Elementary probability theory Lecture-1

Learning Objectives

After careful study of this Lecture, you should be able to do the following:

1. Examine the basic ideas and approaches to probability and its calculation.
2. Calculate the probabilities of combined events
3. Use counting theory involving permutations and combinations under some circumstances of probabilities
4. The same ideas can be applied to somewhat more complex situations.

Random Experiment

An experiment is a procedure that is carried out under controlled conditions, and executed to discover an unknown result.

An experiment that results in different outcomes even when repeated in the same manner every time is a random experiment.

Sample Spaces

- The set of all possible outcomes of a random experiment is called the sample space, S.
- S is discrete if it consists of a finite or countable infinite set of outcomes.
- S is continuous if it contains an interval of real numbers.

Example 1-1: Defining Sample Spaces

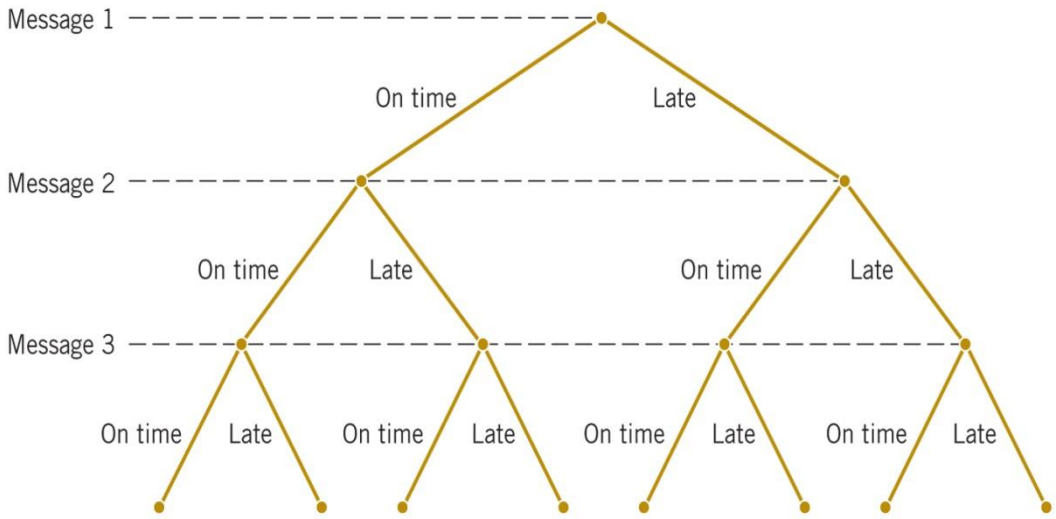
- Randomly select a camera and record the recycle time of a flash $S = R^+ = \{x \mid x > 0\}$, the positive real numbers.
- Suppose it is known that all recycle times are between 1.5 and 5 seconds. Then $S = \{x \mid 1.5 < x < 5\}$ is continuous.
- It is known that the recycle time has only three values (low, medium or high). Then $S = \{low, medium, high\}$ is discrete.

- Does the camera conform to minimum recycle time specifications? S = {*yes, no*} is discrete.

Sample Space Defined by a Tree Diagram

Example 1-2: Messages are classified as on-time (o) or late (l). Classify the next 3 messages.

S = {ooo, ool, olo, oll, loo, lol, llo, lll}



Events are Sets of Outcomes

- An event (E) is a subset of the sample space of a random experiment.
- Event combinations
  - ✓ The Union of two events consists of all outcomes that are contained in one event or the other, denoted as $E_1 \cup E_2$.
  - ✓ The Intersection of two events consists of all outcomes that are contained in one event and the other, denoted as $E_1 \cap E_2$.
  - ✓ The Complement of an event is the set of outcomes in the sample space that are not contained in the event, denoted as E'.

Example 1-3 Discrete Events

Suppose that the recycle times of two cameras are recorded. Consider only whether or not the cameras conform to the manufacturing

specifications. We abbreviate *yes* and *no* as *y* and *n*. The sample space is $S = \{yy, yn, ny, nn\}$.

Suppose, E_1 denotes an event that at least one camera conforms to specifications, then $E_1 = \{yy, yn, ny\}$

Suppose, E_2 denotes an event that no camera conforms to specifications, then $E_2 = \{nn\}$

Suppose, E_3 denotes an event that at least one camera does not conform. Then $E_3 = \{yn, ny, nn\}$,

- ✓ Then $E_1 \cup E_3 = S$
- ✓ Then $E_1 \cap E_3 = \{yn, ny\}$
- ✓ Then $E_1' = \{nn\}$

Example 1-4: Continuous Events

Measurements of the thickness of a part are modeled with the sample space: $S = R^+$

