Lecture # 02

Flexural Analysis and Design of Reinforced Concrete Beam

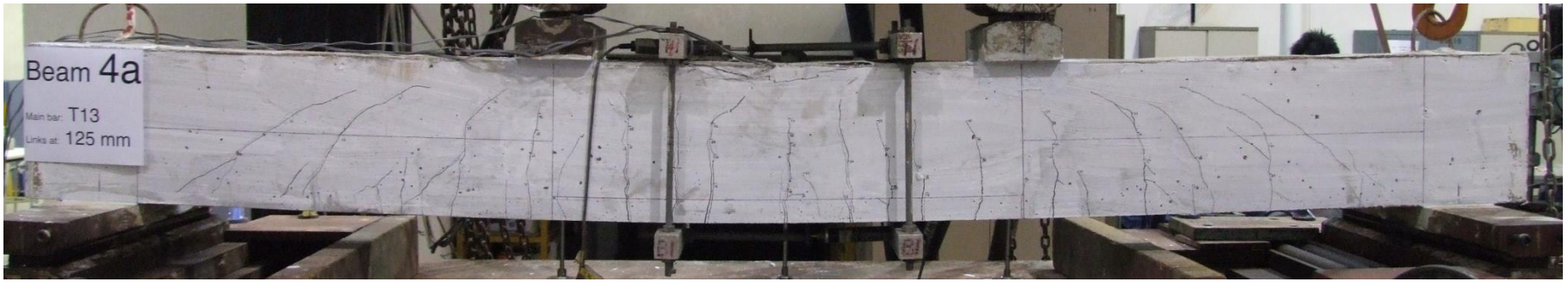
CE315: DESIGN OF CONCRETE STRUCTURES I

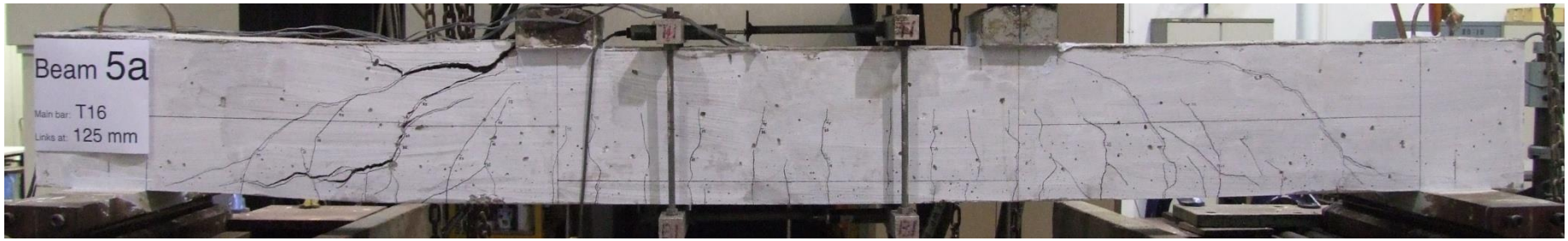
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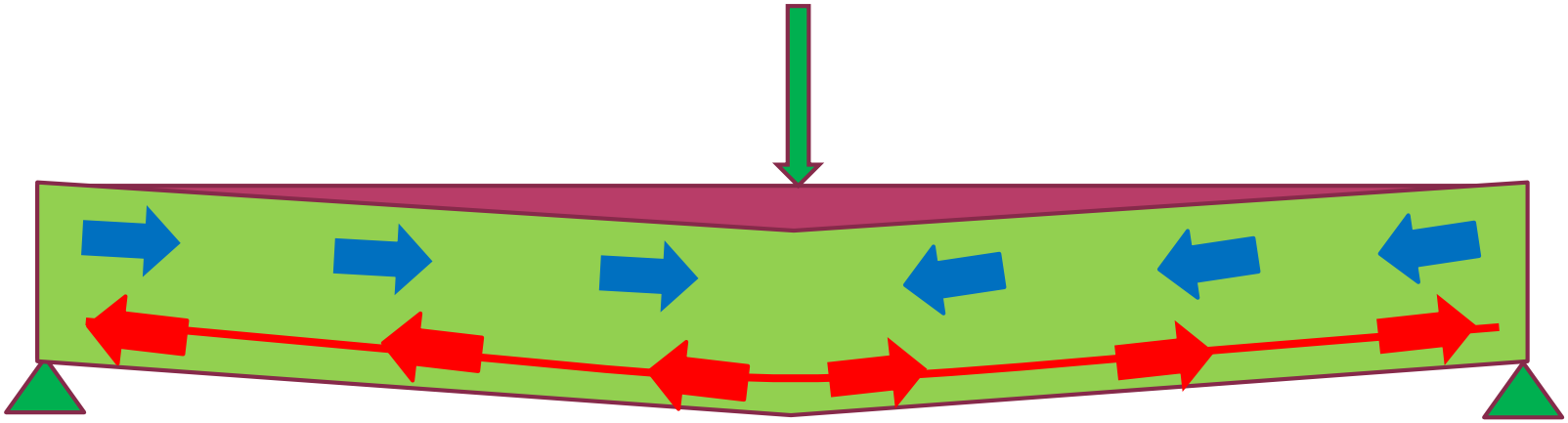
RCC BEAM BEHAVIOR





REINFORCED CONCRETE (RCC) BEAM BEHAVIOR

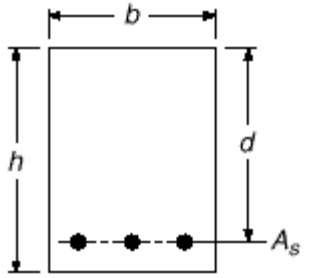
- When a simply supported plain concrete beam subjected a vertically downward load it deforms downward.
- In such a beam, tension is developed at the bottom and compression is developed at the top.



REINFORCED CONCRETE (RCC) BEAM BEHAVIOR

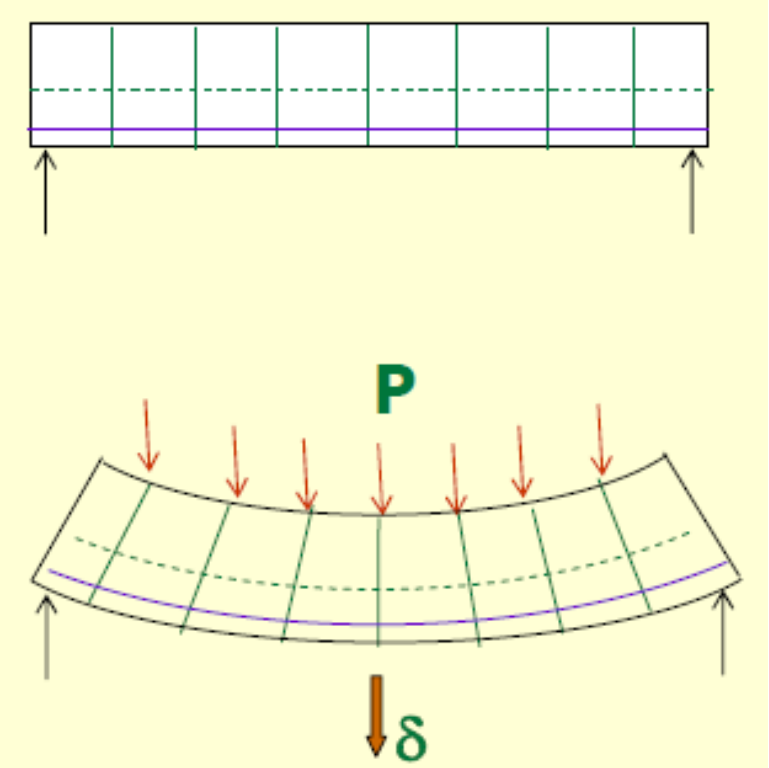
- Plain concrete beams are inefficient as flexural member because they are weak in tension during bending.
- For this reason, steel reinforcing bars are placed on the tension side as close to the extreme tension fiber.
- In such a RCC beam, the tension caused by the bending moments is chiefly resisted by the steel reinforcement, while the concrete alone is capable of resisting the corresponding compression.

