Case 1: When roots are different and differing not by an integer.

1. Solve in series the differential equation

$$4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0 \quad \text{——} (1)$$

Soln: Let $y = x^\lambda (c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots)$

where $c_0 \neq 0$.

$$y = c_0x^\lambda + c_1x^{\lambda+1} + c_2x^{\lambda+2} + c_3x^{\lambda+3} + \cdots \quad (2)$$

Then $\frac{dy}{dx} = \lambda c_0x^{\lambda-1} + (\lambda+1)c_1x^\lambda + (\lambda+2)c_2x^{\lambda+1} + (\lambda+3)c_3x^{\lambda+2} + \cdots$

and $\frac{d^2y}{dx^2} = \lambda(\lambda-1)c_0x^{\lambda-2} + (\lambda+1)\lambda c_1x^{\lambda-1} + (\lambda+2)(\lambda+1)c_2x^\lambda + (\lambda+3)(\lambda+2)c_3x^{\lambda+1} + \cdots$

Putting the values of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in (1) we obtain

$$4\lambda(\lambda-1)c_0x^{\lambda-1} + 4(\lambda+1)\lambda c_1x^\lambda + 4(\lambda+2)(\lambda+1)c_2x^{\lambda+1} + 4(\lambda+3)(\lambda+2)c_3x^{\lambda+2} + \cdots$$
$$+ 2\lambda c_0x^{\lambda-1} + 2(\lambda+1)c_1x^\lambda + 2(\lambda+2)c_2x^{\lambda+1} + 2(\lambda+3)c_3x^{\lambda+2} + \cdots + c_0x^\lambda + c_1x^{\lambda+1} + c_2x^{\lambda+2} + c_3x^{\lambda+3} + \cdots = 0$$

$$\Rightarrow \{4\lambda(\lambda-1)c_0 + 2\lambda c_0\}x^{\lambda-1} + \{4(\lambda+1)\lambda c_1 + 2(\lambda+1)c_1 + c_0\}x^\lambda + \{4(\lambda+2)(\lambda+1)c_2 + 2(\lambda+2)c_2 + c_1\}x^{\lambda+1} + \{4(\lambda+3)(\lambda+2)c_3 + 2(\lambda+3)c_3 + c_2\}x^{\lambda+2} + \cdots = 0$$

The lowest power of x is $x^{\lambda-1}$ and its coefficient equated to zero gives

$$4\lambda(\lambda-1)c_0 + 2\lambda c_0 = 0$$

$$\Rightarrow 4\lambda(\lambda-1) + 2\lambda = 0 \quad | \text{ since } c_0 \neq 0$$

$$\Rightarrow 2\lambda(2\lambda - 2 + 1) = 0 \quad \text{(Indicial equation)}$$

$$\lambda = 0 \quad 2\lambda - 1 = 0$$

Thus the roots of indicial equation are

$\lambda = 0$, $\lambda = \frac{1}{2}$ (Distinct roots).

Equating the co-efficient of x^λ to zero, we get

$$4(\lambda+1)\lambda c_1 + 2(\lambda+1)c_1 + c_0 = 0$$

or $$2c_1(\lambda+1)(2\lambda+1) = -c_0$$

$$c_1 = -\frac{c_0}{2(\lambda+1)(2\lambda+1)}$$

Equating the co-efficient of $x^{\lambda+1}$ to zero we get

$$4(\lambda+2)(\lambda+1)c_2 + 2(\lambda+2)c_2 + c_1 = 0$$

$$\Rightarrow 2c_2(\lambda+2)(2\lambda+3) = -c_1$$

$$\therefore c_2 = -\frac{c_1}{2(\lambda+2)(2\lambda+3)}$$

$$c_2 = \frac{c_0}{4(\lambda+1)(\lambda+2)(2\lambda+1)(2\lambda+3)}$$

Equating the co-efficient of $x^{\lambda+2}$ ~~we get~~ to zero, we get

$$4(\lambda+3)(\lambda+2)c_3 + 2(\lambda+3)c_3 + c_2 = 0$$

$$\Rightarrow 2c_3(\lambda+3)(2\lambda+5) = -c_2$$

$$c_3 = -\frac{c_2}{2(\lambda+3)(2\lambda+5)}$$

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$$c_3 = -\frac{c_0}{8(\lambda+1)(\lambda+2)(\lambda+3)(2\lambda+1)(2\lambda+3)(2\lambda+5)}$$

and so on.

Substituting the values of c_1, c_2, c_3 -- in (1) we get

$$y = c_0 x^{\lambda} - \frac{c_0}{2(\lambda+1)(2\lambda+1)} x^{\lambda+1} + \frac{c_0 x^{\lambda+2}}{4(\lambda+1)(\lambda+2)(2\lambda+1)(2\lambda+3)}$$

$$- \frac{c_0 x^{\lambda+3}}{8(\lambda+1)(\lambda+2)(\lambda+3)(2\lambda+1)(2\lambda+3)(2\lambda+5)} + \cdots$$

When $\lambda = 0$, we have

$$y_1 = c_0 - \frac{c_0}{2 \cdot 1} x + \frac{c_0}{4 \cdot 1 \cdot 2 \cdot 1 \cdot 3} x^2 - \frac{c_0}{8 \cdot 1 \cdot 2 \cdot 3 \cdot 1 \cdot 3 \cdot 5} x^3 + \cdots$$

When $\lambda = \frac{1}{2}$, we have

$$y_2 = c_0 x^{1/2} - \frac{c_0 x^{3/2}}{2 \cdot 3/2 \cdot 2} + \frac{c_0 x^{5/2}}{4 \cdot 3/2 \cdot 5/2 \cdot 2 \cdot 4} +$$

$$\frac{c_0}{8 \cdot 3/2 \cdot 5/2 \cdot 7/2 \cdot 2 \cdot 4 \cdot 6} x^{7/2} + \cdots$$

Thus $y = y_1 + y_2$ is the required series solution.