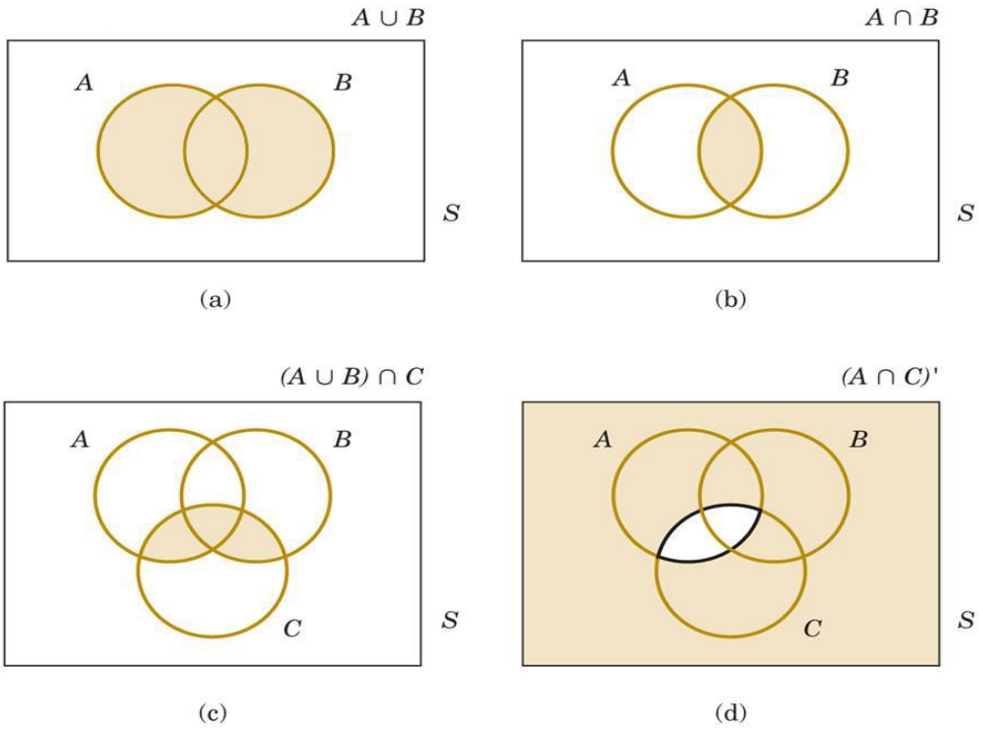
Let $E_1 = \{x \mid 10 \leq x < 12\}$,

Let $E_2 = \{x \mid 11 < x < 15\}$

- ✓ Then $E_1 \cup E_2 = \{x \mid 10 \leq x < 15\}$
- ✓ Then $E_1 \cap E_2 = \{x \mid 11 < x < 12\}$
- ✓ Then $E_1' = \{x \mid 0 < x < 10 \text{ or } x \geq 12\}$
- ✓ Then $E_1' \cap E_2 = \{x \mid 12 \leq x < 15\}$

Venn Diagrams

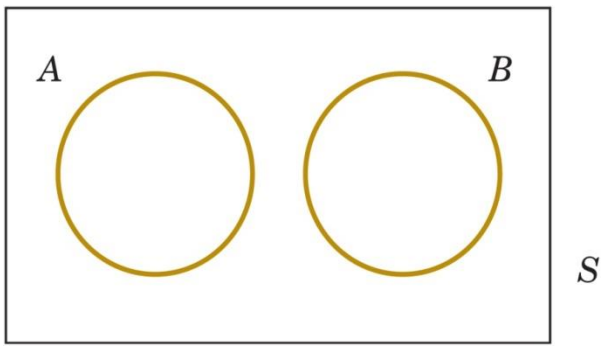
Events *A* & *B* contain their respective outcomes. The shaded regions indicate the event relation of each diagram.



Mutually Exclusive Events

- Events *A* and *B* are mutually exclusive because they share no common outcomes.
- The occurrence of one event precludes the occurrence of the other.

Symbolically, $A \cap B = \emptyset$



Mutually Exclusive Events - Laws

- Commutative law (event order is unimportant):
  - ✓ $A \cap B = B \cap A$ and $A \cup B = B \cup A$
- Distributive law (like in algebra):

- ✓ $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
- ✓ $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

- Associative law (like in algebra):
  - ✓ $(A \cup B) \cup C = A \cup (B \cup C)$
  - ✓ $(A \cap B) \cap C = A \cap (B \cap C)$

Mutually Exclusive Events - Laws

- De Morgan's law:

✓ $(A \cup B)' = A' \cap B'$ The complement of the union is the intersection of the complements.

✓ $(A \cap B)' = A' \cup B'$ The complement of the intersection is the union of the complements.

✓ Complement law: $(A')' = A$.

Counting Techniques

- There are three special rules, or counting techniques, used to determine the number of outcomes in events.
- They are:
  1. Multiplication rule
  2. Permutation rule
  3. Combination rule
- Each has its special purpose that must be applied properly – the right tool for the right job.

Counting – Multiplication Rule

- Multiplication rule:
  - ✓ Let an operation consist of k steps and there are
    - n_1 ways of completing step 1,
    - n_2 ways of completing step 2, … and

- n_k ways of completing step *k*.

✓ Then, the total number of ways to perform *k* steps is:

- $n_1 \cdot n_2 \cdot \ldots \cdot n_k$

Example 1-5: Web Site Design

- In the design for a website, we can choose to use among:
  - ✓ 4 colors,
  - ✓ 3 fonts, and
  - ✓ 3 positions for an image.

How many designs are possible?

- Answer via the multiplication rule: 4 · 3 · 3 = 36

Counting – Permutation Rule

- A permutation is a unique sequence of distinct items.
- If *S* = {*a, b, c*}, then there are 6 permutations
  - ✓ Namely: *abc*, *acb*, *bac*, *bca*, *cab*, *cba* (order matters)
- Number of permutations for a set of *n* items is *n*!
- *n*! = *n*·(*n*-1)·(*n*-2)·…·2·1
- 7! = 7·6·5·4·3·2·1 = 5,040 = FACT (7) in Excel
- By definition: 0! = 1
- For a sequence of *r* items from a set of *n* items:

$$P_r^n = n(n-1)(n-2)...(n-r+1) = \frac{n!}{(n-r)!}$$

Example 1-6: Printed Circuit Board

A printed circuit board has eight different locations in which a component can be placed. If four different components are to be placed on the board, how many designs are possible?

Answer: Order is important, so use the permutation formula with *n* = 8, *r* = 4.

$$P_4^8 = \frac{8!}{(8-4)!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4!}{4!} = 8 \cdot 7 \cdot 6 \cdot 5 = 1,680$$

Counting - Similar Item Permutations

- Used for counting the sequences when some items are identical.
- The number of permutations of:

  $n = n_1 + n_2 + \ldots + n_r$ items of which

  $n_1, n_2, \ldots, n_r$ are identical.

is calculated as: $\dfrac{n!}{n_1!n_2!\ldots n_r!}$

Example 1-7: Hospital Schedule

- In a hospital, an operating room needs to schedule three knee surgeries and two hip surgeries in a day. The knee surgery is denoted as *k* and the hip as *h*.
  - ✓ How many sequences are there?

Since there are 2 identical hip surgeries and 3 identical knee surgeries, then

- ✓ What is the set of sequences?

{*kkkhh*, *kkhkh*, *kkhhk*, *khkkh*, *khkhk*, *khhkk*, *hkkkh*, *hkkhk*, *hkhkk*, *hhkkk*}

Counting – Combination Rule

- A combination is a selection of *r* items from a set of *n* where order does not matter.
- If *S* = {*a, b, c*}, *n* =3, then
  - ✓ If *r* = 3, there is 1 combination, namely: *abc*
  - ✓ If *r* = 2, there are 3 combinations, namely *ab*, *ac*, and *bc*
- # of permutations ≥ # of combinations
- Since order does not matter with combinations, we are dividing the # of permutations by *r*!, where *r*! is the # of arrangements of *r* elements.

$$C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example 1-8: Sampling w/o Replacement

- A bin of 50 parts contains 3 defectives and 47 non-defective parts. A sample of 6 parts is selected from the 50 without replacement. How many samples of size 6 contain 2 defective parts?
- First, how many ways are there for selecting 2 parts from the 3 defective parts? $C_2^3 = \frac{3!}{2!\cdot 1!} = 3$ different ways
- In Excel: 3 = COMBIN(3,2)
- Now, how many ways are there for selecting 4 parts from the 47 non-defective parts?

$$C_4^{47} = \frac{47!}{4!\cdot 43!} = \frac{47\cdot 46\cdot 45\cdot 44\cdot 43!}{4\cdot 3\cdot 2\cdot 1\cdot 43!} = 178,365 \text{ different ways}$$

- In Excel: 178,365 = COMBIN(47,4)
- Now, how many ways are there to obtain:
  - 2 from 3 defectives, and
  - 4 from 47 non-defectives?

