more
Let's do analysis

1 of prev 3 math in final compulsory.

~~Legendre~~

24.02

Legendre differential eqn and legendre polynomial

The differential eqn,

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\cdot\frac{dy}{dx} + n(n+1)y = 0$$

is known as the legendre differential eqn.

The resulting soln of legendre's eqn is called the legendre polynomial of ~~dg~~ degree n and is denoted by,

$$P_n(x) = \sum_{r=0}^{N} (-1)^r \frac{(2n-2r)!}{2^n\, r!\,(n-r)!\,(n-2r)!}\, x^{n-2r}$$

where,

$N = \frac{n}{2}$ if n be an even number

$N = \frac{n-1}{2}$ if n be an odd number

Application

1. Determination of wave func of electron in the orbitals
2. seismic wave analysis



Finding P_0, P_1, P_2, P_3, P_4 by using legendre polynomial

By the defn

$$P_n(x) = \sum_{r=0}^{N} (-1)^r \frac{(2n-2r)!}{2^n\, r!\,(n-r)!\,(n-2r)!}\, x^{n-2r}$$

when $n=0$, then $N=0$, $r=0$

$$P_0(x) = \sum_{r=0}^{0} (-1)^r \frac{(0-2r)!}{2^0 \cdot (0-r)!\,(0-2r)!}\, x^{0-2r}$$

$$= (-1)^0 \frac{0!}{1 \cdot 0!\,0!}\, x^0 = 1$$

$$P_1(x) = \sum_{r=0}^{0} (-1)^r \frac{(2-2r)!}{2^1\, r!\,(1-r)!\,(1-2r)!}\, x^{1-2r}$$

$$= (-1)^0 \frac{2!}{2\; 0!\,1!\,1!}\, x^1 = x$$

When, $n=2$, $N=1$, $r=0,1$

$$P_2(x) = \sum_{r=0}^{1} (-1)^r \frac{(4-2r)!}{2^2\, r!\,(2-r)!\,(2-2r)!}\, x^{2-2r}$$

$$= (-1)^0 \frac{4!}{4\; 0!\,2!\,1!}\, x^2 + (-1)^1 \frac{2!}{2^2\, 1!\,1!\,0!}\, x^0$$

$$= \frac{1}{2}(3x^2 - 1)$$

$$P_3 = \frac{1}{2}(5x^3 - 3x)$$

$$P_4 = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

legenre — 3rd math, Fourier — 3rd math
Bessel —

Q state and prove Rodrigue's form of legenre polynomial.

Statement: $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$

proof: from legendre polynomial, we have

$$P_n(x) = \sum_{r=0}^{N} (-1)^n \frac{(2n-2r)!}{2^n r!(n-r)!(n-2r)!} x^{n-2r} \quad \text{---(1)}$$

Now,

$$(x^2-1)^n = (x^2)^n - {}^nC_1 (x^2)^{n-1} + {}^nC_2 (x^2)^{n-2} - {}^nC_3 (x^2)^{n-3} + \dots$$

$$= \sum_{r=0}^{n} (-1)^r {}^nC_r (x^2)^{n-r}$$

$$= \sum_{r=0}^{n} (-1)^r {}^nC_r x^{2n-2r}$$

$$\therefore \frac{d^n}{dx^n}(x^2-1)^n = \sum_{r=0}^{n} (-1)^r {}^nC_r \frac{d^n}{dx^n}\{x^{2n-2r}\}$$

$\frac{d^n}{dx^n}(x^m) = \frac{m!}{(m-n)!} x^{m-n}$

$$= \sum_{r=0}^{n} (-1)^r \frac{n!}{r!(n-r)!} \frac{(2n-2r)!}{(2n-2r-n)!} x^{2n-2r-n}$$

$$= \sum_{r=0}^{n} \frac{(-1)^r n!\,(2n-2r)!}{r!(n-r)!(n-2r)!} x^{n-2r}$$

$$\therefore \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n = \sum_{r=0} \frac{(-1)^r (2n-2r)!}{2^n r!(n-r)!(n-2r)!} x^{n-2r}$$