(2) $2x(1-x)\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + 3y = 0$ ——— (1)

Let $y = x^\lambda (c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots)$

where $c_0 \neq 0$

or, $y = c_0 x^\lambda + c_1 x^{\lambda+1} + c_2 x^{\lambda+2} + c_3 x^{\lambda+3} + c_4 x^{\lambda+4} + \cdots$ ——— (2)

Then

$\frac{dy}{dx} = \lambda c_0 x^{\lambda-1} + (\lambda+1) c_1 x^\lambda + (\lambda+2) c_2 x^{\lambda+1}$

$+ (\lambda+3) c_3 x^{\lambda+2} + (\lambda+4) c_4 x^{\lambda+3} + \cdots$

and $\frac{d^2y}{dx^2} = \lambda(\lambda-1) c_0 x^{\lambda-2} + \lambda(\lambda+1) c_1 x^{\lambda-1} + (\lambda+1)(\lambda+2) c_2$

$x^\lambda + (\lambda+3)(\lambda+2) c_3 x^{\lambda+1} + (\lambda+4)(\lambda+3) c_4 x^{\lambda+2}$

$+ \cdots$

putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1) we get

$(2x - 2x^2)\{\lambda(\lambda-1) c_0 x^{\lambda-2} + \lambda(\lambda+1) c_1 x^{\lambda-1}$

$+ (\lambda+1)(\lambda+2) c_2 x^\lambda + (\lambda+3)(\lambda+2) c_3 x^{\lambda+1}$

$+ (\lambda+4)(\lambda+3) c_4 x^{\lambda+2} + \cdots\}$

$+ (1-x)\{\lambda c_0 x^{\lambda-1} + (\lambda+1) c_1 x^\lambda + (\lambda+2) c_2 x^{\lambda+1}$

$+ (\lambda+3) c_3 x^{\lambda+2} + (\lambda+4) c_4 x^{\lambda+3} + \cdots\}$

$+ 3 c_0 x^\lambda + 3 c_1 x^{\lambda+1} + 3 c_2 x^{\lambda+2} + 3 c_3 x^{\lambda+3}$

$+ 3 c_4 x^{\lambda+4} + \cdots = 0$

$\Rightarrow \{2\lambda(\lambda-1) c_0 + \lambda c_0\} x^{\lambda-1} + \{2\lambda(\lambda+1) c_1 - 2\lambda(\lambda-1) c_1$

$+ (\lambda+1) c_1 - \lambda c_0 + 3 c_0\} x^\lambda + \{2(\lambda+2)(\lambda+1) c_2$

$- 2\lambda(\lambda+1) c_1 + (\lambda+2) c_2 - (\lambda+1) c_1 + 3 c_1\} x^{\lambda+1}$

$$+\{2(\lambda+2)(\lambda+3)c_3 - 2(\lambda+2)(\lambda+1)c_2 + (\lambda+3)c_3 - (\lambda+2)c_2 + 3c_2\} x^{\lambda+2}$$

$$+ \{ \text{- - - -} \} = 0 .$$

The lowest power of x is $x^{\lambda-1}$ and its co-efficient equated to zero gives.

$$2\lambda(\lambda-1)c_0 + \lambda c_0 = 0 \qquad c_0 \neq 0$$

$$\Rightarrow \lambda(2\lambda - 2 + 1) = 0$$

$$\Rightarrow \lambda \cdot (2\lambda - 1) = 0$$

$$\lambda = 0, \quad \lambda = \frac{1}{2}$$

Equating the co-efficient of x^{λ} to zero. we get

$$2\lambda(\lambda+1)c_1 - 2\lambda(\lambda-1)c_0 + (\lambda+1)c_1 - \lambda c_0 + 3c_0 = 0$$

$$\Rightarrow \{2\lambda(\lambda+1) + (\lambda+1)\} c_1 = \{2\lambda(\lambda-1) + (\lambda-3)\} c_0$$

$$\Rightarrow \{(\lambda+1)(2\lambda+1)\} c_1 = \{2\lambda^2 - 2\lambda + \lambda - 3\} c_0$$

$$c_1 = \frac{(2\lambda^2 - \lambda - 3)\, c_0}{(\lambda+1)(2\lambda+1)}$$

$$= \frac{(2\lambda^2 - 3\lambda + 2\lambda - 3)}{(\lambda+1)(2\lambda+1)} c_0$$

$$= \frac{\{\lambda(2\lambda-3) + 1(2\lambda-3)\}}{(\lambda+1)(2\lambda+1)} c_0$$