$C_2^3 C_4^{47} = 3\cdot 178,365 = 535,095$ different ways

✓ In Excel: 535,095 = COMBIN(3,2)*COMBIN(47,4)

Probability

- Probability is the likelihood or chance that a particular outcome or event from a random experiment will occur.
- In this lecture, we consider only discrete (finite or countably infinite) sample spaces.
- Probability is a number in the [0, 1] interval.
- A probability of:

✓ 1 means certainty

✓ 0 means impossibility

Probability Based on Equally-Likely Outcomes

✓ Whenever a sample space consists of N possible outcomes that are equally likely, the probability of each outcome is $1/N$.

For example, in a batch of 100 diodes, 1 is laser diode. A diode is randomly selected from the batch. Random means each diode has an equal chance of being selected. The probability of choosing the laser diode is 1/100 or 0.01, because each outcome in the sample space is equally likely.

Probability of an Event

- For a discrete sample space, the probability of an event E, denoted by $P(E)$, equals the sum of the probabilities of the outcomes in E.
- The discrete sample space may be:
  - ✓ A finite set of outcomes
  - ✓ A countably infinite set of outcomes.

Example 1-9: Probabilities of Events

- A random experiment has a sample space {*a, b, c, d*}. These outcomes are not equally-likely; their probabilities are: 0.1, 0.3, 0.5, 0.1
- Let Event A = {*a, b*}, B = {*b, c, d*}, and C = {*d*}
  - ✓ $P(A) = 0.1 + 0.3 = 0.4$
  - ✓ $P(B) = 0.3 + 0.5 + 0.1 = 0.9$
  - ✓ $P(C) = 0.1$
  - ✓ $P(A') = 0.6$ and $P(B') = 0.1$ and $P(C') = 0.9$
  - ✓ Since event $A \cap B$ = {*b*}, then $P(A \cap B) = 0.3$
  - ✓ Since event $A \cup B$ = {a, *b, c, d*}, then $P(A \cup B) = 1.0$
  - ✓ Since event $A \cap C$ = {null}, then $P(A \cap C) = 0$

Axioms of Probability

- Probability is a number that is assigned to each member of a collection of events from a random experiment that satisfies the following properties:

If S is the sample space and E is any event in the random experiment, $P(S) = 1$

- ✓ $0 \le P(E) \le 1$
- ✓ For any two events E_1 and E_2 with $E_1 \cap E_2 = \emptyset$, $P(E_1 \cup E_2) = P(E_1) + P(E_2)$

- The axioms imply that:
  - ✓ $P(\emptyset) = 0$ and $P(E') = 1 - P(E)$
  - ✓ If E_1 is contained in E_2, then $P(E_1) \le P(E_2)$.

Addition Rules

- Joint events are generated by applying basic set operations to individual events, specifically:
  - ✓ Unions of events, $A \cup B$
  - ✓ Intersections of events, $A \cap B$
  - ✓ Complements of events, A'
- Probabilities of joint events can often be determined from the probabilities of the individual events that comprise them.

Example 1-10: Semiconductor Wafers

A wafer is randomly selected from a batch that is classified by contamination and location as shown below.

Contamination	Location of Tool		Total
	Center	Edge	
Low	514	68	582
High	112	246	358
Total	626	314	940

- ✓ Let H be the event of high concentrations of contaminants. Then *P* (H) = 358/940.
- ✓ Let *C* be the event of the wafer being located at the center of a sputtering tool. Then *P*(*C*) = 626/940.
- ✓ *P*(H∩C) = 112/940
- ✓ *P*(H ⋃ *C*) = *P*(*H*) + *P*(*C*) – *P*(*H*∩*C*) = (358 + 626 – 112)/940 = 0.928.

This is the addition rule.

Probability of a Union

- For any two events *A* and *B*, the probability of union is given by:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- If events *A* and *B* are mutually exclusive, then

$P(A \cap B) = \varphi,$
and therefore:
$P(A \cup B) = P(A) + P(B)$

Addition Rule: 3 or More Events

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Note the alternating signs.

Conditional Probability

- *P*(*B*|*A*) is the probability of event *B* occurring, given that event *A* has already occurred.
- A communications channel has an error rate of 1 per 1000 bits transmitted. Errors are rare, but do tend to occur in bursts. If a bit is in error, the probability that the next bit is also in error is greater than 1/1000.

Conditional Probability Rule

- The conditional probability of an event *B* given an event *A*, denoted as *P*(*B*|*A*), is:

P(*B*|*A*) = *P*(*A*∩*B*)/*P*(*A*) for *P*(*A*) > 0.

- From a relative frequency perspective of n equally likely outcomes:
  - ✓ *P*(*A*) = (number of outcomes in *A*)/n
  - ✓ *P*(*A*∩*B*) = (number of outcomes in *A*∩*B*)/*n*
  - ✓ *P*(*B*|*A*) = number of outcomes in *A*∩*B* / number of outcomes in *A*

Example 1-11:

There are 4 probabilities conditioned on flaws in the below table.

Parts Classified			
Defective	Surface Flaws		Total
	Yes (*F*)	No (*F'*)	
Yes (*D*)	10	18	28
No (*D'*)	30	342	372
Total	40	360	400

$P(F) = 40/400$ and $P(D) = 28/400$

$P(D \mid F) = P(D \cap F)/P(F) = \frac{10}{400} / \frac{40}{400} = \frac{10}{40}$

$P(D' \mid F) = P(D' \cap F)/P(F) = \frac{30}{400} / \frac{40}{400} = \frac{30}{40}$

$P(D \mid F') = P(D \cap F')/P(F') = \frac{18}{400} / \frac{360}{400} = \frac{18}{360}$

$P(D' \mid F') = P(D' \cap F')/P(F') = \frac{342}{400} / \frac{360}{400} = \frac{342}{360}$

Random Samples

- Random means each item is equally likely to be chosen. If more than one item is sampled, random means that every sampling outcome is equally likely.
  - ✓ 2 items are taken from *S* = {*a, b, c*} without replacement.
  - ✓ Ordered sample space: *S* = {*ab, ac, bc, ba, ca, cb*}
  - ✓ Unordered sample space: *S* = {*ab, ac, bc*}

Example 1-12: Sampling Without Enumeration

- A batch of 50 parts contains 10 made by instrument 1 and 40 made by instrument 2. If 2 parts are selected randomly*,

a) What is the probability that the 2nd part came from instrument 2, given that the 1st part came from instrument 1?