We know, $\frac{d^n}{dx^n}(x^m) = \frac{m!}{(m-n)!} x^{m-n}$

Finding P_0, P_1, P_2, P_3, P_4 by Rodrigue's form.

$$P_4(x) = \frac{1}{2^4 4!} \frac{d^4}{dx^4}(x^2-1)^4$$

$$= \frac{1}{16 \times 24} \frac{d^4}{dx^4}(x^8 + 4x^6 + 6x^4 - 4x^2 + 1)$$

$$= \frac{1}{384} \frac{d^3}{dx^3}(8x^7 - 24x^5 + 24x^3 - 8x)$$

$$= \frac{1}{384} \frac{d^2}{dx^2}(56x^6 - 120x^4 + 72x^2 - 8)$$

$$= \frac{1}{384} \frac{d}{dx}(336x^5 - 480x^3 + 144x)$$

$$= \frac{1}{384}(1680x^4 - 1440x^2 + 144)$$

$$= \frac{1}{8}(35x^4 - 30x^2 + 3)$$

Q Using Rodrigues's formula, show that

$$\int_{-1}^{1} x^n P_n(x)\,dx = \frac{\sqrt{\pi}\,\sqrt{n+1}}{2^n\,\sqrt{n+3/2}}$$

→ From legendre polynomials, we have

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$\Rightarrow \int_{-1}^{1} x^n P_n(x)\,dx = \int_{-1}^{1} \frac{1}{2^n n!}\, x^n \frac{d^n}{dx^n}(x^2-1)^n\,dx$$

$$= \frac{1}{2^n n!}\left[x^n \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n - n\,x^{n-1}\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n + n(n-1)\,x^{n-2}\frac{d^{n-3}}{dx^{n-3}}(x^2-1)^n - \cdots + (-1)^{n-1} n(n-1)\cdots 4\cdot 3\cdot 2\cdot x^{n-(n-1)}\frac{d^{n-n}}{dx^{n-n}}(x^2-1)^n + (-1)^n n(n-1)\cdots 4\cdot 3\cdot 2\int x^0 (x^2-1)^n\,dx \right]_{-1}^{1}$$

$$= \frac{1}{2^n n!}(-1)^n\, n! \int_{-1}^{1}(x^2-1)^n\,dx$$

$$= \frac{(-1)^n}{2^n}\int_{-1}^{1}(-1)^n (1-x^2)^n\,dx$$

$$= \frac{(-1)^{2n}}{2^n}\,2\int_0^1 (1-x^2)^n\,dx \qquad \left[\because (1-x^2)^n = \text{even function}\right]$$

$$= \frac{1}{2^{n-1}}\int_0^1 (1-x^2)^n\,dx$$

$$= \frac{1}{2^{n-1}}\int_0^{\pi/2} (1-\sin^2\theta)^n \cos\theta\,d\theta$$

let, $x = \sin\theta \Rightarrow$
$dx = \cos\theta\,d\theta$
$x \to 0 \to \theta \to 0$
$x \to 1 \to \theta \to \pi/2$

$$= \frac{1}{2^{n+1}} \int_0^{\pi/2} \cos^{2n+1}\theta \, d\theta$$

$$= \frac{1}{2^{n-1}} \cdot 2 \, \frac{\left\lfloor \frac{0+1}{2} \right\rfloor \left\lfloor \frac{2n+1+1}{2} \right\rfloor}{2 \left\lfloor \frac{0+2n+1+2}{2} \right\rfloor}$$

$$= \frac{1}{2^n} \, \frac{\sqrt{\pi} \cdot \lfloor n+1}{\lfloor n+3/2}$$

$$\therefore \int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \, d\theta = \frac{\left\lfloor \frac{p+1}{2} \right\rfloor \left\lfloor \frac{q+1}{2} \right\rfloor}{2 \left\lfloor \frac{p+q+2}{2} \right\rfloor}$$

Final

legenre, fourier
3ta math 3ta math

polynomial
Bessel

Final type

④ Express $x^4+2x^3+2x^2-x-3$ in terms of Legendre polynomial.