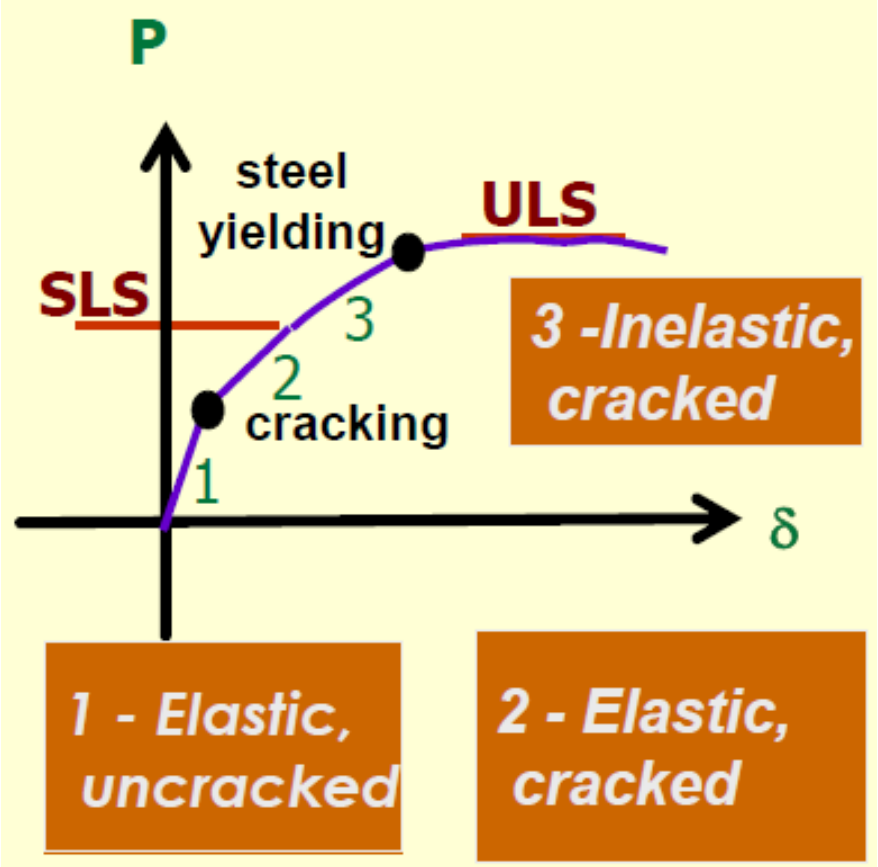




RCC BEAM BEHAVIOR

- When the load in a beam is gradually increased from zero to the magnitude that will cause the beam to fail, several stages of behavior can be clearly distinguished.

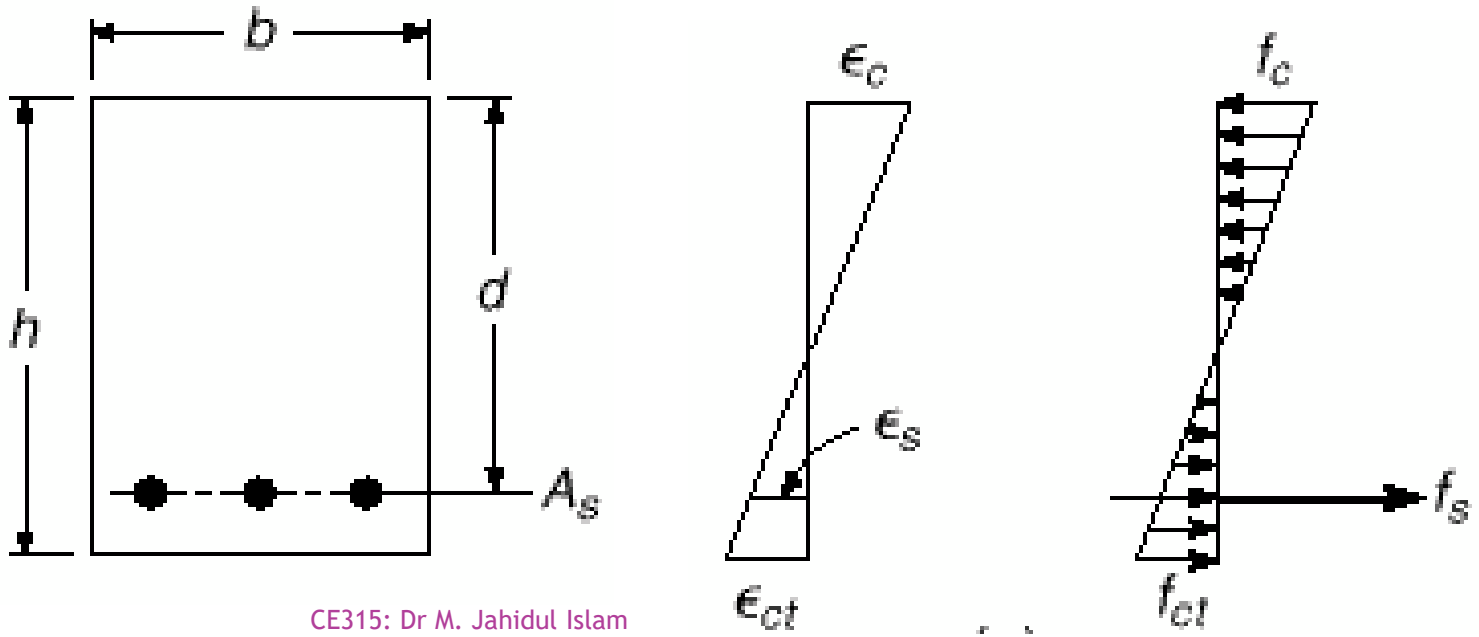




RCC BEAM BEHAVIOR



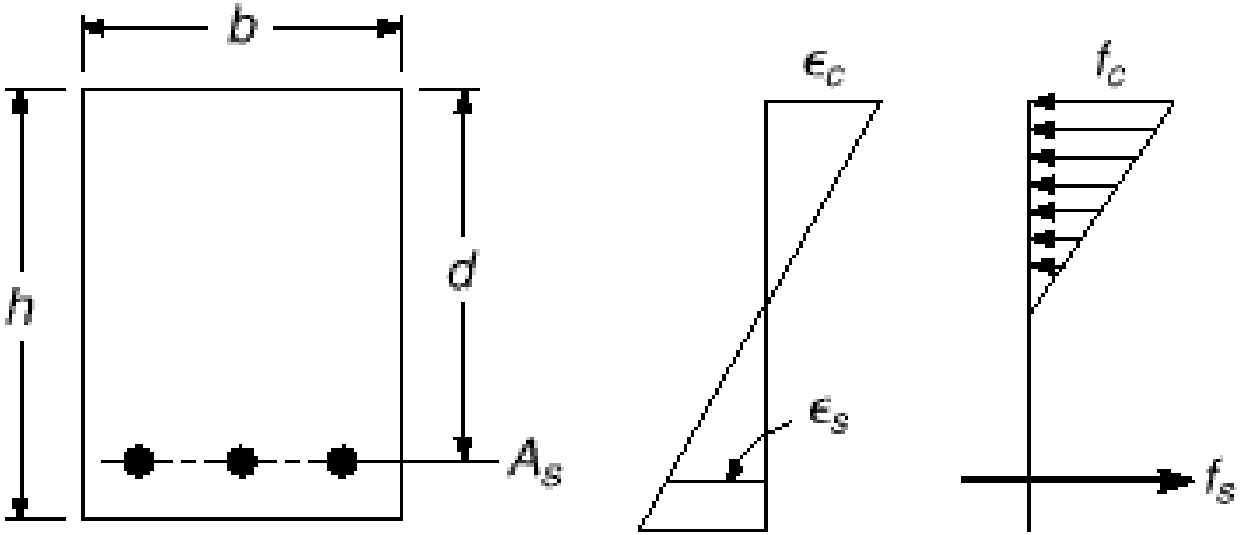
- **Stage1 (Stresses Elastic & Section Uncracked):**
  - At low loads, as long as the tensile stress in the concrete is smaller than the modulus of rupture, the entire concrete is effective in resisting stress.
  - At this stage, all stresses in the concrete are of small magnitude and are proportional to the strains.