- $P(E_1)$= P(1st part came from instrument 1) = 10/50
- $P(E_2 \mid E_1)$ = P (2nd part came from instrument 2 given that 1st part came from instrument 1) = 40/49.

b) What is the probability that the 1st part came from instrument 1 and the 2nd part came from instrument 2?

- $P(E_1 \cap E_2)$ = P(1st part came from instrument 1 and 2nd part came from instrument 2) = (10/50)·(40/49) = 8/49

*Selected randomly implies that at each step of the sample, the items remain in the batch are equally likely to be selected.

Multiplication Rule

- The conditional probability can be rewritten to generalize a multiplication rule.

  $P(A \cap B) = P(B|A) \cdot P(A) = P(A|B) \cdot P(B)$
- The last expression is obtained by exchanging the roles of A and B.

Example 1-13: Machining Stages

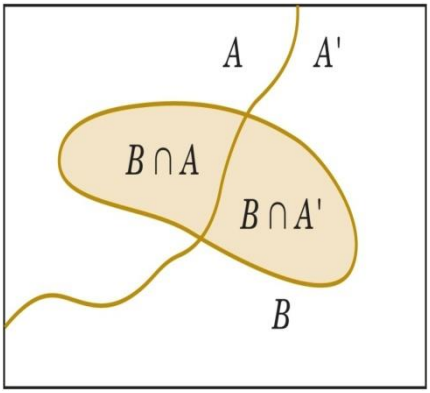
The probability that a part made in the 1st stage of a machining operation meets specifications is 0.90. The probability that it meets specifications in the 2nd stage, given that met specifications in the first stage is 0.95.

What is the probability that both stages meet specifications?

- Let A and B denote the events that the part has meet 1st and 2nd stage specifications, respectively.
- $P(A \cap B) = P(B \mid A) \cdot P(A) = 0.95 \cdot 0.90 = 0.855$

Two Mutually Exclusive Subsets

- A and A' are mutually exclusive.
- $A \cap B$ and $A' \cap B$ are mutually exclusive
- $B = (A \cap B) \cup (A' \cap B)$



Total Probability Rule

- For any two events *A* and *B*

$$P(B) = P(B \cap A) + P(B \cap A')$$
$$= P(B \mid A) \cdot P(A) + P(B \mid A') \cdot P(A')$$

Example 1-14: Semiconductor Contamination

Information about product failure based on chip manufacturing process contamination is given below. Find the probability of failure.

Probability of Failure	Level of Contamination	Probability of Level
0.1	High	0.2
0.005	Not High	0.8

Let *F* denote the event that the product fails.

✓ Let *H* denote the event that the chip is exposed to high contamination during manufacture. Then

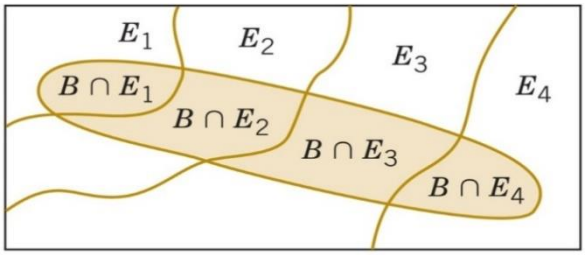
- ✓ $P(F \mid H) = 0.100$ and $P(H) = 0.2$, so $P(F \cap H) = 0.02$
- ✓ $P(F \mid H') = 0.005$ and $P(H') = 0.8$, so $P(F \cap H') = 0.004$
- ✓ $P(F) = P(F \cap H) + P(F \cap H')$ (Using Total Probability rule)

$$= 0.020 + 0.004 = 0.024$$

Total Probability Rule (Multiple Events)

- A collection of sets $E_1, E_2, \ldots E_k$ such that $E_1 \cup E_2 \cup \ldots\ldots \cup E_k = S$ is said to be exhaustive.
- Assume $E_1, E_2, \ldots E_k$ are *k* mutually exclusive and exhaustive. Then

$$P(B) = P(B \cap E_1) + P(B \cap E_2) + \ldots + P(B \cap E_k)$$
$$= P(B \mid E_1) \cdot P(E_1) + P(B \mid E_2) \cdot P(E_2) + \ldots + P(B \mid E_k) \cdot P(E_k)$$

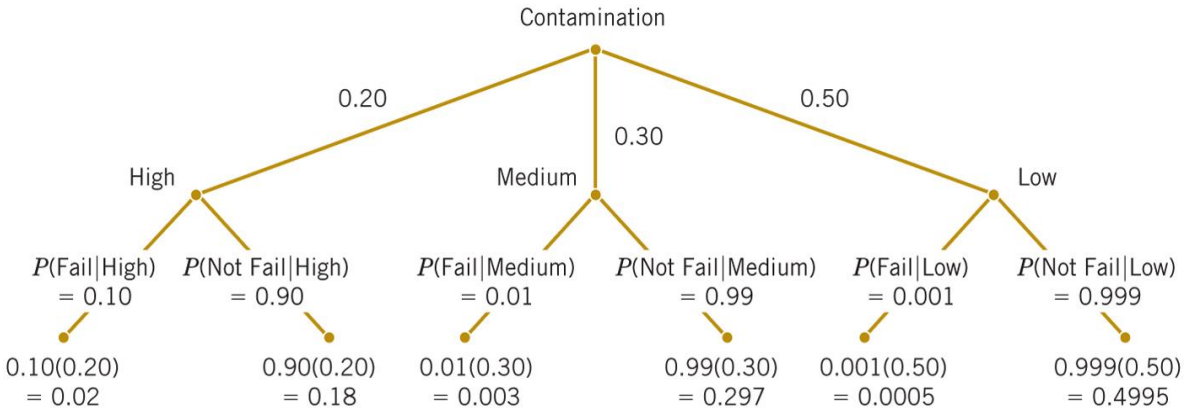


$$B = (B \cap E_1) \cup (B \cap E_2) \cup (B \cap E_3) \cup (B \cap E_4)$$

Example 1-15: Semiconductor Failures-1

Continuing the discussion of contamination during chip manufacture, find the probability of failure.

