERENTIAL EQUATIONS

Example 37. Given the function $U(x,t)$ defined for $a \le x \le b$, $t > o$. Find.

(i) $\mathcal{L}\left\{\frac{\partial U}{\partial t}\right\}$ (ii) $\mathcal{L}\left\{\frac{\partial U}{\partial x}\right\}$

(iii) $\mathcal{L}\left\{\frac{\partial^2 U}{\partial t^2}\right\}$ (iv) $\mathcal{L}\left\{\frac{\partial^2 U}{\partial x^2}\right\}$

Solution :

(i) $\mathcal{L}\left\{\frac{\partial U}{\partial t}\right\} = \int_0^\infty e^{-st} \frac{\partial U}{\partial t}\, dt$

$= \lim_{p\to\infty} \int_0^p e^{-st} \frac{\partial U}{\partial t}\, dt$

$= \lim_{P\to\infty} \left[e^{-st} U(x,t)\right]_0^p + \lim_{p\to\infty} s \int_0^p e^{-st} U(x,t)\, dt$

$= o - U(x, o) + \lim_{p\to\infty} s \int_0^p e^{-st} U(x,t)\, dt$

$= s \int_0^\infty e^{-st} U(x,t)\, dt - U(x,o)$

$= s\,\mathcal{L}\{U(x,t)\} - U(x,o)$

$= s\,u(x, s) - U(x,o)$

$\therefore \quad \mathcal{L}\left\{\frac{\partial U}{\partial t}\right\} = s\,u(x, s) - U(x,o)$

where $u = u(x, s) = \mathcal{L}\{U(x,t)\}$.

(ii) $\mathcal{L}\left\{\frac{\partial U}{\partial x}\right\} = \int_0^\infty e^{-st} \frac{\partial U}{\partial x}\, dt$

$= \frac{d}{dx} \int_0^\infty e^{-st} U(x,t)\, dt$

$= \frac{d}{dx}\,\mathcal{L}\{U(x,t)\} = \frac{du}{dx}$

$\therefore \quad \mathcal{L}\left\{\frac{\partial U}{\partial x}\right\} = \frac{du}{dx}.$

Method–11

(iii) Let $V = \frac{\partial U}{\partial t}$.

Then using (i), we get

$$\mathcal{L}\left\{\frac{\partial^2 U}{\partial t^2}\right\} = \mathcal{L}\left\{\frac{\partial V}{\partial t}\right\}$$

$$= s\,\mathcal{L}\{V\} - V(x, 0)$$

$$= s\,\mathcal{L}\left\{\frac{\partial U}{\partial t}\right\} - V(x, o)$$

$$= s\,[s\,u(x,s) - U(x,o)] - V(x,o)$$

$$= s^2\,u - s\,U(x,o) - U_t(x,o)$$

Since $V = \frac{\partial U}{\partial t} = U_t$.

$$\therefore \quad \mathcal{L}\left\{\frac{\partial^2 U}{\partial x^2}\right\} = s^2 u - s\,U(x,o) - U_t(x,o).$$

where $U_t(x,o) = \left.\frac{\partial U}{\partial x}\right|_{t=o}$ and $u = u(x,s) = \mathcal{L}\{U(x,t)\}$.

(iv) Let $V = \frac{\partial U}{\partial x}$ **. Then using (ii), we have**

$$\mathcal{L}\left\{\frac{\partial^2 U}{\partial x^2}\right\} = \mathcal{L}\left\{\frac{\partial V}{\partial x}\right\} = \frac{d}{dx}\,\mathcal{L}\{V\}$$

$$= \frac{d}{dx}\mathcal{L}\left\{\frac{\partial U}{\partial x}\right\} = \frac{d}{dx}\left(\frac{du}{dx}\right) = \frac{d^2u}{dx^2}$$

where $u = u(x,s) = \mathcal{L}\{U(x,t)\}$.

$$\therefore \quad \mathcal{L}\left\{\frac{\partial^2 U}{\partial x^2}\right\} = \frac{d^2u}{dx^2}.$$

Example 38. Using Laplace transform solve the following boundary value problem :

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, \qquad U(x,0) = 3 \sin 2\pi x$$

$$U(0,t) = 0, \qquad U(1,t) = 0$$

where $0 < x < 1, t > 0$.

Solution :

Given $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$ (1)

Taking the Laplace transform of the partial differential equation (1), we have

$$\mathcal{L}\left\{\frac{\partial U}{\partial t}\right\} = \mathcal{L}\left\{\frac{\partial^2 U}{\partial x^2}\right\}$$

$$s\,u - U(x,0) = \frac{d^2u}{dx^2}$$

where $u = u(x, s) = \mathcal{L}\{U(x, t)\}$.