$$= \frac{(\lambda+1)(2\lambda-3)}{(\lambda+1)(2\lambda+1)} c_0$$

$$c_1 = \frac{2\lambda - 3}{2\lambda + 1} \cdot c_0$$

Equating the co-efficient of $x^{\lambda+1}$ to zero we get

$$2(\lambda+2)(\lambda+1)c_2 - 2\lambda(\lambda+1)c_1 + (\lambda+2)c_2 - (\lambda+1)c_1 + 3c_1 = 0$$

$$\Rightarrow \{(\lambda+2)(2\lambda+2+1)\}c_2 = 2\lambda(\lambda+1)c_1 + (\lambda+1)c_1 - 3c_1$$

$$\Rightarrow (\lambda+2)(2\lambda+3)c_2 = \{2\lambda^2+2\lambda+\lambda+1-3\}c_1$$

$$= \{2\lambda^2+3\lambda-2\}c_1$$

$$= \{2\lambda^2+4\lambda-\lambda-2\}c_1$$

$$= \{2\lambda(\lambda+2)-1(\lambda+2)\}c_1$$

$$= \{(\lambda+2)(2\lambda-1)\}c_1$$

$$\therefore c_2 = \frac{2\lambda-1}{2\lambda+3}c_1$$

$$= \frac{(2\lambda-1)(2\lambda-3)}{(2\lambda+1)(2\lambda+3)}c_0$$

$$c_2 = \frac{(2\lambda-3)(2\lambda-1)}{(2\lambda+1)(2\lambda+3)}c_0$$

Equating the co-efficient of $x^{\lambda+2}$ to zero, we get

$$2(\lambda+3)(\lambda+2)c_3 - 2(\lambda+2)(\lambda+1)c_2 + (\lambda+3)c_3 - (\lambda+2)c_2 + 3c_2 = 0$$

$$\Rightarrow (\lambda+3)\{2\lambda+4+1\}c_3 = \{2(\lambda+2)(\lambda+1) + (\lambda+2) - 3\}c_2$$

$$(\lambda+3)(2\lambda+5)c_3 = \{2\lambda^2+7\lambda+3\}c_2$$

$$= \{2\lambda^2+6\lambda+\lambda+3\}c_2$$

$$= \{2\lambda(\lambda+3)+1(\lambda+3)\}c_2$$

$$= (\lambda+3)(2\lambda+1)c_2$$

$$c_3 = \frac{(\lambda+3)(2\lambda+1)}{(\lambda+3)(2\lambda+5)} \cdot \frac{(2\lambda-3)(2\lambda-1)}{(2\lambda+1)(2\lambda+3)} c_0$$

$$= \frac{(2\lambda-3)(2\lambda-1)}{(2\lambda+3)(2\lambda+5)} c_0$$

Substituting the values of c_1, c_2 and c_3 in (2)

$$y = c_0 x^\lambda + \frac{2\lambda-3}{2\lambda+1} c_0 x^{\lambda+1} + \frac{(2\lambda-3)(2\lambda-1)}{(2\lambda+1)(2\lambda+3)} c_0 x^{\lambda+2}$$

$$+ \frac{(2\lambda-3)(2\lambda-1)}{(2\lambda+3)(2\lambda+5)} c_0 x^{\lambda+3} + \cdots$$

When $\lambda = 0$ we have

$$y_1 = c_0\left[1 - \frac{3x}{1} + \frac{3x^2}{1\cdot 3} + \frac{3x^3}{3\cdot 5} + \cdots\right]$$

~~$= c_0(1 - 3x \cdots$~~

When $\lambda = \frac{1}{2}$ we have

$$y_2 = c_0 x^{1/2} - \frac{2}{2} c_0 x^{3/2} + 0 = c_0 x^{1/2}(1-x)$$

$\therefore y = y_1 + y_2$ is the required series soln.

3. $9x(1-x)\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0$ ——①

Let $y = x^\lambda(c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots)$, where $c_0 \neq 0$

$y = c_0x^\lambda + c_1x^{\lambda+1} + c_2x^{\lambda+2} + c_3x^{\lambda+3} + \cdots$ ②

Then $\frac{dy}{dx} = \lambda c_0x^{\lambda-1} + (\lambda+1)c_1x^\lambda + (\lambda+2)c_2x^{\lambda+1} + (\lambda+3)c_3x^{\lambda+2} + \cdots$

and $\frac{d^2y}{dx^2} = \lambda(\lambda-1)c_0x^{\lambda-2} + (\lambda+1)\lambda c_1x^{\lambda-1} + (\lambda+2)(\lambda+1)c_2x^\lambda + (\lambda+3)(\lambda+2)c_3x^{\lambda+1} + \cdots$

Putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in ① we get

$(9x-9x^2)\{\lambda(\lambda-1)c_0x^{\lambda-2} + (\lambda+1)\lambda c_1x^{\lambda-1} + (\lambda+2)(\lambda+1)c_2x^\lambda + (\lambda+3)(\lambda+2)c_3x^{\lambda+1} + \cdots\} - \{12\lambda c_0x^{\lambda-1} + 12(\lambda+1)c_1x^\lambda + 12(\lambda+2)c_2x^{\lambda+1} + 12(\lambda+3)c_3x^{\lambda+2} + \cdots\} + 4c_0x^\lambda + 4c_1x^{\lambda+1} + 4c_2x^{\lambda+2} + 4c_3x^{\lambda+3} + \cdots = 0$