Probability of Failure	Level of Contamination	Probability of Level
0.100	High	0.2
0.010	Medium	0.3
0.001	Low	0.5



$P(\text{Fail}) = 0.02 + 0.003 + 0.0005 = 0.0235$

- Let *F* denote the event that a chip fails
- Let *H* denote the event that a chip is exposed to high levels of contamination
- Let *M* denote the event that a chip is exposed to medium levels of contamination
- Let *L* denote the event that a chip is exposed to low levels of contamination.
- Using Total Probability Rule,

$P(F) = P(F \cap H) + P(F \cap M) + P(F \cap L)$

$= P(F \mid H)P(H) + P(F \mid M)P(M) + P(F \mid L)P(L)$

$= (0.1)(0.2) + (0.01)(0.3) + (0.001)(0.5)$

$= 0.0235.$

Event Independence

- Two events are independent if any one of the following equivalent statements is true:
    1. $P(A \mid B) = P(A)$
    2. $P(B \mid A) = P(B)$
    3. $P(A \cap B) = P(A) \cdot P(B)$
- This means that occurrence of one event has no impact on the probability of occurrence of the other event.

Example 1-16: Flaws and Functions

Table 1 provides an example of 400 parts classified by surface flaws and as (functionally) defective. Suppose that the situation is different and follows Table 2. Let *F* denote the event that the part has surface flaws. Let *D* denote the event that the part is defective. The data shows whether the events are independent.

TABLE 1 Parts Classified

	Surface Flaws		
Defective	Yes (F)	No (F')	Total
Yes (D)	10	18	28
No (D')	30	342	372
Total	40	360	400
	$P(D \mid F) =$	10/40 =	0.25
	$P(D) =$	28/400 =	0.10
			not same
	Events D & F are **dependent**		

TABLE 2 Parts Classified (data chg'd)

	Surface Flaws		
Defective	Yes (F)	No (F')	Total
Yes (D)	2	18	20
No (D')	38	342	380
Total	40	360	400
	$P(D \mid F) =$	2/40 =	0.05
	$P(D) =$	20/400 =	0.05
			same
	Events D & F are **independent**		

Independence with Multiple Events

The events $E_1, E_2, \ldots, E_k$ are independent if and only if, for any subset of these events:

$P(E_{i1} \cap E_{i2} \cap \ldots, \cap E_{ik}) = P(E_{i1}) \cdot P(E_{i2}) \cdot \ldots \cdot P(E_{ik})$

Example 1-17: Semiconductor Wafers

Assume the probability that a wafer contains a large particle of contamination is 0.01 and that the wafers are independent; that is, the probability that a wafer contains a large particle does not depend on the characteristics of any of the other wafers. If 15 wafers are analyzed, what is the probability that no large particles are found?

Solution:

Let E_i denote the event that the i^{th} wafer contains no large particles, i = 1, 2, …,15

Then, $P(E_i) = 0.99$.

The required probability is $P(E_1 \cap E_2 \cap \ldots \cap E_{15})$.

From the assumption of independence,

$$P(E_1 \cap E_2 \cap \ldots \cap E_{15}) = P(E_1) \cdot P(E_2) \cdot \ldots \cdot P(E_{15})$$
$$= (0.99)^{15}$$
$$= 0.86.$$

Example 1-18:

A bin contains 5 defectives (that immediately fail when put in use), 10 partially defective (that fail after a couple of hours of use), and 25 acceptable transistors. A transistor is chosen at random from the bin and put into use. If it does not immediately fail, what is the probability it is acceptable?

SOLUTION

Since the transistor did not immediately fail, we know that it is not one of the 5 defectives and so the desired probability is:

P{acceptable | not defective} = P{acceptable, not defective} / P{not defective}

= P{acceptable} / P {not defective}

where the last equality follows since the transistor will be both acceptable and not defective if it is acceptable. Hence, assuming that each of the 40 transistors is equally likely to be chosen, we obtain that

P {acceptable | not defective} = (25/40) / (35/40) = 5/7

It should be noted that we could also have derived this probability by working directly with the reduced sample space. That is, since we know that the chosen transistor is not defective, the problem reduces to computing the probability that a transistor, chosen at random from a bin containing 25 acceptable and 10 partially defective transistors, is acceptable. This is clearly equal to 25 / 35.

Example 1-19:

A laboratory blood test is 99 percent effective in detecting a certain disease when it is, in fact, present. However, the test also yields a "false positive" result for 1 percent of the healthy persons tested. (That is, if a healthy person is tested, then, with probability .01, the test result will imply he or she has the disease.) If .5 percent of the population actually has the disease, what is the probability a person has the disease given that his test result is positive?

SOLUTION

Let D be the event that the tested person has the disease and E the event that his test result is positive. The desired probability P(D|E) is obtained by

$P(D|E) = P(DE) / P(E)$

$= P(E|D) P(D) / \{P(E|D) P(D)+P(E|D^c) P(D^c)\}$

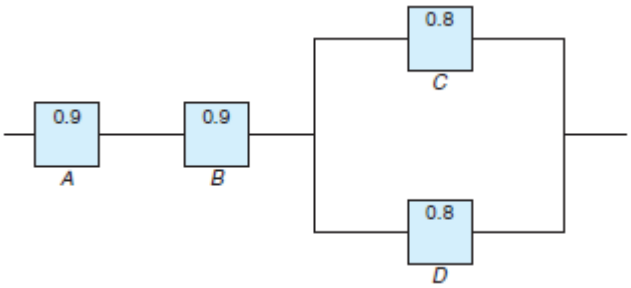
$= (.99) (.005) / \{(.99) (.005) +(.01) (.995)\}$

$= .3322$

Thus, only 33 percent of those persons whose test results are positive actually have the disease.

Example 1-20:

An electrical system consists of four components as illustrated in the following figure. The system works if components A and B work and either of the components C or D works. The reliability (probability of working) of each component is also shown in the figure. Find the probability that (a) the entire system works and (b) the component C does not work, given that the entire system works. Assume that the four components work independently.



Solution: In this configuration of the system, A, B, and the subsystem C and D constitute

a serial circuit system, whereas the subsystem C and D itself is a parallel circuit system.