Or, $\frac{d^2u}{dx^2} - su = -U(x,0)$

Or, $\frac{d^2u}{dx^2} - su = -3 \sin 2\pi x$ (2)

Since $U(x, 0) = 3 \sin 2\pi x$.

A.E is $m^2 - s = 0$

Or, $m^2 - (\sqrt{s})^2 = 0$ $\quad \therefore \; m = \pm\sqrt{s}$.

$\therefore$ Complementary function (C.F) is

$$u_c = y_c = C_1 e^{\sqrt{s}x} + C_2 e^{-\sqrt{s}x}.$$

Particular Integral (P.I) is

$$u_p = y_p = \frac{1}{D^2 - s} (-3 \sin 2\pi x)$$

$$= -3. \frac{1}{D^2 - s} (\sin 2\pi x)$$

$$= -3. \frac{1}{-(2\pi)^2 - s} \sin 2\pi x$$

$$= \frac{-3}{-4\pi^2 - s} \sin 2\pi x = \frac{3 \sin 2\pi x}{s + 4\pi^2}$$

Hence the general solution of (2) is

$$u = u_c + u_p = C_1 e^{\sqrt{s}x} + C_2 e^{-\sqrt{s}x} + \frac{3}{s + 4\pi^2} \sin 2\pi x \quad (3)$$

Taking the Laplace transform of those boundary conditions which involve t, we have

$\mathcal{L}\{U(o,t)\} = u(o, s) = 0$ and

$\mathcal{L}\{U(1,t)\} = u(1,s) = 0.$

Using the first condition $u(o,s) = 0$ in (3)

we have $0 = C_1 e^0 + C_2 e^0 + 0$

Since $\sin 0 = 0$

Or, $C_1 + C_2 = 0$ (4)

Again using the second condition

$U(1, s) = 0$ in (3), we get

$$0 = C_1 e^{\sqrt{s}} + C_2 e^{-\sqrt{s}} + 0$$

Since $\sin 2\pi = 0$

or, $C_1 e^{\sqrt{s}} + C_2 e^{-\sqrt{s}} = 0$ (5)

Solving the above two equations (4) & (5) for C_1 and C_2 we have $C_1 = 0, C_2 = 0$.

Puting $C_1 = 0, C_2 = 0$ in (3), we get

$$u = \frac{3}{s + 4\pi^2} \sin 2\pi x \quad (6)$$

Now taking the inverse Laplace transform of both sides of (6), we have

$$\mathcal{L}^{-1}\{u\} = \mathcal{L}^{-1}\left\{ \frac{3}{s + 4\pi^2} \sin 2\pi x \right\}$$

$$= 3 \sin 2\pi x \; \mathcal{L}^{-1}\left\{ \frac{1}{s + 4\pi^2} \right\}$$

Or, $U(x,t) = 3 \sin 2\pi x \, e^{-4\pi^2 t}$ Since $\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$

Or, $U(x, t) = 3 e^{-4\pi^2 t} \sin 2\pi x$.

This problem has the following physical interpretation :

If we consider a solid bounded by the infinite plane faces $x = 0$ and $x = 1$, the equation $\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}$ is the equation for heat conduction in this solid where $U = U(x,t)$ is the temperature at any plane face x at any time t and k is a constant called the **diffusivity** (k=1) which depends on the material of the solid. The boundary conditions $U(0,t) = 0$ and $U(1,t) = 0$ indicate that the temperatures at $x = 0$ and $x = 1$ are kept at temperature zero, while $U(x,0) = 3 \sin 2\pi x$ represents the initial temperature everywhere in $0 < x < 1$. The result $U(x,t) = 3 e^{-4\pi^2 t} \sin 2\pi x$. represents the temperature every where in the solid at time $t > 0$.

Example 39. Using Laplace transform solve

$$\frac{\partial U}{\partial t} = 2\frac{\partial^2 U}{\partial x^2},\ U(o, t) = o,\ U(5, t) = o,$$

$$U(x, o) = 10 \sin 4\pi x - 5 \sin 6\pi x.$$

Solution : Given $\frac{\partial U}{\partial t} = 2\frac{\partial^2 U}{\partial x^2}$ (1)

Taking the Laplace transform of the partial differential equation (1), we have

$$\mathcal{L}\left\{\frac{\partial U}{\partial t}\right\} = \mathcal{L}\left\{2\frac{\partial^2 U}{\partial x^2}\right\} = 2\,\mathcal{L}\left\{\frac{\partial^2 U}{\partial x^2}\right\}$$

Or, $su - U(x, o) = 2\frac{d^2u}{dx^2}$

where $u = u(x, s) = \mathcal{L}\{U(x, t)\}$

Or, $2\frac{d^2u}{dx^2} - su = -U(x, o)$

Or, $\frac{d^2u}{dx^2} - \frac{s}{2}u = -\frac{1}{2}U(x, o)$

Or, $\frac{d^2u}{dx^2} - \frac{s}{2}u = -\frac{1}{2}(10\sin 4\pi x - 5\sin 6\pi x)$ (2)