⟶ Given that, $x^4+2x^3+2x^2-x-3$

Now $P_0 = 1$

$P_1 = x$

$P_2 = \frac{1}{2}(3x^2-1)$

$\Rightarrow 2P_2 = 3x^2-1$

$\Rightarrow x^2 = \frac{1}{3}(2P_2+P_0)$

$P_3 = \frac{1}{2}(5x^3-3x)$

$\Rightarrow 2P_3 = 5x^3-3x$

$\Rightarrow x^3 = \frac{1}{5}\{2P_3+3P_1\}$

$P_4 = \frac{1}{8}(35x^4-30x^2+3)$

~~$\Rightarrow 8P_4 = 35x^4-30x^2+3$~~

$\Rightarrow 8P_4 = 35x^4 - 30\cdot\frac{1}{3}(2P_2+P_0)+3P_0$

$\therefore x^4 = \frac{1}{35}(8P_4+20P_2+7P_0)$

Hence expansion becomes.

$= \frac{1}{35}\left[8P_4+20P_2+7P_0\right] + \frac{2}{5}\{2P_3+3P_1\}$

$+ \frac{2}{3}\{2P_2+P_0\} - P_1 - 3P_0$

$= \frac{8}{35}P_4 + \frac{4}{5}P_5 + \frac{40}{21}P_2 + \frac{1}{5}P_1 - \frac{32}{15}P_0$

Legendre series

General form.

$$f(x) = \sum_{r=0}^{n} c_r P_r(x)$$

where, $c_r = (r+\frac{1}{2}) \int_{-1}^{1} f(x) P_r(x)\, dx$

Q2

Find the first four terms of legendre series of

$$f(x) = \begin{cases} 0 & \text{when } -1 < x < 0 \\ x & \text{when } 0 < x < 1 \end{cases}$$

$$\longrightarrow f(x) = \begin{cases} 0 & \text{when } -1 < x < 0 \\ x & \text{when } 0 < x < 1 \end{cases} \longrightarrow (1)$$

By the defn of leg. ser.

$$f(x) = \sum_{r=0}^{n} c_r P_r = c_0 P_0 + c_1 P_1 + c_2 P_2 + c_3 P_3 + \dots$$

$$c_r = (r+\tfrac{1}{2}) \int_{-1}^{1} f(x) P_r\, dx$$

$$= (r+\tfrac{1}{2}) \left[\int_{-1}^{0} 0 \cdot P_r\, dx + \int_{0}^{1} x \cdot P_r\, dx \right]$$

Now,

$$c_0 = \frac{1}{2} \int_0^1 x P_0\, dx$$

$$= \frac{1}{2} \int_0^1 x\, dx$$

$$= \frac{1}{2} \left[\frac{x^2}{2} \right]_0^1$$

$$= 1/4$$

$$c_1 = \frac{3}{2} \int_0^1 x \cdot x\, dx$$

$$= \frac{3}{2} \left[\frac{x^3}{3} \right]_0^1$$

$$= \frac{1}{2}$$

$$c_2 = \frac{5}{2}\int_0^1 x \cdot \frac{1}{2}(3x^2-1)\,dx$$

$$= \frac{5}{4}\left[\frac{3x^4}{4} - \frac{x^2}{2}\right]_0^1$$

$$= \frac{5}{4} \times \frac{1}{4} = \frac{5}{16}$$

$$c_3 = \frac{7}{2}\int_0^1 x \cdot \frac{1}{2}(5x^3-3x)$$

$$= \frac{7}{4}\left[\frac{5x^5}{5} - \frac{3x^3}{3}\right]_0^1$$

$$= 0$$

$$\therefore f(x) = \frac{1}{4}P_0 + \frac{1}{2}P_1 + \frac{5}{16}P_2 + \cdots$$

Q3 find the first four terms of Legendre series of

$$f(x) = \begin{cases} 0 & \text{when } -1 < x < 0 \\ x^2 & \text{when } 0 < x < 1 \end{cases}$$

Q4 $f(x) = \begin{cases} 0 & \text{when } -1 < x < 0 \\ 1 & \text{when } 0 < x < 1 \end{cases}$