(a) Clearly the probability that the entire system works can be calculated as follows:

$P[A \cap B \cap (C \cup D)] = P(A)\,P(B)\,P(C \cup D) = P(A)P(B)[1 - P(C^c \cap D^c)]$

$= P(A)P(B)[1 - P(C^c)P(D^c)]$

$= (0.9)\,(0.9)\,[1 - (1 - 0.8)\,(1 - 0.8)] = 0.7776.$

The equalities above hold because of the independence among the four components.

(b) To calculate the conditional probability in this case, notice that

P = P (the system works but C does not work) / P(the system works)

$= P(A \cap B \cap C^c \cap D) / P(\text{the system works})$

$= (0.9)(0.9)(1 - 0.8)(0.8)/ 0.7776 = 0.1667.$

Example 1-21:

A sample of four electronic components is taken from the output of a production line. The probabilities of the various outcomes are calculated to be: Pr [0 defectives] = 0.6561, Pr [1 defective] = 0.2916, Pr [2 defectives] = 0.0486, Pr [3 defectives] = 0.0036, Pr [4 defectives] = 0.0001. What is the probability of at least one defective?

Answer: It would be perfectly correct to calculate as follows:

Pr [at least one defective] = Pr [1 defective] + Pr [2 defectives] +Pr [3 defectives] + Pr [4 defectives] = 0.2916 + 0.0486 + 0.0036 + 0.0001 = 0.3439.

but it is easier to calculate instead:

Pr [at least one defective] = 1 – Pr [0 defectives] = 1 – 0.6561

= 0.3439 or 0.344.

Example 1-22:

The class registrations of 120 students are analyzed. It is found that: 30 of the students do not take any of Applied Mechanics, Chemistry, or Computers. 15 of them take only Applied Mechanics. 25 of them take Chemistry and Computers but not Applied Mechanics. 20 of them take Applied Mechanics and Computers but not Chemistry. 10 of them take all three of Applied Mechanics, Chemistry, and Computers. A total of 45 of them take Chemistry. 5 of them take only Chemistry.

a) How many of the students take Applied Mechanics and Chemistry but not Computers?

b) How many of the students take only Computers?

c) What is the total number of students taking Computers?

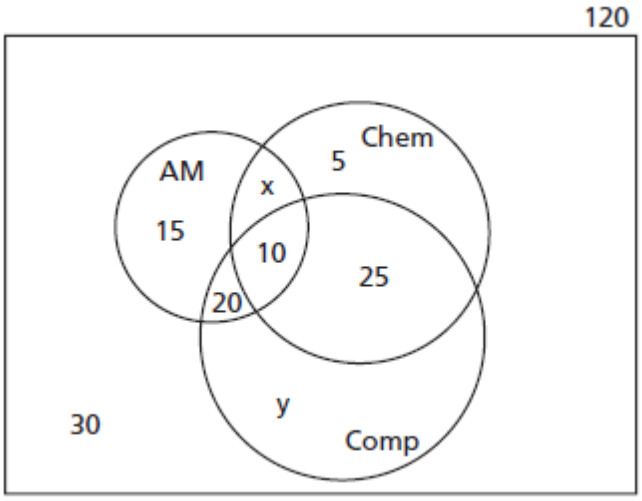
d) If a student is chosen at random from those who take neither Chemistry nor Computers, what is the probability that he or she does not take Applied Mechanics either?

e) If one of the students who take at least two of the three courses is chosen at random, what is the probability that he or she takes all three courses?

Answer: Let's abbreviate the courses as AM, Chem, and Comp. The number of items in the sample space, which is the total number of items under consideration, is often marked just above the upper right-hand corner of the rectangle. In this example that number is 120. Then the Venn diagram incorporating the given information for this problem is shown below. Two of the simple areas on the diagram correspond to unknown numbers. One of these is $(AM \cap Chem \cap \overline{Comp})$, which is taken by x students. The other is $(\overline{AM} \cap \overline{Chem} \cap Comp)$, so only Computers but not the other courses, and that is taken by y students. In terms of quantities corresponding to simple areas on the Venn diagram, the given information that a total of 45 of the students take Chemistry requires that

$x + 10 + 25 + 5 = 45$

Then $x = 5$.



Let n(...) be the number of students who take a specified course or combination of courses. Then from the total number of students and the number who do not take any of the three courses we have n(AM ∪ Chem ∪ Comp) = 120 – 30 = 90. But from the Venn diagram and the knowledge

of the total taking Chemistry we have n(AM∪ Chem∪ Comp) = n(Chem) + n(AM∩ Chem ∩ $\overline{\text{Comp}}$) +n(AM∩ $\overline{\text{Chem}}$ ∩ Comp) +n($\overline{AM}$ ∩ Chem ∩ Comp) = 45 + 15 + 20 + y = 80 + y Then y = 90 – 80 = 10.

Now we can answer the specific questions.

a) The number of students who take Applied Mechanics and Chemistry but not Computers is 5.

b) The number of students who take only Computers is 10.

c) The total number of students taking Computers is 10 + 20 + 10 + 25 = 65.

d) The number of students taking neither Chemistry nor Computers is 15 + 30 = 45. Of these, the number who do not take Applied Mechanics is 30. Then if a student is chosen randomly from those who take neither Chemistry nor Computers, the probability that he or she does not take Applied Mechanics either is 30 / 45 = 2 / 3.

e) The number of students who take at least two of the three courses is n(AM ∩ Chem ∩ $\overline{\text{Comp}}$) + n(AM ∩ $\overline{\text{Chem}}$ ∩ Comp) + n($\overline{\text{AM}}$ ∩ Chem ∩ Comp) + n(AM ∩ Chem ∩ Comp) = 5 + 20 + 25 + 10 = 60.

Of these, the number who take all three courses is 10. If a student taking at least two courses is chosen randomly, the probability that he or she takes all three courses is 10/60 = 1/6.

Example 1-23:

A box contains 8 red, 3 white, and 9 blue balls. If 3 balls are drawn at random, determine the probability that (i) all 3 are red (ii) all 3 are white (iii) 2 are red and 1 is white (iv) at least 1 is white.