$\Rightarrow \{9\lambda(\lambda-1)c_0 - 12\lambda c_0\}x^{\lambda-1} + \{9c_1(\lambda+1)\lambda - 9\lambda(\lambda-1)c_0 - 12(\lambda+1)c_1 + 4c_0\}x^\lambda + \{9(\lambda+2)(\lambda+1)c_2 - 9(\lambda+1)\lambda c_1 - 12(\lambda+2)c_2 + 4c_1\}x^{\lambda+1} + \{9(\lambda+3)(\lambda+2)c_3 - 9(\lambda+2)(\lambda+1)c_2 - 12(\lambda+3)c_3 + 4c_2\}x^{\lambda+2} + \cdots = 0$,

The lowest power of x is $x^{\lambda-1}$ and its co-efficient equated to zero gives

$9\lambda(\lambda-1)c_0 - 12\lambda c_0 = 0$ since $c_0 \neq 0$

$3\lambda(\lambda-1) - 4\lambda = 0$

$\Rightarrow \lambda(3\lambda-7) = 0$

$\lambda = 0, \quad \lambda = \frac{7}{3}$

Equating the co-efficient of x^λ to zero. get

$$9(\lambda+1)\lambda c_1 - 9\lambda(\lambda-1)c_0 - 12(\lambda+1)c_1 + 4c_0 = 0$$

$$\Rightarrow 9(\lambda+1)\lambda c_1 - 12(\lambda+1)c_1 = 9\lambda(\lambda-1)c_0 - 4c_0$$

$$\Rightarrow (\lambda+1)(9\lambda-12)c_1 = (9\lambda^2 - 9\lambda - 4)c_0$$

$$\Rightarrow 3(\lambda+1)(3\lambda-4)c_1 = (9\lambda^2 - 12\lambda + 3\lambda - 4)c_0$$

$$= \{3\lambda(3\lambda-4) + 1(3\lambda-4)\}c_0$$

$$= (3\lambda+1)(3\lambda-4)c_0$$

$$\therefore c_1 = \frac{3\lambda+1}{3(\lambda+1)}c_0$$

Equating the co-efficient of $x^{\lambda+1}$ to zero, we get

$$9(\lambda+2)(\lambda+1)c_2 - 9(\lambda+1)\lambda c_1 - 12(\lambda+2)c_2 + 4c_1 = 0$$

$$\Rightarrow 3(\lambda+2)\{3\lambda+3-4\}c_2 = \{9(\lambda+1)\lambda - 4\}c_1$$

$$\Rightarrow 3(\lambda+2)(3\lambda-1)c_2 = \{9\lambda^2 + 9\lambda - 4\}c_1$$

$$= \{9\lambda^2 + 12\lambda - 3\lambda - 4\}c_1$$

$$= \{3\lambda(3\lambda+4) - 1(3\lambda+4)\}c_1$$

$$= (3\lambda+4)(3\lambda-1)c_1$$

$$\therefore c_2 = \frac{3\lambda+4}{3(\lambda+2)} \cdot \frac{(3\lambda+1)}{3(\lambda+1)}c_0$$

$$c_2 = \frac{(3\lambda+4)(3\lambda+1)}{3^2(\lambda+1)(\lambda+2)}c_0$$

Equating the co-efficient of $x^{\lambda+2}$ to zero, we get,

$$9(\lambda+3)(\lambda+2)c_3 - 9(\lambda+2)(\lambda+1)c_2 - 12(\lambda+3)c_3 + 4c_2 = 0$$

$$\Rightarrow 3(\lambda+3)\{3(\lambda+2)-4\}c_3 = \{9\lambda^2+27\lambda+18-4\}c_2$$

$$\Rightarrow 3(\lambda+3)(3\lambda+2)c_3 = \{9\lambda^2+27\lambda+14\}c_2$$

$$= \{9\lambda^2+21\lambda+6\lambda+14\}c_2$$

$$= \{3\lambda(3\lambda+7)+2(3\lambda+7)\}c_2$$

$$c_3 = \frac{(3\lambda+2)(3\lambda+7)}{3(\lambda+3)(3\lambda+2)}c_2$$

$$c_3 = \frac{(3\lambda+1)(3\lambda+4)(3\lambda+7)}{3^3(\lambda+1)(\lambda+2)(\lambda+3)}c_0 \quad \text{and so on.}$$

Substituting the values of $c_1, c_2, c_3 \cdots$ in (2) we get

$$y = c_0 x^{\lambda} + \frac{(3\lambda+1)}{3(\lambda+1)}c_0 x^{\lambda+1} + \frac{(3\lambda+1)(3\lambda+4)}{3^2(\lambda+1)(\lambda+2)}c_0 x^{\lambda+2} + \frac{(3\lambda+1)(3\lambda+4)(3\lambda+7)}{3^3(\lambda+1)(\lambda+2)(\lambda+3)}c_0 x^{\lambda+3} + \cdots$$