Since $U(x, o) = 10\sin 4\pi x - 5 \sin 6\pi x$

Auxiliary equation is $m^2 - \frac{s}{2} = o$ or, $m = \pm\sqrt{\frac{s}{2}}$.

$$\therefore\quad C.F = u_c = C_1 e^{\sqrt{\frac{s}{2}}x} + C_2 e^{-\sqrt{\frac{s}{2}}x}$$

Particular integral is

$$u_p = \frac{1}{D^2 - \frac{s}{2}}\left\{-\frac{1}{2}(10 \sin 4\pi x - 5\sin 6\pi x)\right\}$$

$$= -5.\frac{1}{-16\pi^2 - \frac{s}{2}}\sin 4\pi x + \frac{5}{2}.\frac{1}{-36\pi^2 - \frac{s}{2}}\sin 6\pi x$$

$$= \frac{10\sin 4\pi x}{s + 32\pi^2} - \frac{5\sin 6\pi x}{s + 72\pi^2}$$

Hence the general solution of (2) is

$$u = u_c + u_p = C_1 e^{\sqrt{\frac{s}{2}}x} + C_2 e^{-\sqrt{\frac{s}{2}}x} + \frac{10\sin 4\pi x}{s + 32\pi^2}$$

$$- \frac{5\sin 6\pi x}{s + 72\pi^2} \quad (3)$$

Taking the Laplace transform of those boundary conditions which involve t, we have

$$\mathcal{L}\{U(o,t)\} = u(o, s) = o$$

$$\mathcal{L}\{U(5, t)\} = u(5, s) = o$$

Using the first condition u(o, s) = o in (3), we have

$$o = C_1 e^o + C_2 e^o + o$$

or, $C_1 + C_2 = o$ (4)

Again using the second condition u (5, s) = o, we have

$$o = C_1 e^{5\sqrt{\frac{s}{2}}} + C_2 e^{-5\sqrt{\frac{s}{2}}} + o$$

or, $C_1 e^{5\sqrt{\frac{s}{2}}} + C_2 e^{-5\sqrt{\frac{s}{2}}} = o$ (5)

Solving the above two equations (4) & (5) for C_1 and C_2, we have $C_1 = o$, $C_2 = o$,

Putting $C_1 = o$, $C_2 = o$ in (3), we have

$$u = \frac{10\sin 4\pi x}{s + 32\pi^2} - \frac{5\sin 6\pi x}{s + 72\pi^2} \quad (6)$$

Now taking the inverse Laplace transform on both sides of (6), we have

$$\mathcal{L}^{-1}\{u\} = \mathcal{L}^{-1}\left\{\frac{10\sin 4\pi x}{s + 32\pi^2} - \frac{5\sin 6\pi x}{s + 72\pi^2}\right\}$$

$$= 10\sin 4\pi x\, \mathcal{L}^{-1}\left\{\frac{1}{s + 32\pi^2}\right\} - 5\sin 6\pi x\, \mathcal{L}^{-1}\left\{\frac{1}{s + 72\pi^2}\right\}$$

$$= 10\sin 4\pi x.\, e^{-32\pi^2 t} - 5\sin 6\pi x.\, e^{-72\pi^2 t}$$

Since $\mathcal{L}^{-1}\left\{\frac{1}{s + a}\right\} = e^{-at}$.

$$\therefore \quad U(x, t) = 10e^{-32\pi^2 t}\sin 4\pi x - 5e^{-72\pi^2 t}\sin 6\pi x.$$

which is the required solution.

Example 40. Solve using Laplace transform the following boundary value problem :

$$\frac{\partial^2 Y}{\partial t^2} = 9\frac{\partial^2 Y}{\partial x^2},\ Y(o, t) = o,\ Y(2, t) = o,$$

$$Y(x, o) = 20\sin 2\pi x - 10 \sin 5\pi x.$$

Solution : Given $\frac{\partial^2 Y}{\partial t^2} = 9\frac{\partial^2 Y}{\partial x^2}$ (1)