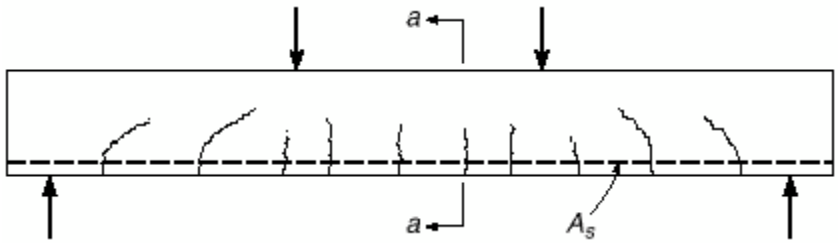
RCC BEAM BEHAVIOR



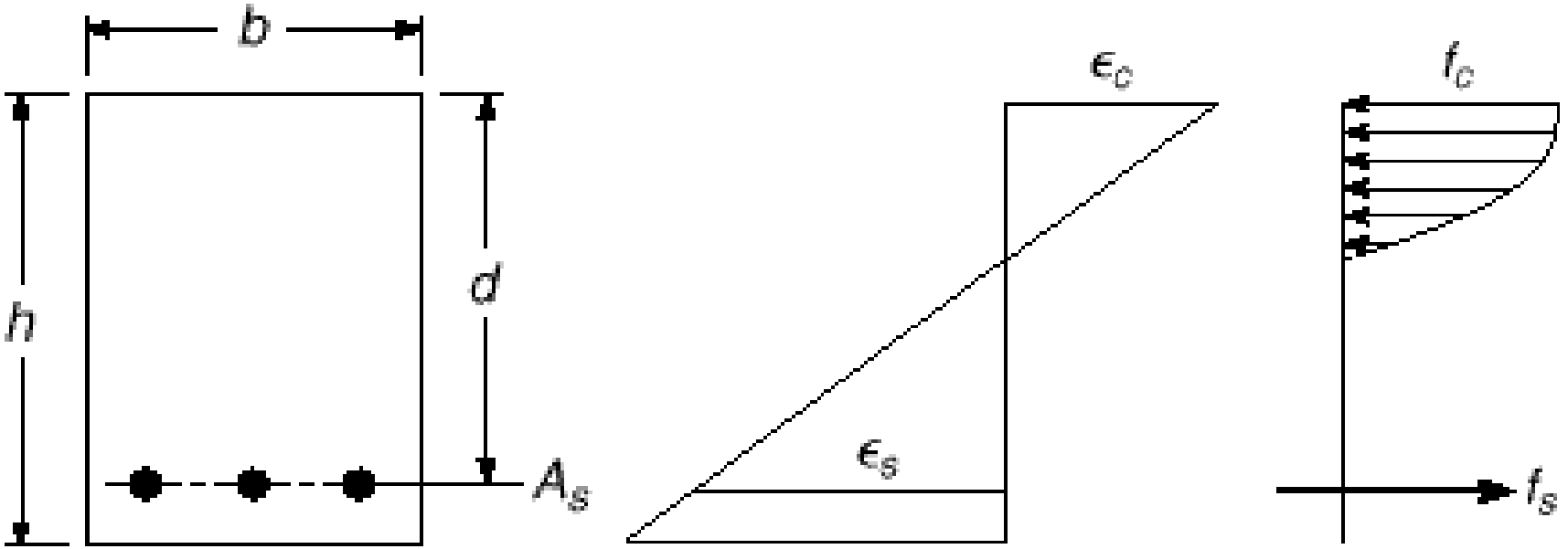
- **Stage2 (Elastic & Cracked):**
  - When the load is further increased the tensile strength of concrete is reached and tension crack is developed.
  - In a cracked section, concrete does not transmit any tensile stresses and steel resists the entire tension.
  - At moderate load, if the concrete stresses do not exceed approximately $f_c'/2$, stresses and strains continue to be proportional.



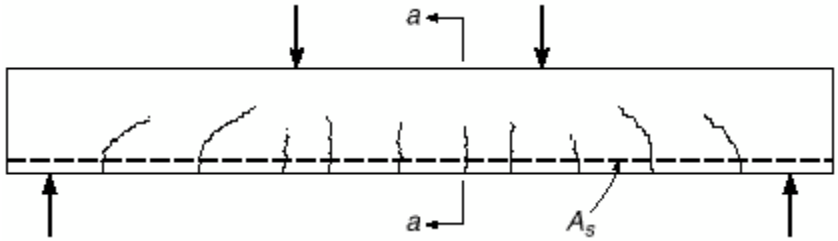
RCC BEAM BEHAVIOR



- **Stage3 (Inelastic & Cracked):**
  - When the load is further increased, stress and strain rise correspondingly and no longer proportional.
  - The resulting nonlinear relation between stresses and strains is that given by the concrete stress-strain curve.

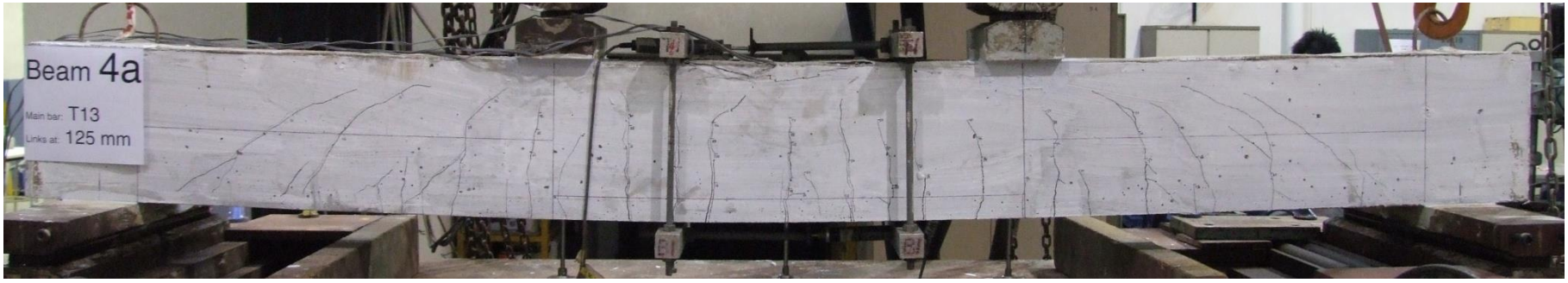


RCC BEAM BEHAVIOR

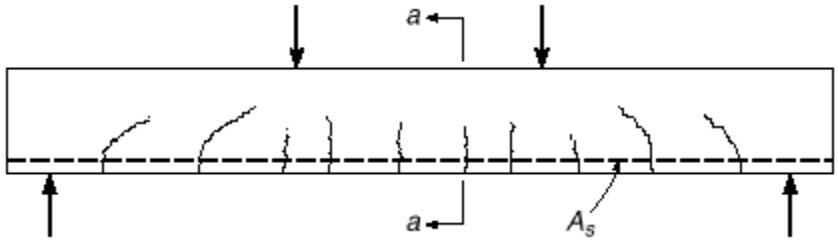


- **Stage4 (Failure):**
  - Failure can be caused in one of two ways.
  - With moderate amount of reinforcement, when steel reach its yield point, reinforcement stretches a large amount, and the tension cracks widen visibly and propagate upward, with simultaneous significant deflection of the beam. This is a gradual failure.

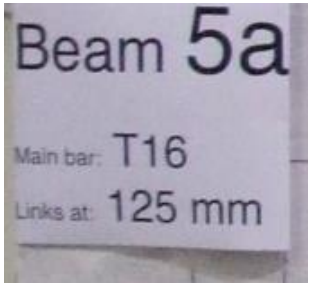


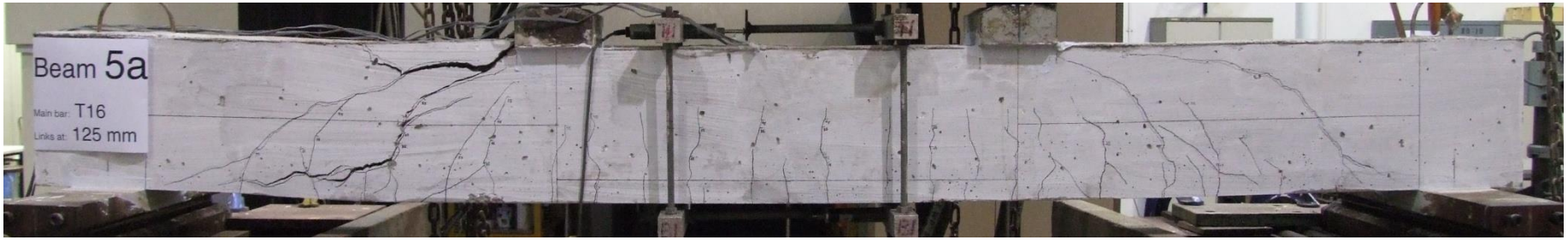


RCC BEAM BEHAVIOR



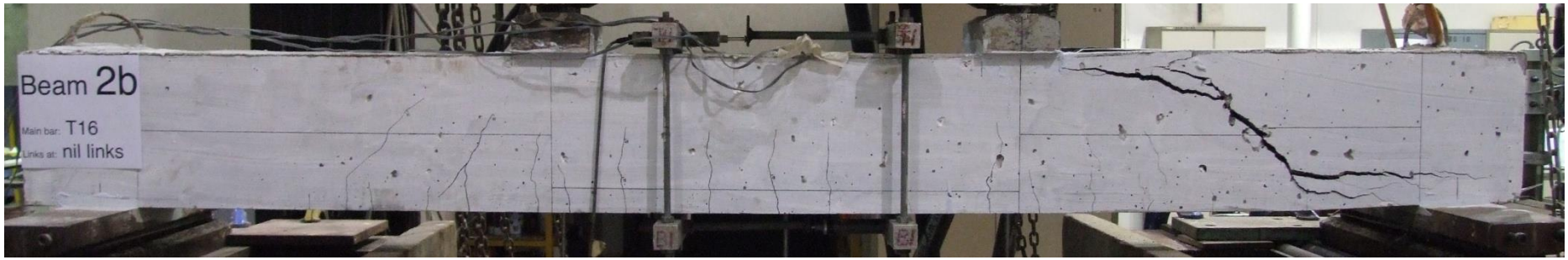
- **Stage4 (Failure):**
  - Failure can be caused in one of two ways.
  - If large amount of reinforcement are used, the compressive strength of concrete may be failed before the steel starts yielding. Compressive failure of concrete is sudden and explosive in nature.