Solution:

Number of red balls = 8

Number of white balls = 3

Number of blue balls = 9

Number of Total balls = 20

(i) The Probability of 3 balls will be red $= \frac{\binom{8}{3}}{\binom{20}{3}} = \frac{\frac{8.7.6}{3!}}{\frac{18.19.20}{3!}} = \frac{6.7.8}{18.19.20} = \frac{14}{285}$

(ii) The Probability of 3 balls will be white $= \frac{\binom{3}{3}}{\binom{20}{3}} = \frac{1}{\frac{18.19.20}{3!}} = \frac{1}{1140}$

(iii) The Probability of 2 are red 1 is white will be $= \frac{\binom{8}{2}\binom{3}{1}}{\binom{20}{3}} = \frac{\frac{7.8}{2}.3}{\frac{18.19.20}{3!}}$

$= \frac{3.4.7}{3.19.20} = \frac{7}{95}$

(iv) The probability of at least 1 is white will be the following way

(1) 1 is white and 2 are others

(2) 2 are white 1 is other

(3) 3 are white

For case (1), $\frac{\binom{3}{1}\binom{17}{2}}{\binom{20}{3}} = \frac{3\frac{16.17}{2}}{\frac{18.19.20}{6}} = \frac{3.8.17}{3.19.20} = \frac{34}{95}$

For case (2), $\frac{\binom{3}{2}\binom{17}{1}}{\binom{20}{3}} = \frac{3.17}{3.19.20} = \frac{17}{380}$

For case (3), $\frac{\binom{3}{3}}{\binom{20}{3}} = \frac{1}{1140}$

So the probability of at least 1 is white will be $\frac{34}{95} + \frac{17}{380} + \frac{1}{1140} = \frac{23}{57}$

Example 1-24:

The probability that a married man watches a certain TV show is 0.4 and that a married woman watches the show is 0.5. The probability that a man watches the show, given that his wife does, is 0.7. Find

(i) The probability that a married couple watches the show.

(ii) The probability that a wife watches the show given that her husband does.

(iii) The probability that at least one person of a married couple will watch the show.

Solution: Let us define two events H and W as follows:

H: husband watches the show

W: wife watches the show

It is given that $P(H) = 0.4, P(W) = 0.5 \; and \;\; P(H/W) = 0.7$

(i) P (married couple watches the show) = $P(W \cap H) = P(W)P(H/W) = 0.5 \times 0.7 = 0.35$

(ii) P (wife watches given her husband does) = $P(W/H) = \frac{P(W \cap H)}{P(H)} = \frac{0.35}{0.4} = 0.875$.

(iii) P (at least one them watches) = $P(W \cup H) = P(W) + P(H) - P(W \cap H) = 0.4 + 0.5 - 0.35 = 0.55$.

Example 1-25:

A married couple (husband and wife) appear for an interview for two vacancies against the same post. The probability of husband's selection is $\frac{1}{6}$ and the probability of wife's selection is $\frac{2}{5}$. What is the probability that

i. both of them will be selected
ii. only one of them will be selected
iii. none of them will be selected
iv. at least one of them will be selected?

Solution:

Let E be the event "husband is selected" and F denote the event "wife is selected'. We are given that $P(E) = \frac{1}{6}$ and $P(F) = \frac{2}{5}$.

Since there are two vacancies, selection of one does not affect the other. Hence E and F are independent events. Then

i. $P(\text{both of them are selected}) = P(E \cap F)$
$= P(E)P(F)$ since E and F are independent.
$= \frac{1}{6} \cdot \frac{2}{5} = \frac{1}{15}$

ii. Since $E \cap \bar{F}$ and $\bar{E} \cap F$ are exclusive, we have
$P(\text{only one of them is selected})$
$= P(E \cap \bar{F}) \cup P(\bar{E} \cap F)$
$= P(E \cap F) + (E \cap F)$; (Exclusive events)
$= P(E)P(\bar{F}) + P(\bar{E})P(F)$, Since E and F are independent.
$= P(E)\big(1 - P(F)\big) + \big(1 - P(E)\big)P(F)$

$$= \frac{1}{6}\left(1 - \frac{2}{5}\right) + \left(1 - \frac{1}{6}\right)\frac{2}{5} = \frac{1}{10} + \frac{1}{3} = \frac{13}{30}$$

iii. We have

P (none of them is selected) =P (not E not F)

$= P(\bar{E} \cap \bar{F}) = P(\bar{E})P(\bar{F})$, Since E and F are independent.

$$= \big(1 - P(E)\big)\big(1 - P(F)\big)$$

$$= \left(1 - \frac{1}{6}\right)\left(1 - \frac{2}{5}\right) = \frac{5}{6} \cdot \frac{3}{2} = \frac{1}{2}$$

iv. We have

P (at least one of them gets selected)

$$= P(E \text{ or } F) = P(E \cup F)$$
$$= P(E) + P(F) - P(E \cap F)$$
$$= \frac{1}{6} + \frac{2}{5} - \frac{1}{15}$$
$$= \frac{1}{2}.$$

Bayes' Theorem

- Thomas Bayes (1702-1761) was an English mathematician and Presbyterian minister.
- His idea was that we observe conditional probabilities through prior information.
- Bayes' theorem states that,

$$P(A \mid B) = \frac{P(B \mid A) \bullet P(A)}{P(B)} \quad \text{for } P(B) > 0$$

Example 1-26:

The conditional probability that a high level of contamination was present when a failure occurred is to be determined. The information from Example 4-14 is summarized here.

Probability of Failure	Level of Contamination	Probability of Level
0.1	High	0.2
0.005	Not High	0.8

Solution:

Let *F* denote the event that the product fails, and let *H* denote the event that the chip is exposed to high levels of contamination. The requested probability is *P*(*F*).

$$P(H \mid F) = \frac{P(F \mid H) \cdot P(H)}{P(F)} = \frac{0.10 \cdot 0.20}{0.024} = 0.83$$

$$P(F) = P(F \mid H) \cdot P(H) + P(F \mid H') \cdot P(H')$$
$$= 0.1 \cdot 0.2 + 0.005 \cdot 0.8 = 0.024$$

Example 1-27:

The probability that a student speaks French fluently is 0.20, the probability that a student speaks Greek is 0.10 and the probability that a student speaks neither language is 0.75. Use Venn diagram to calculate the probability that a student speaks both.

Solution:

Let F be the set of students that speaks French fluently and G be the set of students that speaks Greek fluently.