Taking the Laplace transform on both sides of (1), we have

$$\mathcal{L}\left\{\frac{\partial^2 Y}{\partial t^2}\right\} = 9\,\mathcal{L}\left\{\frac{\partial^2 Y}{\partial x^2}\right\}$$

or, $s^2 y - sY(x,0) - Y_t(x, o) = 9\frac{d^2 y}{dx^2}$

where $y = y(x, s) = \mathcal{L}\{Y(x, t)\}$

or, $9\frac{d^2 y}{dx^2} - s^2 y = - sY(x, 0) - Y_t(x, o)$

or, $\frac{d^2 y}{dx^2} - \frac{s^2}{3^2} y = -\frac{1}{9} sY(x, o) - \frac{1}{9} Y_t(x, o)$

$$= -\frac{20}{9} s \sin 2\pi x + \frac{10}{9} s \sin 5\pi x \quad (2)$$

Since $Y(x, o) = 20 \sin 2\pi x - 10\sin 5\pi x$ and

$Y_t(x, o) = o$

A. E is $m^2 - \left(\frac{s}{3}\right)^2 = 0, \therefore m = \pm\frac{s}{3}$.

$$C.F = y_c = C_1 e^{\frac{s}{3}x} + C_2 e^{-\frac{s}{3}x} \quad (3)$$

$$P.I. = y_p = \frac{1}{D^2 - \left(\frac{s}{3}\right)^2}\left\{-\frac{20}{9}s \sin 2\pi x + \frac{10}{9} s \sin 5\pi x\right\}$$

$$= \frac{-\frac{20}{9}s \sin 2\pi x}{-4\pi^2 - \frac{s^2}{9}} + \frac{\frac{10}{9}s \sin 5\pi x}{-25\pi^2 - \frac{s^2}{9}}$$

$$= \frac{20s \sin 2\pi x}{s^2 + 36\pi^2} - \frac{10s \sin 5\pi x}{s^2 + 225\pi^2} \quad (4)$$

Hence the general solution of (2) is

$$y = y_c + y_p$$

$$= C_1 e^{\frac{s}{3}x} + C_2 e^{-\frac{s}{3}x} + \frac{20s \sin 2\pi x}{s^2 + (6\pi)^2} - \frac{10s \sin 5\pi x}{s^2 + (15\pi)^2} \quad (5)$$

Taking the Laplace transform of those boundary conditions which involve t, we have

$$\mathcal{L}\{Y(o, t)\} = y(o, s) = o$$

$$\mathcal{L}\{Y(2, t)\} = y(2, s) = o$$

Using the first condition y(o, s) in (5), we have

$$o = C_1 e^o + C_2 e^o + o, \text{ or, } C_1 + C_2 = o \quad (6)$$

Again using the second condition y(2, s) = o in (5), we have

$$o = C_1 e^{\frac{2}{3}s} + C_2 e^{-\frac{2}{3}s} + 0 \quad \text{Or, } C_1 e^{\frac{2}{3}s} + C_2 e^{-\frac{2}{3}s} = 0 \quad (7)$$

Solving the above two equations (6) & (7) for C_1 and C_2, we have

$C_1 = 0, C_2 = 0$.

Putting $C_1 = 0, C_2 = 0$ in (5), we get

$$y = \frac{20s \sin 2\pi x}{s^2 + (6\pi)^2} - \frac{10s \sin 5\pi x}{s^2 + (15\pi)^2} \quad (8)$$

Now taking the inverse Laplace transform of both sides of (8) we get

$$\mathcal{L}^{-1}\{y\} = \mathcal{L}^{-1}\left\{\frac{20s \sin 2\pi x}{s^2 + (6\pi)^2} - \frac{10s \sin 5\pi x}{s^2 + (15\pi)^2}\right\}$$

or, $Y(x, t) = 20\sin 2\pi x\, \mathcal{L}^{-1}\left\{\frac{s}{s^2 + (6\pi)^2}\right\}$

$$- 10\sin 5\pi x \mathcal{L}^{-1}\left\{\frac{s}{s^2 + (15\pi)^2}\right\}$$

or, $Y(x, t) = 20\sin 2\pi x \cos 6\pi t - 10 \sin 5\pi x \cos 15\pi t$.

which is the required solution.

Example 41. Using Laplace transform solve the following boundary value problem :

$$\frac{\partial^2 Y}{\partial t^2} = 16\frac{\partial^2 Y}{\partial x^2},\ Y_x(0, t) = 0,\ Y(\tfrac{3}{2}, t) = 0,\ Y(x, 0) = 0,$$

$$Y_t(x, 0) = 12 \cos \pi x + 16 \cos 3\pi x - 8 \cos 5\pi x.$$

Solution : Given $\frac{\partial^2 Y}{\partial t^2} = 16\frac{\partial^2 Y}{\partial x^2}$ (1)

Taking Laplace transform on both sides of (1), we have

$$\mathcal{L}\left\{\frac{\partial^2 Y}{\partial t^2}\right\} = 16\,\mathcal{L}\left\{\frac{\partial^2 Y}{\partial x^2}\right\}$$

or, $s^2 y - sY(x, 0) - Y_t(x, 0) = 16\frac{d^2 y}{dx^2}$

or, $16\frac{d^2 y}{dx^2} - s^2 y = - sY(x, 0) - Y_t(x,0)$

or, $\frac{d^2 y}{dx^2} - \frac{s^2}{16} y = -\frac{1}{16} s\, Y(x, 0) - \frac{1}{16} Y_t(x, 0)$

43. Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} - 2k\frac{dy}{dt} + k^2y(t) = f(t).$$

Answer : $y = y(t) = [A + (B - kA)t]\, e^{kt} + \int_0^t (t-u)\, e^{(t-u)}\, f(u)du.$

ORDINARY DEFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

44. Solve the following differential equation by using the Laplace transform;

$$Y'' + tY' - Y = 0;\ Y(0) = 0,\ Y'(0) = 1.$$

Answer : $Y = t.$

45. Solve the following differential equation by the Laplace transform;

$$ty'' + 2y' + ty = o;\ y(0+) = 1,\ y(\pi) = 0.$$

Answer : $y = \frac{\sin t}{t}.$

46. Find the bounded solution of the equation

$t^2Y'' + tY' + (t^2 - 1)Y = 0$ which is such that $Y(1) = 2$.

Answer : $Y = \frac{2J_1(t)}{J_1(1)}$

47. Solve the following differential equation by the Laplace transform:

$$ty''(t) + (1 - a - t)\, y'(t) + ay(t) = t - 1;$$

$y(0) = 0$, $a > 0$ and $a \neq 1$

if $y(t)$ and its derivatives have transforms.

Answer : $y(t) = \frac{t}{a-1} + ct^a.$

48. Solve the following differential equation by the Laplace transform;

$y''(t) + at\,y'(t) - 2a\,y(t) = 1;\quad y(0) = y'(0) = 0,\ a > 0$

if $y(t)$ and its derivatives have transforms.

Answer : $y = \frac{1}{2}t^2$.

49. Prove that the Laplace transform of $J_0(t)$ and $J_0(at)$ are respectively $\frac{C}{\sqrt{s^2+1}}$ and $\frac{C}{\sqrt{s^2+a^2}}$

and using this solve the differential equation

$tY''(t) + Y'(t) + 4t\,Y(t) = 0;\ Y(0) = 3,\ Y'(0) = 0.$

Answer : $Y(t) = 3J_0(2t)$.

SIMULTANEOUS ORDINARY DIFFERETIAL EQUATIONS

Solve the following simultaneous differential equations by the Laplace transform ;

50. $\begin{cases} \frac{dx}{dt} - y = e^t \\ \frac{dy}{dt} + x = \sin t \end{cases}$ subject to the conditions $x(0)=1,\ y(0)=0$

Answer : $\begin{cases} x = \frac{1}{2}(e^t + \cos t + 2\sin t - t\cos t) \\ y = \frac{1}{2}(t\sin t - e^t + \cos t - \sin t) \end{cases}$

51. $\begin{cases} \frac{dx}{dt} - \frac{dy}{dt} - 2x + 2y = 1 - 2t \\ \frac{d^2x}{dt^2} + 2\frac{dy}{dt} + x = 0 \end{cases}$

when $x(0) = 0,\ y(0) = 0,\ x'(0) = 0$

Answer : $\begin{cases} x = 2(1 - e^{-t} - te^{-t}) \\ y = -t(1 + 2e^{-t}) + 2(1 - e^{-t}) \end{cases}$

52. $\begin{cases} (D-2)x - (D+1)y = 6e^{3t} \\ (2D-3)x + (D-3)y = 6e^{3t} \end{cases}$ if $x(0) = 3,\ y(0) = 0$

Answer : $\begin{cases} x = (1 + 2t)e^t + 2e^{3t} \\ y = (2 - t)e^t - e^{3t} \end{cases}$

53. $\begin{cases} (D^2 + 2)x - Dy = 1 \\ Dx + (D^2 + 2)y = 0 \end{cases}$ with $t > o$;

$x = 0$, $Dx = Dy = y = 0$ when $t = o$

Answer : $\begin{cases} x = \frac{1}{2} - \frac{1}{3}\cos t - \frac{1}{6}\cos 2t \\ y = -\frac{1}{3}\sin t + \frac{1}{6}\sin 2t. \end{cases}$

54. $\begin{cases} X'' + Y' + 3X = 15e^{-t} \\ Y'' - 4X' + 3Y = 15\sin 2t \end{cases}$;

$X(0) = 35$, $X'(0) = -48$

$Y(0) = 27$, $Y'(0) = -55$

Answer : $\begin{cases} X = 30\cos t - 15\sin 3t + 3e^{-t} + 2\cos 2t \\ Y = 30\cos 3t - 6o\sin t - 3e^{-t} + \sin 2t \end{cases}$

55. $\begin{cases} x'' - 3x - 4y = 0 \\ y'' + y + x = 0 \end{cases}$ if $x(0) = y(0) = y'(o) = 0$ and $x'(0) = 2$