When $\lambda = 0$ we get

$$y_1 = c_0 + \frac{1}{3}c_0 x + \frac{1\cdot 4}{3^2\cdot 1\cdot 2}c_0 x^2 + \frac{1\cdot 4\cdot 7}{3^3\cdot 1\cdot 2\cdot 3}c_0 x^3 + \cdots$$

When $\lambda = 7/2$ we have

$$y_2 = c_0 x^{7/3} + \frac{8}{3\cdot\frac{10}{3}}c_0 x^{7/3+1} + \cdots$$

Thus $y = y_1 + y_2$ is the required solution.

$$-(\lambda+1)\lambda c_1 + (\lambda+2)c_2 - (\lambda+1)c_1 - c_1 \} x^{\lambda+1}$$

$$+ \{ (\lambda+3)(\lambda+2)c_3 - (\lambda+2)(\lambda+1)c_2 + (\lambda+3)c_3$$

$$- (\lambda+2)c_2 - c_2 \} x^{\lambda+2} + \cdots = 0.$$

The lowest power of x is $x^{\lambda-1}$ and its co-efficient equated to zero gives

$$\lambda(\lambda-1)c_0 + \lambda c_0 = 0 \quad \text{Since } c_0 \neq 0$$

$$\lambda^2 - \lambda + \lambda = 0$$

$$\Rightarrow \lambda^2 = 0 \Rightarrow \lambda = 0, \lambda = 0.$$

Equating the co-efficient of x^{λ} to zero, we get.

$$(\lambda+1)\lambda c_1 - \lambda(\lambda-1)c_0 + (\lambda+1)c_1 - \lambda c_0 - c_0 = 0$$

$$(\lambda+1)^2 c_1 = (\lambda^2 - \lambda + \lambda + 1)c_0$$

$$c_1 = \frac{\lambda^2+1}{(\lambda+1)^2} c_0$$

Equating the co-efficient of $x^{\lambda+1}$ to zero. we get

$$(\lambda+2)(\lambda+1)c_2 - (\lambda+1)\lambda c_1 - (\lambda+2)c_2 - (\lambda+1)c_1 - c_1 = 0$$

$$\Rightarrow (\lambda+2)^2 c_2 = (\lambda^2 + \lambda + \lambda + 1 + 1)c_1$$

$$\Rightarrow c_2 = \frac{\lambda^2 + 2\lambda + 2}{(\lambda+2)^2} c_1.$$

$$c_2 = \frac{(\lambda^2+1)\{(\lambda+1)^2+1\}}{(\lambda+1)^2(\lambda+2)^2} c_0$$

Equating the co-efficient of $x^{\lambda+2}$ to zero, we get

$$(\lambda+3)(\lambda+2)c_3 - (\lambda+2)(\lambda+1)c_2 + (\lambda+3)c_3 - (\lambda+2)c_2 - c_2 = 0$$

$$\Rightarrow (\lambda+3)^2 c_3 = \{(\lambda+2)(\lambda+1) + (\lambda+2) + 1\} c_2$$

$$c_3 = \frac{\{(\lambda+2)^2 + 1\}}{(\lambda+3)^2} c_2$$

$$= \frac{(\lambda^2+1)\{(\lambda+1)^2+1\}\{(\lambda+2)^2+1\}}{(\lambda+1)^2(\lambda+2)^2(\lambda+3)^2} c_0$$

and so on.

Substituting the values of $c_1, c_2, c_3 \cdots$ in (2) we get

$$y = c_0 x^\lambda + \frac{\lambda^2+1}{(\lambda+1)^2} c_0 x^{\lambda+1} + \frac{(\lambda^2+1)\{(\lambda+1)^2+1\}}{(\lambda+1)^2(\lambda+2)^2} c_0 x^{\lambda+2} + \frac{(\lambda^2+1)\{(\lambda+1)^2+1\}\{(\lambda+2)^2+1\}}{(\lambda+1)^2(\lambda+2)^2(\lambda+3)^2} c_0 x^{\lambda+3} + \cdots \quad (3)$$

When $\lambda = 0$, we have

$$y_1 = c_0 + \frac{1}{1^2} x c_0 + \frac{1(1^2+1)}{1^2 2^2} c_0 x^2 + \frac{1 \cdot (1^2+1)(2^2+1)}{1^2 2^2 3^2} c_0 x^3 + \cdots$$

From (3)

$$y = c_0 x^\lambda \left[1 + \frac{\lambda^2+1}{(\lambda+1)^2} x + \frac{(\lambda^2+1)\{(\lambda+1)^2+1\}}{(\lambda+1)^2(\lambda+2)^2} x^2 + \frac{(\lambda^2+1)\{(\lambda+1)^2+1\}\{(\lambda+2)^2+1\}}{(\lambda+1)^2(\lambda+2)^2(\lambda+3)^2} x^3 + \cdots\right]$$

(4)

Differentiating (4) with respect to x

$$\frac{\partial y}{\partial x} = x^{\lambda}\log_e x \cdot c_0\left[1+\frac{\lambda^2+1}{(\lambda+1)^2}x+\frac{(\lambda^2+1)\{(\lambda+1)^2+1\}}{(\lambda+1)^2(\lambda+2)^2}x^2\right.$$
$$\left.+\cdots\right]+x^{\lambda}c_0\left[0+\frac{(\lambda+1)^2\cdot 2\lambda-(\lambda^2+1)2(\lambda+1)}{(\lambda+1)^4}x\right.$$
$$\left.+\cdots\cdots\right] \quad (5)$$

Putting $\lambda=0$ in

$$y_2 = c_0\log_e\left(1+\frac{1}{1^2}x+\cdots\right)$$
$$+c_0\left(\frac{-2}{1}x+\cdots\right)$$

$$= c_0\log_e\left(1+\frac{1}{1^2}x+\cdots\right)+c_0\left(-2x+\cdots\right)$$

$\therefore y = y_1 + y_2$ is the required solution.