RCC BEAM BEHAVIOR

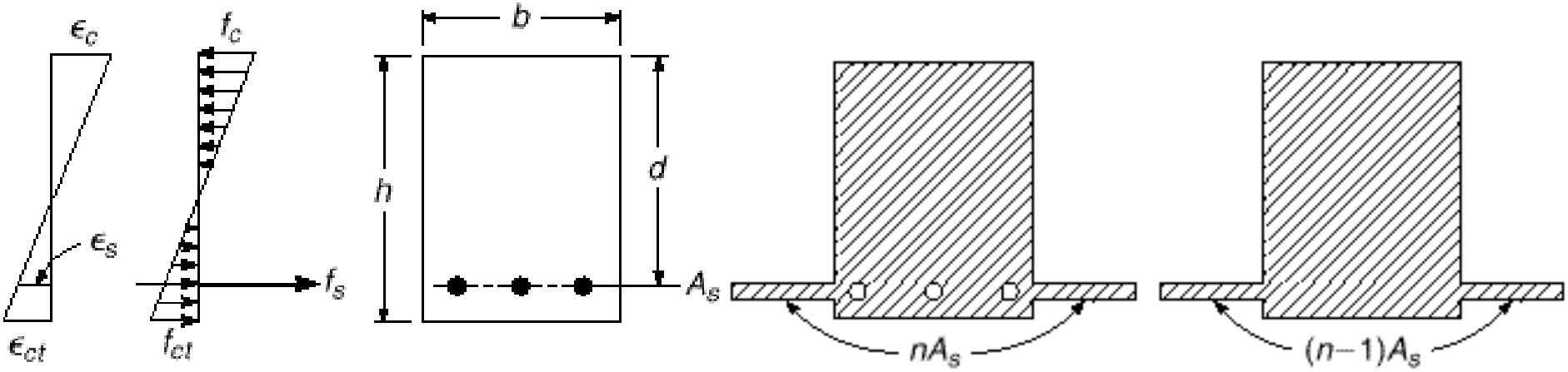




STRESSES ELASTIC & SECTION UNCRACKED

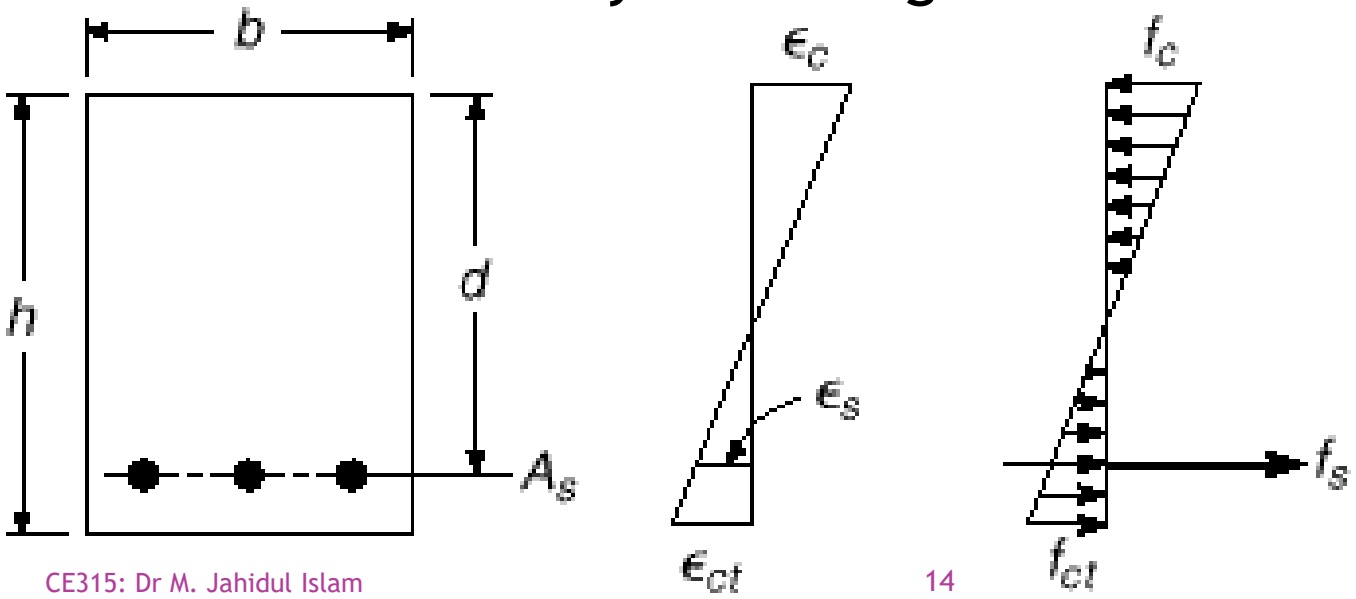
STRESSES ELASTIC & SECTION UNCRACKED

- As long as the tensile stress in the concrete is smaller than the modulus of rupture, so that no tension crack developed, the stress and strain distribution remain linear as shown in the figure.
- This beam can be solved by adopting "transformed section" method, where the actual area of the reinforcement is replaced with an equivalent concrete area to nA_s located at the level of the steel.

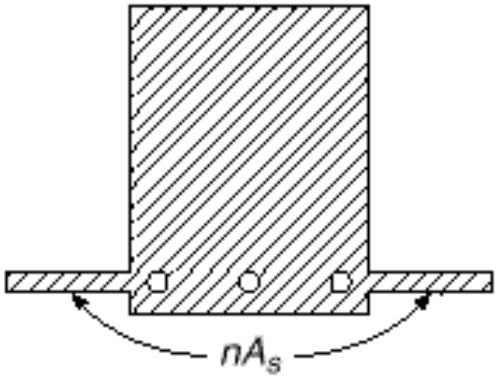


EXAMPLE 2.1

- A rectangular beam has the dimensions b = 10 in., h = 25 in., and d = 23 in. and is reinforced with three No. 8 (No. 25) bars so that A_s = 2.37 in^2.
- The concrete cylinder strength (f_c') is 4,000 psi, and tensile strength in bending (modulus of rupture) is 475 psi. The yield point of the steel (f_y) is 60,000 psi. And n = 8.
- Determine the stresses caused by a bending moment M = 45 ft-kips.

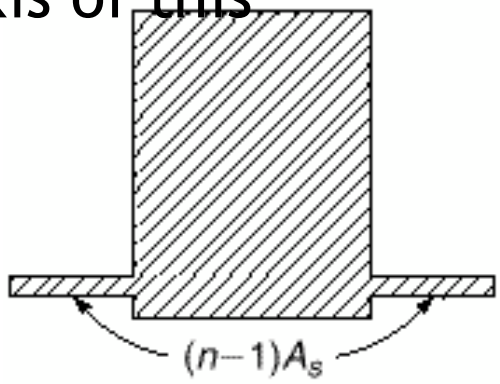


EXAMPLE 2.1 (SOLUTION)



- $n = E_s/E_c$ = 29,000,000 / 3,600,000 = 8
- $(n\text{-}1)A_s$ = (8-1) x 2.37 = 16.59 in^2
- Calculation of the location of the neutral axis of this section:
- From the top,

$$\bar{y} = \frac{10 \times 25 \times 12.5 + 16.59 \times 23}{10 \times 25 + 16.59} = 13.2 \text{ in}$$

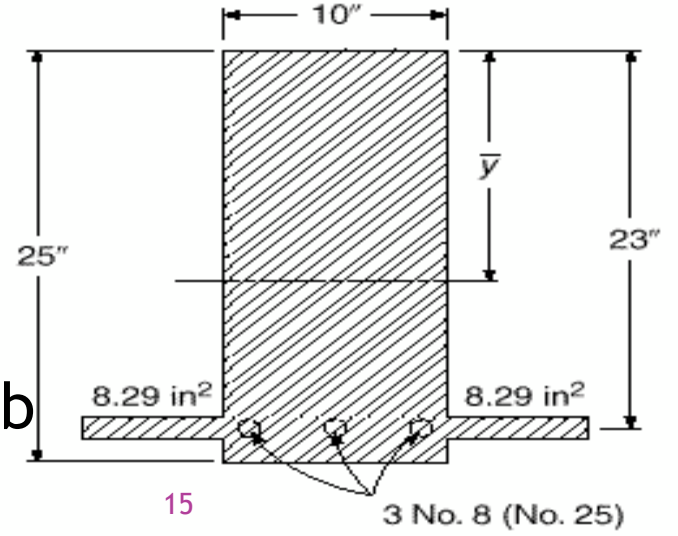


- Moment of inertia:

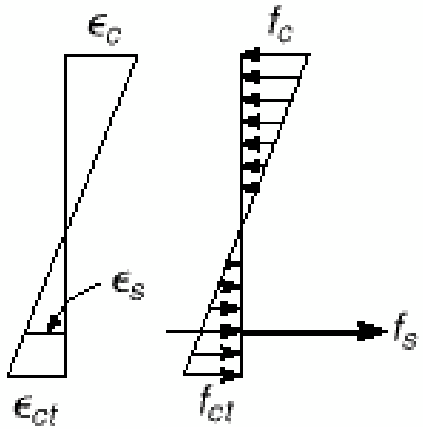
$$I = \frac{10 \times 25^3}{12} + 10 \times 25 \times (13.2 - 12.5)^2$$

$$+16.59 \times (23 - 13.2)^2 = 14,737 \text{ in}^4$$

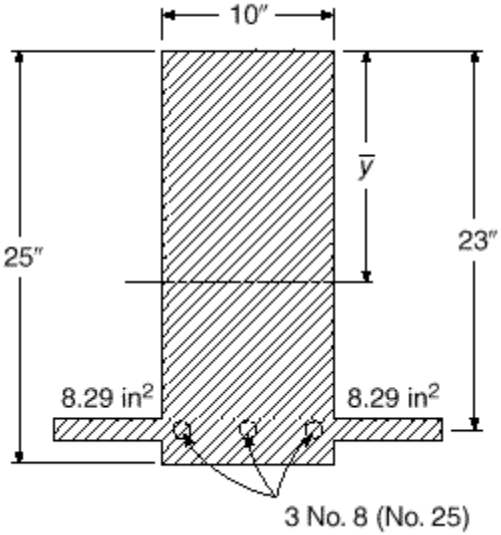
- Moment, M = 45 ft-kips = 540,000 in-lb