Answer : $\begin{cases} x = t(e^t + e^{-t}) \\ y = \frac{1}{2}(1 - t)e^t - \frac{1}{2}(1 + t)e^{-t} \end{cases}$

56. $\begin{cases} x' + y = \sin t \\ y' - z = e^t \\ z' + x + y = 1 \end{cases}$; $x(0) = 0$, $y(0) = 1$, $z(0) = 1$

Answer : $\begin{cases} x = 1 - e^t \\ y = e^t + \sin t \\ z = \cos t \end{cases}$

PARTIAL DIFFERENTIAL EQUATIONS

57. Solve the following boundary value problem by using the Laplace transform:

$\frac{\partial U}{\partial t} = 2\frac{\partial^2 U}{\partial x^2}$, $U(0, t) = 0$ $U(5, t) = 0$,

$U(x, 0) = 10\sin 4\pi x$.

Answer : $U(x, t) = 10e^{-32\pi^2 t}\sin 4\pi x$.

58. Solve the following boundary value problem by using the Laplace transform:

$$\frac{\partial U}{\partial t} = 3\frac{\partial^2 U}{\partial x^2},\ U_x(0, t) = 0\ U\left(\frac{\pi}{2}, t\right) = 0,$$

$U(x,0) = 30 \cos 5x.$

Answer : $U(x, t) = 30e^{-75t} \cos 5x.$

59. Solve the following boundary value problem by using the Laplace transform :

$$\frac{\partial U}{\partial t} = 3\frac{\partial^2 U}{\partial x^2},\ U_x(0, t) = 0\ U\left(\frac{\pi}{2}, t\right) = 0,$$

$U(x,0) = 20 \cos 3x - 5\cos 9x$

Answer : $U(x, t) = 20e^{-27t} \cos 3x - 5e^{-243t} \cos 9x.$

60. Using Laplace transform find the solution of

$$\frac{\partial U}{\partial x} = 2\frac{\partial U}{\partial t} + U,\ U(x, o) = 6e^{-3x}$$

which is bounded for $x>0$, $t > o$.

Answer : $U(x, t) = 6e^{-2t} - 3x.$

61. Find the solution of the following boundary value problem by using the Laplace transform.

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} - 4U,\ U(0, t) = 0, U(\pi, t) = 0,$$

$U(x, 0) = 6\sin x - 4\sin 2x.$

Answer : $U(x,t) = 6e^{-5t} \sin x - 4e^{-8t} \sin 2x.$

62. Solve the following boundary value problem by using the Laplace transform :

$$\frac{\partial^2 y}{\partial t^2} = 9\frac{\partial^2 y}{\partial x^2},\ y(0, t) = 0, y(2, t) = 0, y_t(x,0) = 0,$$

$y(x, o) = 20 \sin 2\pi x - 10\sin 5\pi x.$

Answer : $y(x, t) = 20 \sin 2\pi x \cos 6\pi t$

$-10 \sin 5\pi x \cos 15\pi t.$

63. Solve the following boundary value problem using the Laplace transform:

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} - xt,\ y(x, 0) = 0,\ y_t(x, 0) = 0,\ y(0, t) = 0.$$

Answer. $y(x, t) = -\frac{xt^3}{6}$.

3.7 The complex inversion formula

If $f(s) = \mathcal{L}\{F(t)\}$ then $\mathcal{L}^{-1}\{f(s)\}$ is given by

$$F(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} f(s)ds,\ t > 0 \quad (1)$$

and $F(t) = 0$ for $t < 0$.

This result is called the **complex inversion integral (or formula).** It is also known as **Bromwich's integral formula.** The result provides direct means for obtainng the inverse Laplace transform of a given function f(s).