5. $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$ ——— (1)

Let $y = x^\lambda (c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots)$

where $c_0 \neq 0$

$y = c_0 x^\lambda + c_1 x^{\lambda+1} + c_2 x^{\lambda+2} + c_3 x^{\lambda+3} + \cdots$ (2)

Then $\frac{dy}{dx} = \lambda c_0 x^{\lambda-1} + (\lambda+1) c_1 x^\lambda + (\lambda+2) c_2 x^{\lambda+1} + (\lambda+3) c_3 x^{\lambda+2} + \cdots$

and $\frac{d^2y}{dx^2} = \lambda(\lambda-1) c_0 x^{\lambda-2} + \lambda(\lambda+1) c_1 x^{\lambda-1} + (\lambda+2)(\lambda+1) c_2 x^\lambda + (\lambda+3)(\lambda+2) c_3 x^{\lambda+1} + \cdots$

putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1) we get

$\lambda(\lambda-1) c_0 x^{\lambda-1} + \lambda(\lambda+1) c_1 x^\lambda + (\lambda+2)(\lambda+1) c_2 x^{\lambda+1}$
$+ (\lambda+3)(\lambda+2) c_3 x^{\lambda+2} + \lambda c_0 x^{\lambda-1} + (\lambda+1) c_1 x^\lambda$
$+ (\lambda+2) c_2 x^{\lambda+1} + (\lambda+3) c_3 x^{\lambda+2} + c_0 x^{\lambda+1}$
$+ c_1 x^{\lambda+2} + c_2 x^{\lambda+3} + c_3 x^{\lambda+4} + \cdots$

$\Rightarrow \{\lambda(\lambda-1) c_0 + \lambda c_0\} x^{\lambda-1} + \{\lambda(\lambda+1) c_1 + (\lambda+1) c_1\} x^\lambda + \{(\lambda+2)(\lambda+1) c_2 + (\lambda+2) c_2 + c_0\} x^{\lambda+1}$
$+ \{(\lambda+3)(\lambda+2) c_3 + (\lambda+3) c_3 + c_1\} x^{\lambda+2} + \cdots = 0$

The lowest power of x is $x^{\lambda-1}$ and its co-efficient equated to zero, gives

$\lambda(\lambda-1) c_0 + \lambda c_0 = 0$ since $c_0 \neq 0$

$\lambda(\lambda-1+1) = 0$

$\Rightarrow \lambda^2 = 0$

$\Rightarrow \lambda = 0, 0.$

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Equating the co-efficient of x^{λ} to zero we get

$$(\lambda+1)\lambda c_1 + (\lambda+1)c_1 = 0$$

$$\Rightarrow (\lambda+1)^2 c_1 = 0$$

$$\Rightarrow c_1 = 0 .$$

Equating the co-efficient of $x^{\lambda+1}$ to zero,

$$(\lambda+2)(\lambda+1)c_2 + (\lambda+2)c_2 + c_0 = 0$$

$$\Rightarrow (\lambda+2)^2 c_2 = -c_0$$

$$c_2 = -\frac{c_0}{(\lambda+2)^2}$$

Equating the co-efficient of $x^{\lambda+2}$ to zero, we get

$$(\lambda+3)(\lambda+2)c_3 + (\lambda+3)c_3 + c_1 = 0$$

$$\Rightarrow (\lambda+3)^2 c_3 = -c_1$$

$$c_3 = -\frac{c_1}{(\lambda+3)^2} = 0 \quad | \because c_1 = 0 .$$

Similarly $c_5 = c_7 = c_9 = \cdots = 0$.

Substituting the values of $c_1, c_2, c_3, \ldots$ in (2) we get

$$y = c_0 x^{\lambda} - \frac{c_0}{(\lambda+2)^2} x^{\lambda+2} + \cdots \quad (3)$$

When $\lambda = 0$ we have

$$y_1 = c_0 - \frac{c_0}{2^2} x^2 + \cdots$$

Differentiating (3) w.r.to λ

$$y = c_0 x^{\lambda}\left[1 - \frac{1}{(\lambda+2)^2}x^2 + \cdots\right]$$

$$\frac{\partial y}{\partial \lambda} = c_0 x^{\lambda}\log x\left[1 - \frac{1}{(\lambda+2)^2}x^2 + \cdots\right] + c_0 x^{\lambda}\left[-\frac{2}{(\lambda+2)^3}x^2 + \cdots\right]$$

$$\left.\frac{\partial y}{\partial \lambda}\right|_{\lambda=0} = c_0 \log x\left[1 - \frac{1}{2^2}x^2 + \cdots\right] + c_0\left(\frac{2}{2^2}x^2 + \cdots\right)$$

$$\therefore y_2 = c_0 \log x\left(1 - \frac{x^2}{2^2} + \cdots\right) + c_0\left(\frac{1}{2}x^2 + \cdots\right)$$

Thus $y = y_1 + y_2$ is the required series solution.