Given $P(F) = 0.20 \; and \; n(F) = 20$; $P(G) = 0.10 \; and \; n(G) = 10$

$P(F \cup G)^c = 0.75 \; and \; n(F \cup G)^c = 75$

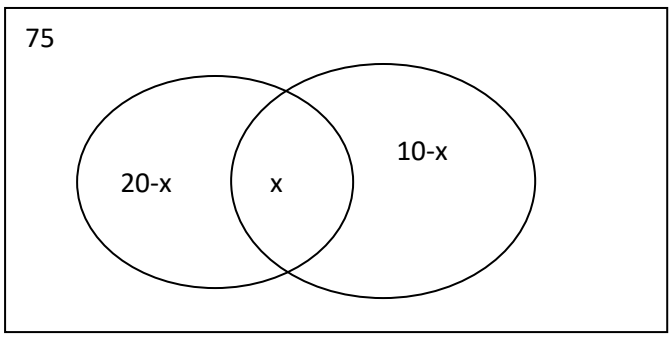
Let $n(F \cap G) = x$

Then from the Venn diagram

$n(F \cup G) = n(F) + n(G) - n(F \cap G) = 20 + 10 - x = 30 - x$

$n(F \cup G)^c = 100 - n(F \cup G) = 100 - 30 + x \Rightarrow 75 = 70 + x \Rightarrow x = 5$

$P(F \cap G) = 0.05.$



Example 1-28

A market survey conducted in four cities pertained to preference for brand A soap. The responses are shown below.

	Delhi	Kolkata	Chennai	Mumbai
Yes	45	55	60	50
No	35	45	35	45
No opinion	5	5	5	5

(i) What is the probability that a consumer selected at random preferred brand A?

(ii) What is the probability that a consumer preferred brand A and was from Chennai?

(iii) What is the probability that a consumer preferred brand A given that he/she was from Chennai?

(v) Given that a consumer preferred brand A what is the probability that he/she was from Mumbai?

Solution:

	Delhi	Kolkata	Chennai	Mumbai	Total
Yes	45	55	60	50	210
No	35	45	35	45	160
No opinion	5	5	5	5	20
Total	85	105	100	100	390

(i) Let E_1 be the event that a consumer preferred brand A randomly

(i) $P(E_1) = \frac{210}{390} = \frac{7}{13} = 0.5385$

(ii) Let E_2 be the event that a consumer preferred brand A and was from Chennai.

$P(E_2) = \frac{60}{390} = \frac{2}{13} = 0.154$

(iii) Let E_3 be the event that a consumer preferred brand A given that he/she was from Chennai.

$P(E_3) = \frac{2/13}{7/13} = \frac{2}{7} = 0.286$

(iv) Let E_4 be the event that a consumer preferred brand A given that he/she was from Mumbai

$P(E_4) = \frac{5/39}{7/13} = \frac{5}{21} = 0.238$

Example 1-29

Two factories manufacture the same machine part. Each part is classified as having 0, 1, 2 or 3 manufacturing defects. The joint probability distribution for this is given below:

	Number of defects			
	0	1	2	3
Manufacturer A	0.1250	0.0625	0.1875	0.1250
Manufacturer B	0.0625	0.0625	0.1250	0.2500

(i) A part is observed to have no defects. What is the probability that it was produced by manufacturer B?
(ii) A part is known to have been produced by manufacturer A. What is the probability that the part has no defects?
(iii) A part is known to have two or more defects. What is the probability that it was manufactured by A?
(vi) A part is known to have one or more defects. What is the probability that it was manufactured by B?

Solution:

	Number of defects				
	0	1	2	3	
Manufacturer A	0.1250	0.0625	0.1875	0.1250	0.5
Manufacturer B	0.0625	0.0625	0.1250	0.2500	0.5
	0.1875	0.125	0.3125	0.375	1.0

Let probability of (i) is E_1, then (i) $P(E_1) = \frac{0.0625}{0.1875} = 0.33$

Let probability of (ii) is E_2, then (ii) $P(E_2) = \frac{0.1250}{0.5} = 0.25$

Let probability of (iii) is E_3, then (iii) $P(E_3) = \frac{0.1875+0.1250}{0.3125+0.375} = \frac{0.3125}{0.6875} = 0.45$

Let probability of (iv) is E_4, then (iv) $P(E_4) = \frac{0.0625+0.1250+0.2500}{0.8125} = \frac{0.4375}{0.8125} = 0.54$

Discussion questions

1. Define and illustrate with examples:

(i) Random experiment; (ii) sample space; (iii) event; (iv) element

2. What do you mean by probability of an event? Define equally likely events, mutually exclusive events and exhaustive events. Give examples in each case.

3. What do you mean by independence of two events? Can two events be mutually exclusive and independent simultaneously? Justify your answer.

4. Explain why there is a mistake in each of the following statements:

(i) The probabilities that a certain truck driver in Dhaka city would have no, one and two or more accidents during the year are 0.90, 0.02, and 0.09 respectively.

(ii) A stockbroker claims that the probability that a stock's price will go up is 0.38, remain unchanged 0.52 and go down 0.12.

(iii) The probabilities that there would be no, one two or three rainy days next week are 0.54, 0.28, 0.14 and 0.04 respectively.

5. Define and derive addition law of probability.

6. What is conditional probability? Show that every unconditional probability is indeed a conditional probability with respect to the sample space.

7. A family has three children. What is the probability that the youngest is a boy if the eldest is known to be a giral?

Practice problems

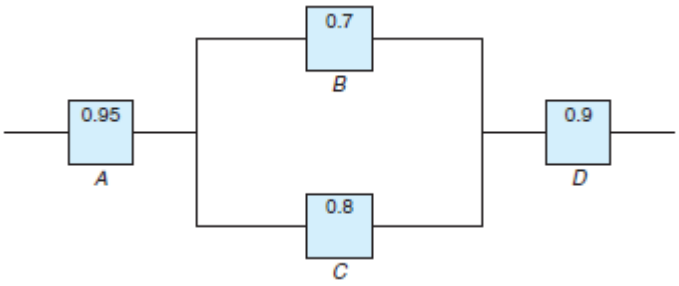


Figure 2.10: Diagram for Exercise 2.92.

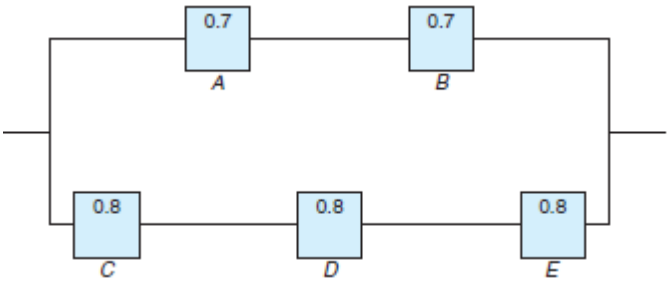


Figure 2.11: Diagram for Exercise 2.93.