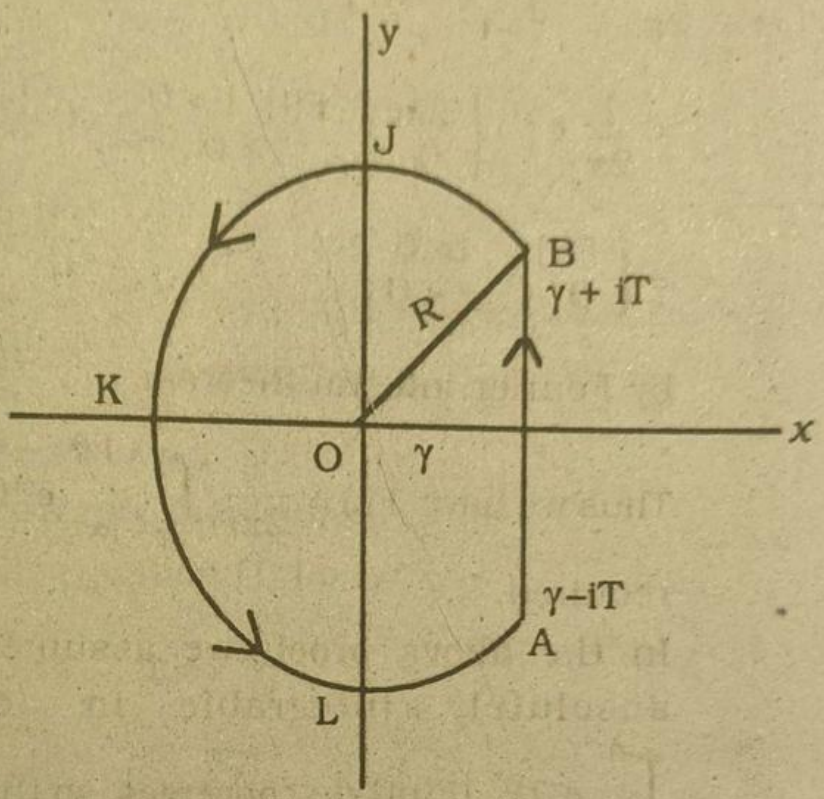
The integration in (1) is to be performed along a line $s = \gamma$ in the complex plane where $s = x + iy$. The real number γ is chosen so that $s = \gamma$ lies to the right of all the singularities (poles, branch points or essential singularities) but is otherwise arbitrary.

3.8 The Bromwich contour.

In Practice, the integral in (1) is evaluated by considering the contour integral

$$\frac{1}{2\pi i}\oint_C e^{st} f(s)ds \quad (2)$$

where C is the contour of the adjacent figure. This contour, some times called. the **Bromwich contour,** is composed of the line AB and the arc BJKLA of a circle of radius R with centre at the origin. If we represent arc BJ K LA by Γ ,then it follows from (1)



that since $T = \sqrt{R^2 - \gamma^2}$

$$F(t) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{\gamma - iT}^{\gamma + iT} e^{st} f(s)ds, \quad (3) \quad .$$

$$= \lim_{R\to\infty} \left\{ \frac{1}{2\pi i} \oint_C e^{st} f(s)\, ds \; - \frac{1}{2\pi i} \int_{\Gamma} e^{st} f(s)ds \right\}.$$

3.9 Validity of the complex inversion formula

By definition of Laplace transform, we have

$f(s) = \int_0^\infty e^{-su} F(u)du$. Then

$$\lim_{T\to\infty} \frac{1}{2\pi i} \int_{\gamma - iT}^{\gamma + iT} e^{st} f(s)ds$$

$$= \lim_{T\to\infty} \frac{1}{2\pi i} \int_{\gamma - iT}^{\gamma + iT} \int_0^\infty e^{st - su} F(u)\, duds \qquad (1)$$

Taking Laplace transform on both sides of (1), we get

$$\mathcal{L}\left\{\frac{\partial U}{\partial t}\right\} = k\mathcal{L}\left\{\frac{\partial^2 U}{\partial x^2}\right\}$$

$$s\,u(x, s) - U(x, o) = k\frac{d^2u}{dx^2}$$

$$s\,u(x, s) - U_0 = k\frac{d^2u}{dx^2}$$

$$\frac{d^2u}{dx^2} - \frac{su}{k} = -\frac{U_0}{k} \quad (2) \text{ where } u(x, s) = u$$

Since $U_x(o, t) = -\alpha U(0, t)$, $|U(x, t)| < M$.

$\therefore$ $U_x(o, s) = -\alpha\, u(o,s)$, $u(x, s)$ is bounded (3)

We will solve the equation (1) for u.

A. E is $m^2 - \frac{s}{k} = o \;\therefore m = \pm\sqrt{\frac{s}{k}}$

$\therefore$ C. F $= C_1 e^{\sqrt{\frac{s}{k}}x} + C_2 e^{-\sqrt{\frac{s}{k}}x}$

$$P.I = \frac{1}{D^2 - \frac{s}{k}}\left(-\frac{U_0}{k}\right) = \frac{U_0}{k}\cdot\frac{1}{\frac{s}{k}(1 - \frac{k}{s}D^2)}\cdot 1$$

$$= \frac{U_0}{s}\left(1 - \frac{k}{s}D^2\right)^{-1}\cdot 1 = \frac{U_o}{s}.$$

Therefore, the general solution of the differential equation is

$$u(x, s) = C_1 e^{\sqrt{\frac{s}{k}}x} + C_2 e^{-\sqrt{\frac{s}{k}}x} + \frac{U_0}{s} \quad (4)$$

Since $U(x, s)$ is bounded, $\therefore C_1 = 0$.