EXAMPLE 2.1 (SOLUTION)



- According to the flexural beam theory, the maximum bending stress occurs at the outer fibers and is equal to $f_{\max} = \dfrac{Mc}{I}$

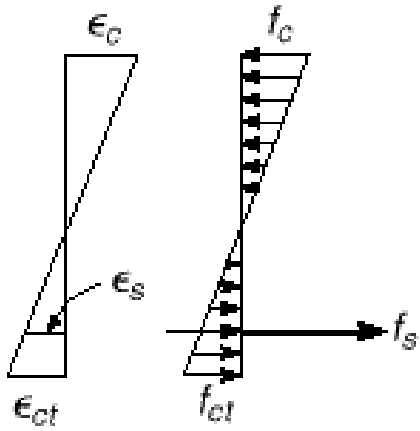


- The concrete compression stress at top fiber is, $f_c = \dfrac{540,000 \times 13.2}{14737} = 484 \text{ psi}$
- The concrete tension stress at bottom fiber is,

$$f_c = \frac{540,000 \times (25 - 13.2)}{14737} = 432 \text{ psi}$$

- Which is less than the tensile strength of the concrete, 475 psi, so no tensile cracks will form.

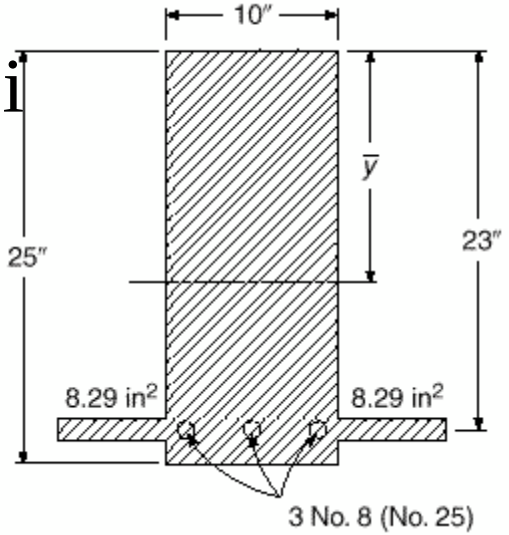
EXAMPLE 2.1 (SOLUTION)



- The stress in the steel reinforcement is,

$$f_s = n\frac{My}{I}$$

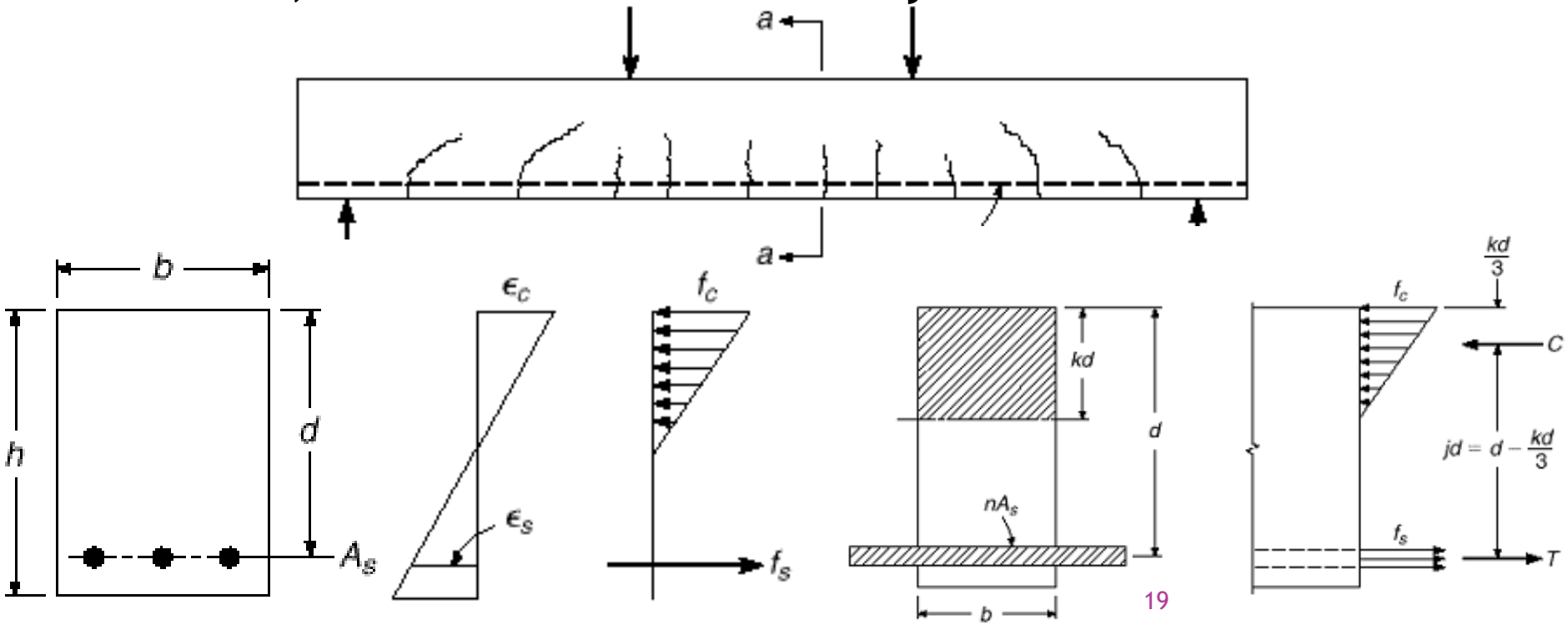
$$f_s = 8 \times \frac{540,000 \times (23 - 13.2)}{14737} = 2873 \text{ psi}$$



STRESSES ELASTIC AND SECTION CRACKED

STRESSES ELASTIC AND SECTION CRACKED

- When the tensile stresses exceeds the modulus of rupture, cracks form. If the concrete compressive stress is less than approximately $f_c'/2$ and the steel has not yet reached the yield point, both material behave elastically.
- All the concrete that is stressed in tension is assumed cracked, and therefore effectively absent.



STRESSES ELASTIC & SECTION CRACKED

- To compute stresses, the transformed section will be used.

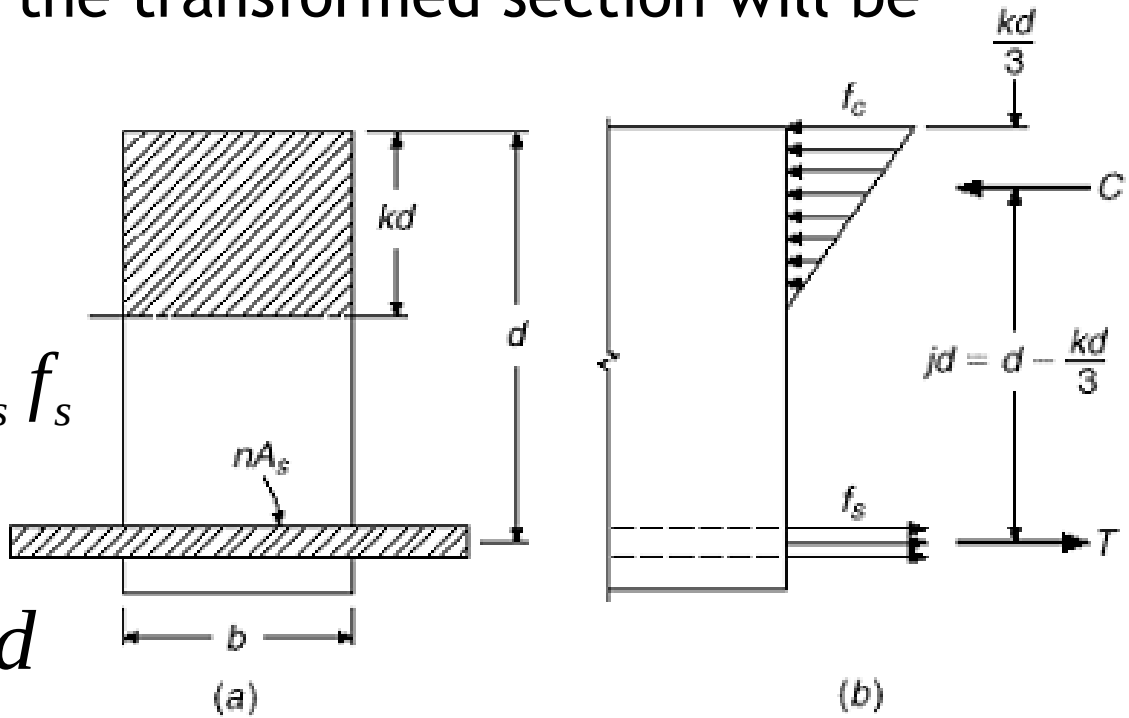
Compression force:

$$C = \frac{f_c}{2} \cdot bkd$$

Tensile force: $T = A_s f_s$

Taking moment about C gives:

$$M = Tjd = A_s f_s jd$$



Therefore, the steel stress is:

$$f_s = \frac{M}{A_s jd}$$

Taking moment about T gives:

$$M = Cjd = \frac{f_c}{2} \cdot bkdjd = \frac{f_c}{2} kjbd^2$$

The concrete stress is

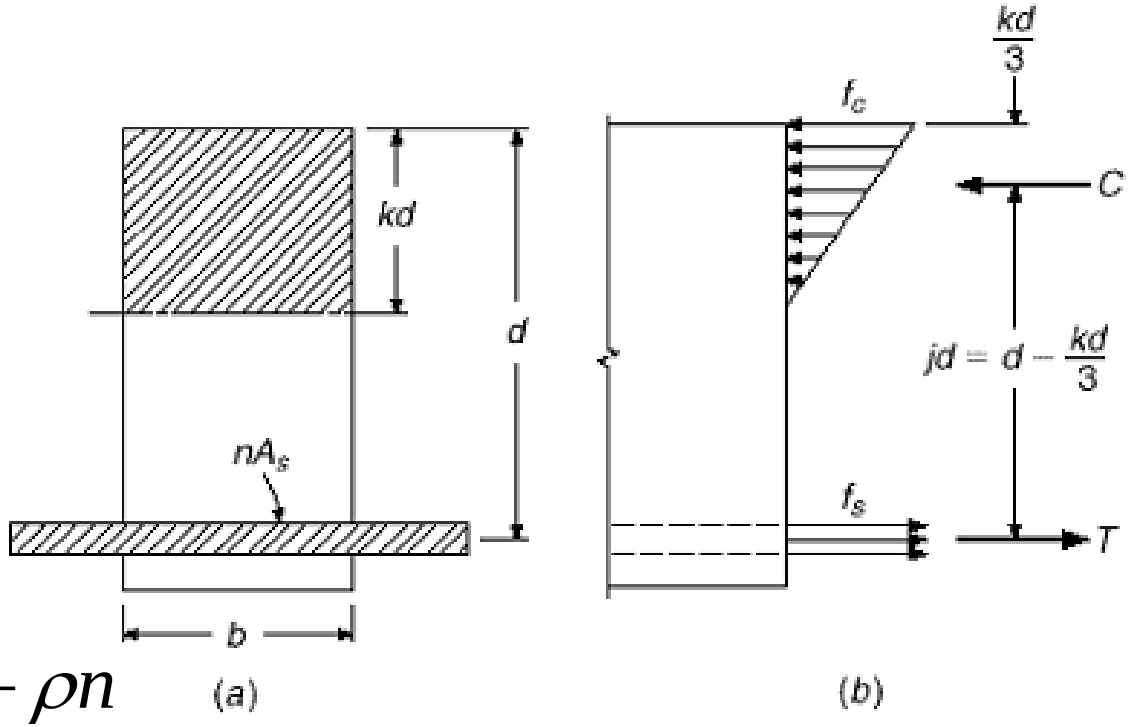
$$f_c = \frac{2M}{kjbd^2}$$

STRESSES ELASTIC & SECTION CRACKED

Reinforcement ratio:

$$\rho = \frac{A_s}{bd}$$

Expression for ***k*** and ***j***



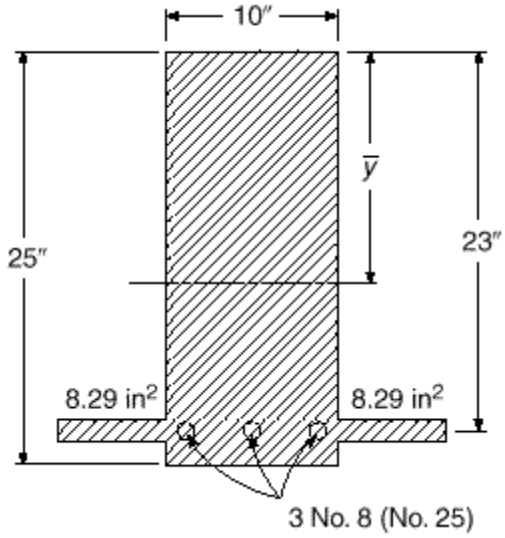
$$k = \sqrt{(\rho n)^2 + 2\rho n} - \rho n$$

$$j = 1 - \frac{k}{3}$$

EXAMPLE 2.2

- A rectangular beam has the dimensions b = 10 in., h = 25 in., and d = 23 in. and is reinforced with three No. 8 (No. 25) bars so that A_s = 2.37 in^2.
- The concrete cylinder strength (f_c') is 4000 psi, and tensile strength in bending (modulus of rupture) is 475 psi. The yield point of the steel (f_y) is 60,000 psi. Given, E_s = 29,000 ksi and E_c = 3,600 ksi.
- Determine the stresses caused by a bending moment M = 90 ft-kips.

EXAMPLE 2.2 (SOLUTION)



- $n = E_s/E_c$ = 29,000,000 / 3,600,000 = 8
- A_s = 3 x 0.79 = 2.37 in²
- Moment, M = 90 ft-kips = 1,080,000 in-lb
- The concrete compression stress at top fiber is,

$$f_c = \frac{1,080,000 \times 13.2}{14737} = 968 \text{ psi}$$

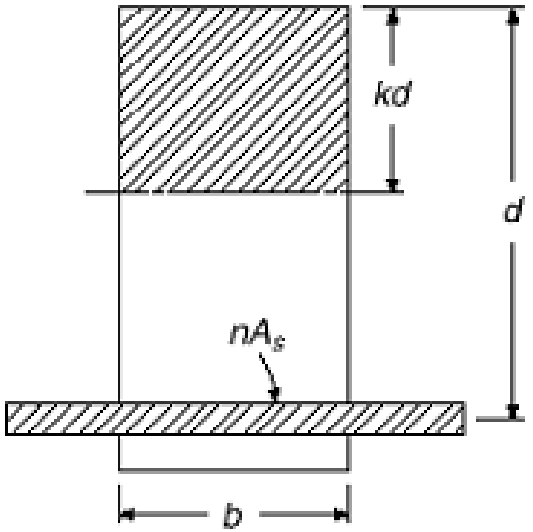
$$f_{\max} = \frac{Mc}{I}$$

- The concrete tension stress at bottom fiber is,

$$f_c = \frac{1,080,000 \times (25 - 13.2)}{14737} = 865 \text{ psi}$$

- Which is greater than the tensile strength of the concrete, 475 psi, so tensile cracks will form and analysis should be adapted considering cracked section.

EXAMPLE 2.2 (SOLUTION)



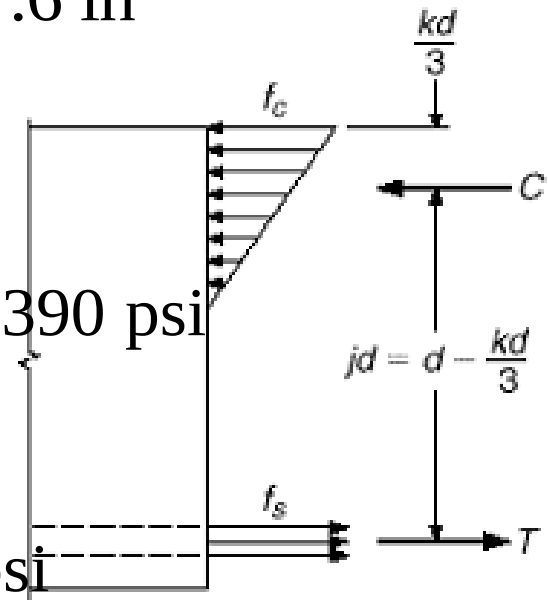
- $$\rho = \frac{A_s}{bd} = \frac{2.37}{10 \times 23} = 0.0103$$

- $$k = \sqrt{(\rho n)^2 + 2\rho n} - \rho n$$

$$= \sqrt{(0.0103 \times 8)^2 + 2 \times 0.0103 \times 8} - 0.0103 \times 8 = 0.33$$

$$kd = 0.33 \times 23 = 7.6 \text{ in}$$

- $$j = 1 - \frac{k}{3} = 1 - \frac{0.33}{3} = 0.89$$

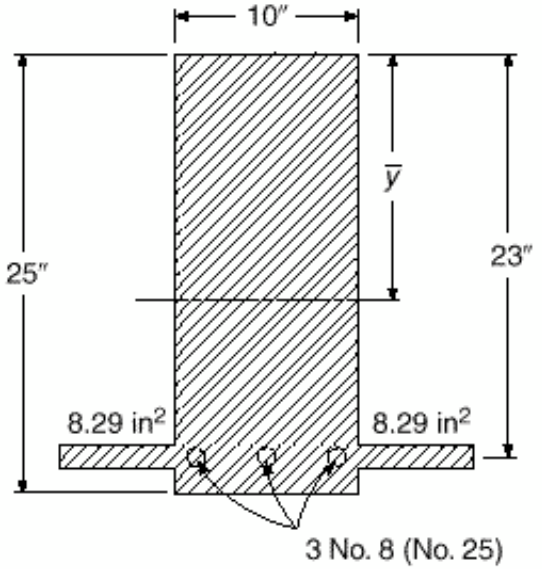


$$f_c = \frac{2M}{kjbd^2} \Rightarrow f_c = \frac{2 \times 1,080,000}{0.33 \times 0.89 \times 10 \times 23^2} = 1,390 \text{ psi}$$