Case III: If the indicial equation has two roots α, β (say $\alpha > \beta$) differing by an integer and if some of the co-efficient of y become infinite when $\lambda = \beta$, we modify the form of y by replacing c_0 by $K(\lambda-\beta)$ then we get two independent solutions by putting $\lambda = \beta$ in the modified form of y and $\frac{\partial y}{\partial \lambda}$. The result of putting $\lambda = \alpha$ in y gives a numerical multiple of that obtained by putting $\lambda = \beta$.

6. $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-4)y = 0$ ——— (1)

Let $y = x^\lambda (c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots)$

$\Rightarrow y = c_0x^\lambda + c_1x^{\lambda+1} + c_2x^{\lambda+2} + c_3x^{\lambda+3} + \cdots$ (2)

Then

$$\frac{dy}{dx} = \lambda c_0x^{\lambda-1} + (\lambda+1)c_1x^\lambda + (\lambda+2)c_2x^{\lambda+1} + (\lambda+3)c_3x^{\lambda+2} + \cdots$$

and

$$\frac{d^2y}{dx^2} = \lambda(\lambda-1)c_0x^{\lambda-2} + (\lambda+1)\lambda c_1x^{\lambda-1} + (\lambda+2)(\lambda+1)c_2x^\lambda + (\lambda+3)(\lambda+2)c_3x^{\lambda+1} + \cdots$$

Putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1) we get

$$\lambda(\lambda-1)c_0x^\lambda + (\lambda+1)\lambda c_1x^{\lambda+1} + (\lambda+2)(\lambda+1)c_2x^{\lambda+2} + (\lambda+3)(\lambda+2)c_3x^{\lambda+3} \cdots + \lambda c_0x^\lambda + (\lambda+1)c_1x^{\lambda+1} + (\lambda+2)c_2x^{\lambda+2} + (\lambda+3)c_3x^{\lambda+3} + \cdots$$

$$c_0x^{\lambda+2} + c_1x^{\lambda+3} + c_2x^{\lambda+4} + c_3x^{\lambda+5} + \cdots$$

$$-4c_0x^\lambda - 4c_1x^{\lambda+1} - 4c_2x^{\lambda+2} - 4c_3x^{\lambda+3} - \cdots$$

$\Rightarrow \{\lambda(\lambda-1)c_0 + \lambda c_0 - 4c_0\}x^\lambda + \{(\lambda+1)\lambda c_1 + (\lambda+1)c_1 - 4c_1\}x^{\lambda+1} + \{(\lambda+2)(\lambda+1)c_2 + (\lambda+2)c_2 + c_0 - 4c_2\}x^{\lambda+2} + \{(\lambda+3)(\lambda+2)c_3 + (\lambda+3)c_3 + c_1 - 4c_2\}x^{\lambda+3}$

The lowest power of x is x^λ and its co-efficient equated to zero gives

$$\lambda(\lambda-1)c_0 + \lambda c_0 - 4c_0 = 0 \quad \text{since } c_0 \neq 0$$

$$\Rightarrow \lambda^2 - \lambda + \lambda - 4 = 0$$

$$\Rightarrow \lambda^2 = 4$$

$$\therefore \lambda = 2, -2.$$

Equating the coefficient of $x^{\lambda+1}$ to zero, we get

$$(\lambda+1)\lambda c_1 + (\lambda+1)c_1 - 4c_1 = 0$$

$$\Rightarrow \{(\lambda+1)\lambda + (\lambda+1) - 4\} c_1 = 0$$

$$\therefore c_1 = 0$$

Equating the co-efficient of $x^{\lambda+2}$ to zero, we get

$$(\lambda+2)(\lambda+1)c_2 + (\lambda+2)c_2 + c_0 - 4c_2 = 0$$

$$\Rightarrow \{(\lambda+2)^2 - 4\} c_2 = -c_0$$

$$c_2 = -\frac{c_0}{(\lambda+2)^2 - 4} = -\frac{c_0}{(\lambda+4)\lambda}$$

Equating the co-efficient of $x^{\lambda+3}$ to zero, we get

$$(\lambda+3)(\lambda+2)c_3 + (\lambda+3)c_3 + c_1 - 4c_3 = 0$$

$$\therefore c_3 = 0$$

Similarly $c_5 = c_7 = c_9 = \cdots = 0$,

Substituting the values of $c_1, c_2, c_3 \cdots$ in (2) we get

$$y = c_0 x^\lambda \left[1 - \frac{1}{\lambda(\lambda+4)}x^2 + \frac{x^4}{\lambda(\lambda+2)(\lambda+4)(\lambda+6)} \cdots\right] \quad \text{--- (3)}$$

If we put $\lambda = -2$ in (3), the co-efficients becomes infinite, so we shall replace c_0 by $K(\lambda+2)$ in (3), then