Thus from (4) we have

$$U(x,s) = C_2\, e^{-\sqrt{\frac{s}{k}}x} + \frac{U_0}{s} \quad (5)$$

Differentiating both sides of (5) with respect to x,

we get $U_x(x, s) = -C_2\sqrt{\frac{s}{k}}\, e^{-\sqrt{\frac{s}{k}}\, x} + 0$

$\therefore \quad U_x(0, s) = -C_2\sqrt{\frac{s}{k}}$

Or, $\quad C_2 = -\sqrt{\frac{k}{s}} \cdot -\alpha u(0, s)$

$$\begin{cases} U_x(0, t) = -\alpha U(0, t) \\ U_x(0, s) = -\alpha U(0, s) \end{cases}$$

$$= \alpha\sqrt{\frac{k}{s}}\, u(0, s) \quad (6)$$

but $u(0, s) = C_2 + \frac{U_0}{s}$

$$\therefore C_2 = \alpha\frac{\sqrt{k}}{\sqrt{s}} \cdot \left(C_2 + \frac{U_0}{s}\right)$$

$$= \frac{\alpha\sqrt{k}}{\sqrt{s}}\, C_2 + \frac{\alpha\sqrt{k}U_0}{s\sqrt{s}}$$

Or, $\quad \left(1 - \frac{\alpha\sqrt{k}}{\sqrt{s}}\right)C_2 = \frac{\alpha\sqrt{k}U_0}{s\sqrt{s}}$

Or, $\quad \left(\frac{\sqrt{s} - \alpha\sqrt{k}}{\sqrt{s}}\right)C_2 = \frac{\alpha\sqrt{k}U_0}{s\sqrt{s}}$

Or, $\quad C_2 = \frac{\alpha\sqrt{k}U_0}{s(\sqrt{s} - \alpha\sqrt{k})}$

Or, $\quad C_2 = \frac{aU_0}{s(\sqrt{s} - a)}$ where $\alpha\sqrt{k} = a$ (say)

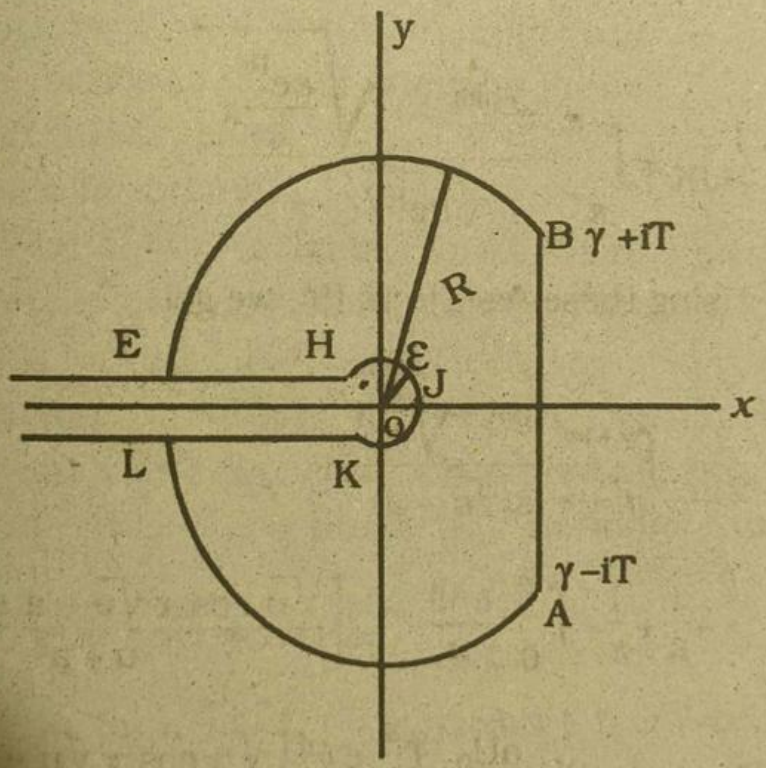
Thus from (5) we have

$$U(x, s) = \frac{aU_0}{s(\sqrt{s} - a)} \cdot e^{-\sqrt{\frac{s}{k}}\, x} + \frac{U_0}{s} \quad (7)$$

Now using the complex inversion formula.

$$\text{we get } U(x, t) = U_0 + aU_0 \mathcal{L}^{-1}\left\{ \frac{e^{-\frac{s}{\sqrt{k}}x}}{s(\sqrt{s} - a)} \right\}$$

$$= U_0 + \frac{a U_0}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st - \sqrt{\frac{s}{k}}x}}{s(\sqrt{s} - a)} ds$$



$$\text{Now } \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st - \sqrt{\frac{s}{k}}x}}{s(\sqrt{s} - a)} ds$$

$$= \lim_{\substack{R \to \infty \\ \varepsilon \to \infty}} \frac{1}{2\pi i} \left\{ \int_{EH} + \int_{Hjk} + \int_{kL} \right\}$$