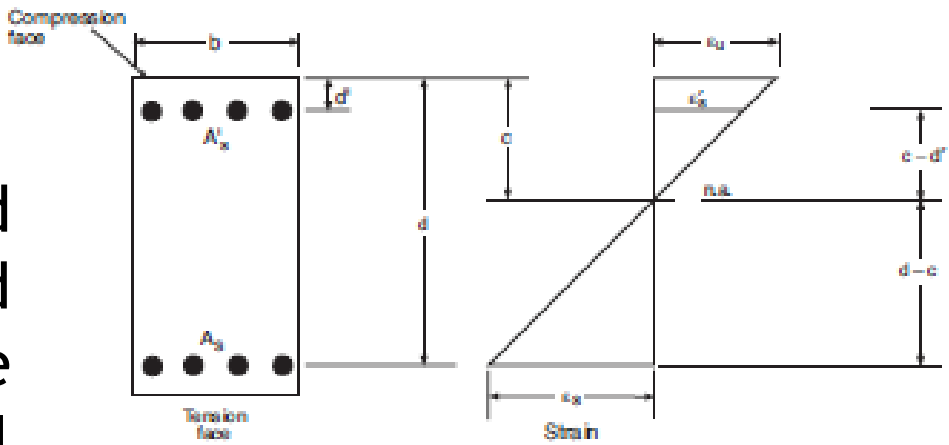
$$f_s = \frac{M}{A_s jd} \Rightarrow f_s = \frac{1,080,000}{2.37 \times 0.89 \times 23} = 22,262 \text{ psi}$$

COMPARISON OF UNCRACKED & CRACKED SECTIONS

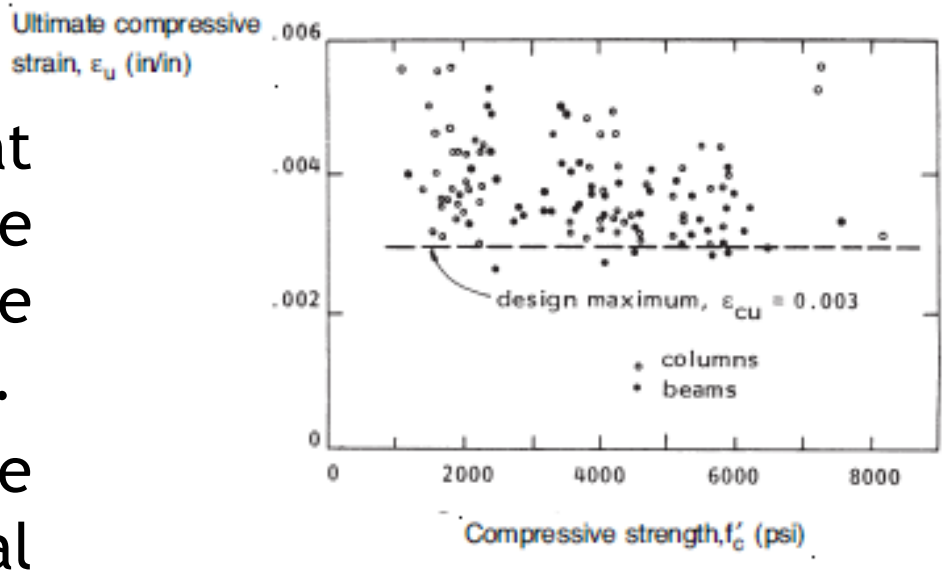
Description	Uncracked Section	Cracked Section
Neutral axis	13.2 in	7.6 in
Moment	45 ft-kips	90 ft-kips
Concrete compression stress	484 psi	1390 psi
Steel stress	2,873 psi	22,262 psi



DESIGN ASSUMPTIONS



1. Strain in reinforcement and concrete shall be assumed directly proportional to the distance from the neutral axis.
2. Maximum usable strain at extreme concrete compression fiber shall be assumed equal to ε_u = 0.003.
3. Tensile strength of concrete shall be neglected in axial and flexural calculations of reinforced concrete.

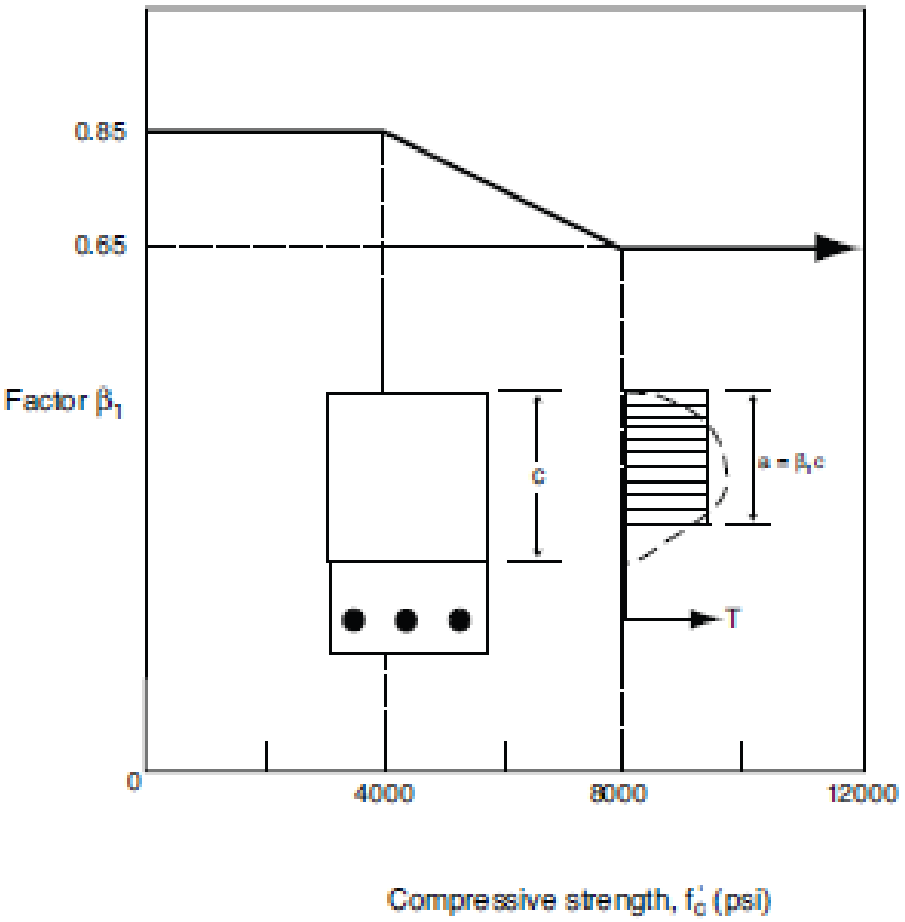


DESIGN ASSUMPTIONS

4. Stress in reinforcement f_s below the yield strength f_y shall be taken as E_s times the steel strain ε_s. For strains greater than f_y/E_s, stress in reinforcement shall be considered independent of strain and equal to f_y.
5. Relationship between concrete compressive stress distribution and concrete strain shall be assumed to be rectangular, trapezoidal, parabolic, or any other shape that results in prediction of strength in substantial agreement with results of compressive tests.