$\lambda(\lambda-1)c_0 - \lambda c_0 = 0$ since $c_0 \neq 0$

$\Rightarrow \lambda^2 - \lambda - \lambda = 0$

$\Rightarrow \lambda(\lambda-2) = 0$

$\lambda = 0, \quad \lambda = 2$

Equating the co-efficient of x^λ to zero we get

$(\lambda+1)\lambda c_1 - \lambda(\lambda-1)c_0 - (\lambda+1)c_1 - 3\lambda c_0 - c_0 = 0$

$\Rightarrow (\lambda+1)(\lambda-1)c_1 = (\lambda^2 - \lambda + 3\lambda + 1)c_0$

$\Rightarrow (\lambda^2-1)c_1 = (\lambda^2 + 2\lambda + 1)c_0$

$\Rightarrow c_1 = \frac{(\lambda+1)^2}{(\lambda-1)(\lambda+1)} c_0 = \frac{\lambda+1}{\lambda-1} c_0$

Equating the co-efficient of $x^{\lambda+1}$ to zero, we get

$(\lambda+2)(\lambda+1)c_2 - (\lambda+1)\lambda c_1 - (\lambda+2)c_2$

$- 3(\lambda+1)c_1 - c_1 = 0$

$\Rightarrow (\lambda+2)(\lambda+1-1)c_2 = (\lambda+1)\lambda c_1 + 3(\lambda+1)c_1 + c_1$

$\Rightarrow \lambda(\lambda+2)c_2 = \{\lambda^2 + \lambda + 3\lambda + 3 + 1\} c_1$

$= (\lambda+2)^2 c_1$

$\therefore c_2 = \frac{\lambda+2}{\lambda} c_1$

$= \frac{(\lambda+1)(\lambda+2)}{\lambda(\lambda-1)} c_0$

Equating the co-efficient of $x^{\lambda+2}$ to zero we get

$$(\lambda+3)(\lambda+2)c_3 - (\lambda+2)(\lambda+1)c_2 - (\lambda+3)c_3 - 3(\lambda+2)c_2 - c_2 = 0$$

$$\Rightarrow (\lambda+3)(\lambda+2-1)c_3 = \{(\lambda+2)(\lambda+1) + 3(\lambda+2) + 1\}c_2$$

$$\Rightarrow (\lambda+3)(\lambda+1)c_3 = \{(\lambda+2)(\lambda+4)+1\}c_2$$

$$\Rightarrow (\lambda+3)(\lambda+1)c_3 = \{\lambda^2+6\lambda+8+1\}c_2$$

$$c_3 = \frac{(\lambda+3)^2}{(\lambda+3)(\lambda+1)}c_2$$

$$= \frac{(\lambda+3)}{(\lambda+1)}c_2$$

$$= \frac{(\lambda+1)(\lambda+2)(\lambda+3)}{\lambda(\lambda-1)(\lambda+1)}c_0$$

$$c_3 = \frac{(\lambda+2)(\lambda+3)}{\lambda(\lambda-1)}c_0$$

Substituting the values of $c_1, c_2, c_3 \cdots$ in (2) we get

$$y = c_0 x^\lambda \left[1 + \frac{\lambda+1}{\lambda-1}x + \frac{(\lambda+1)(\lambda+2)}{\lambda(\lambda-1)}x^2 + \frac{(\lambda+2)(\lambda+3)}{\lambda(\lambda-1)}x^3 + \cdots\right] \quad (3)$$

When $\lambda = 0$, some of the co-efficients become infinite. So we shall replace c_0 by $K(\lambda - 0) = K\lambda$

Then $y = Kx^\lambda \left[\lambda + \frac{\lambda(\lambda+1)}{\lambda-1}x + \frac{(\lambda+1)(\lambda+2)}{\lambda-1}x^2 + \frac{(\lambda+2)(\lambda+3)}{\lambda-1}x^3 + \cdots\right]$ (4)

When $\lambda = 0$, we have

$$y_1 = k\left[0 + 0 + \frac{1\cdot 2}{-1}x^2 + \frac{2\cdot 3}{-1}x^3 + \cdots\right]$$

$$y_1 = k\left(-1\cdot 2x^2 - 2\cdot 3x^3 - \cdots\right)$$

Differentiating (4) w.r.to λ we have

$$\frac{\partial y}{\partial \lambda} = kx^\lambda \log_e x\left[\lambda + \frac{\lambda(\lambda+1)}{\lambda-1}x + \frac{(\lambda+1)(\lambda+2)}{\lambda-1}x^2 + \cdots\right]$$

$$+ kx^\lambda\left[1 - \frac{(\lambda-1)\cdot(2\lambda+1) - (\lambda^2+2)}{(\lambda-1)^2}x + \cdots\right]$$

$$\left.\frac{\partial y}{\partial \lambda}\right|_{\lambda=0} = k\log_e x\left[-1\cdot 2x^2 + \cdots\right] + k\left(1 + \frac{1}{1}x + \cdots\right)$$

$$\therefore y_2 = k\log_e x\left(-1\cdot 2x^2 + \cdots\right) + k\left(1 + x + \cdots\right)$$

$\therefore y = y_1 + y_2$ is the required sol^n.