DESIGN ASSUMPTIONS

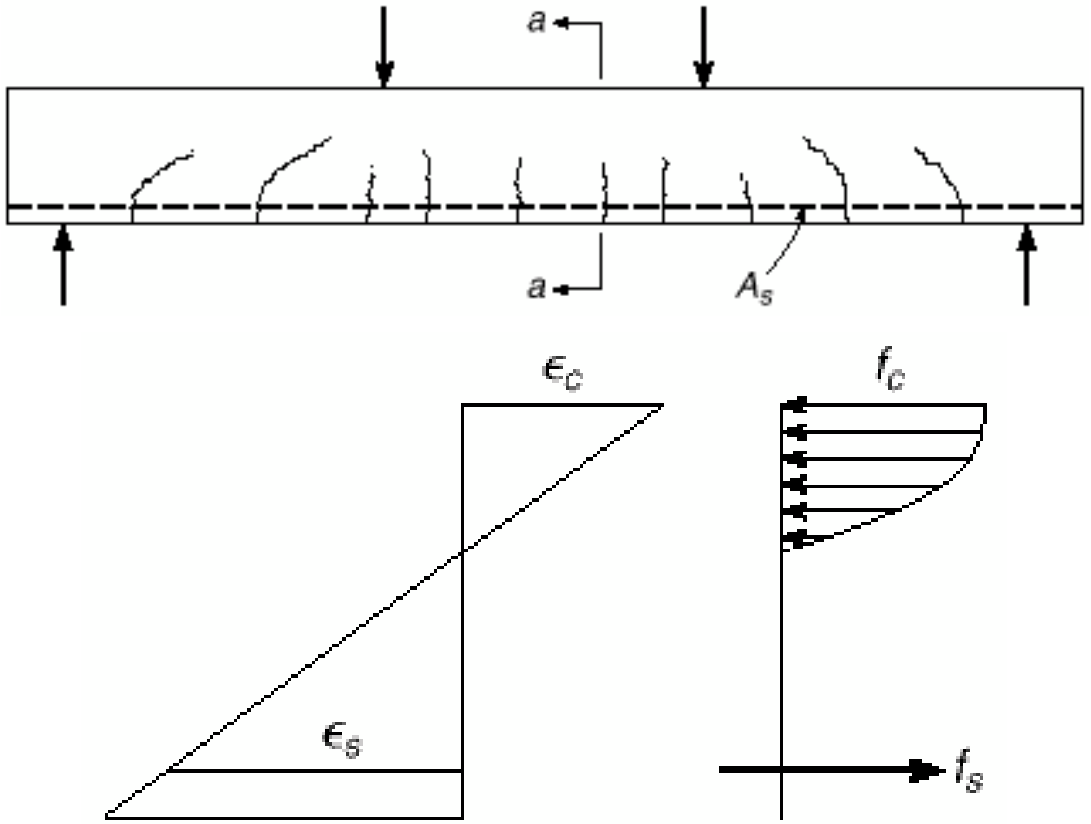
4. Requirements of the above assumption may be considered satisfied by an equivalent rectangular concrete stress distribution defined as follows: A concrete stress of $0.85f_c'$ shall be assumed uniformly distributed over an equivalent compression zone bounded by edges of the cross-section and a straight line located parallel to the neutral axis at a distance $a = \beta_1 c$ from the fiber of maximum compressive strain.



STRESSES INELASTIC & SECTION CRACKED

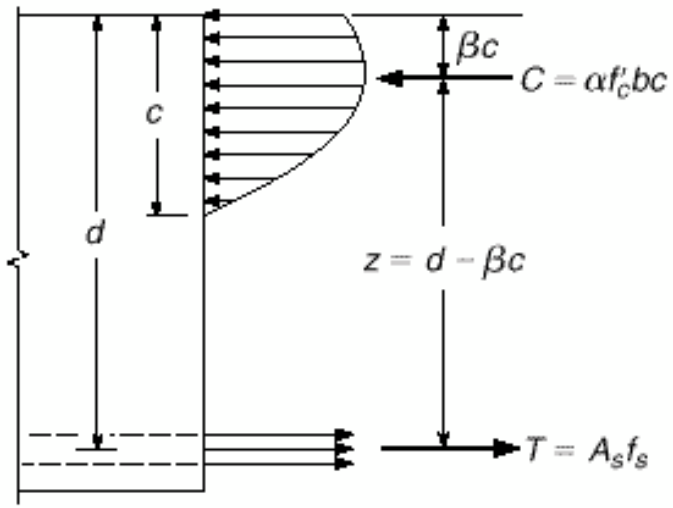
STRESSES INELASTIC & SECTION CRACKED

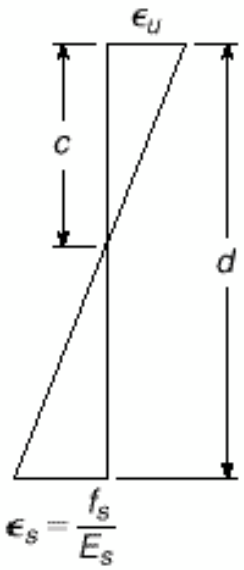
- At high loads, close to failure, the distribution of stresses and strains are no longer elastic distribution.

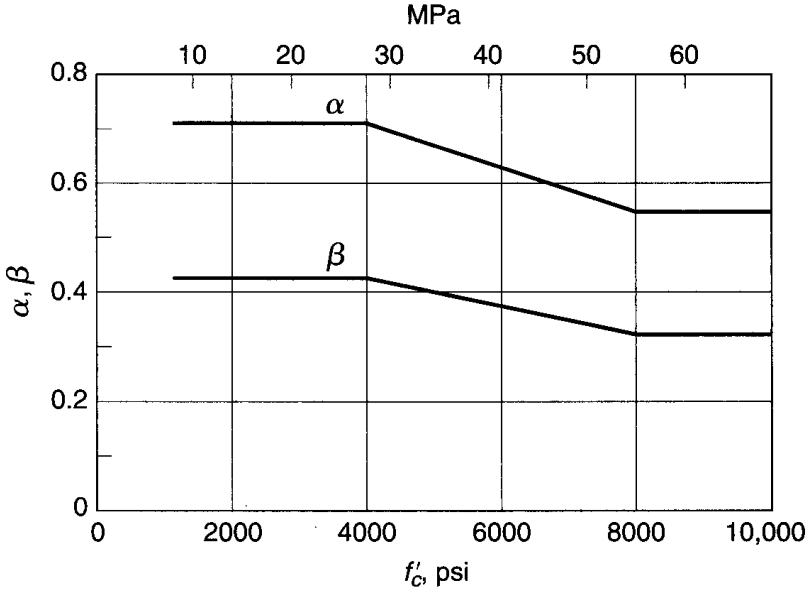


FLEXURAL STRENGTH

- The following figure represents the distribution of internal stresses and strains when the beam is about to fail.



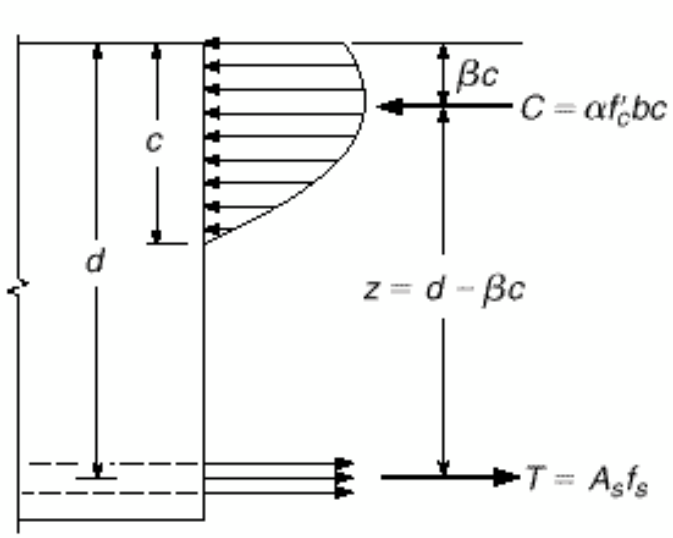


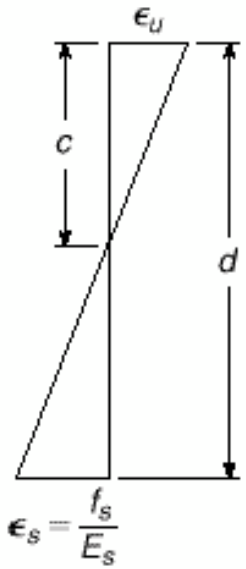


- $\alpha = 0.72$ for $f_c' \leq 4000$ psi and decreases by 0.04 for every 1,000 psi above 4,000 up to 8,000 psi. For $f_c' > 8{,}000$ psi, $\alpha = 0.56$.
- $\beta = 0.425$ for $f_c' \leq 4000$ psi and decreases by 0.025 for every 1000 psi above 4000 up to 8000 psi. For $f_c' > 8000$ psi, $\beta = 0.325$.

FLEXURAL STRENGTH

- For a nominal moment (M_n), the beam will fail either by
  - Tension yielding of the steel
  - Crushing of the concrete in the outer compression fiber.
- For tension yielding of the steel failure, $f_s = f_y$.
- For the second mode of failure, it is assumed that concrete fail when maximum strain ε_u reaches 0.003.





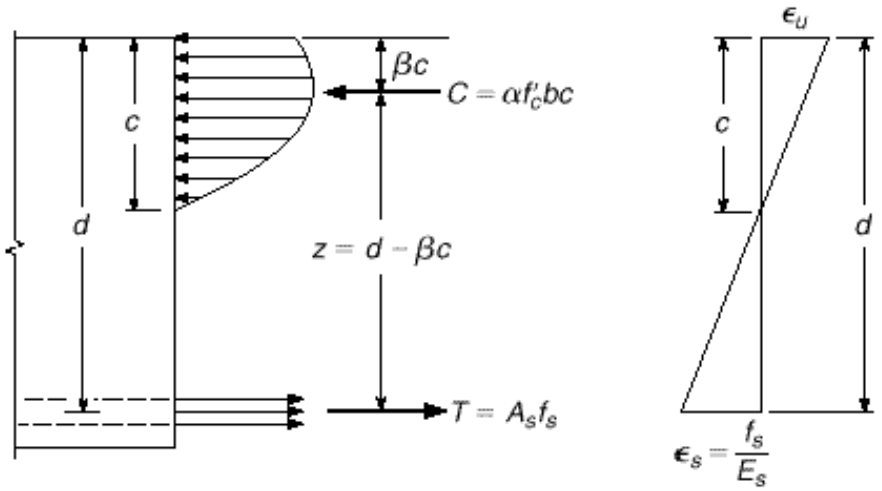
$$C = \alpha f_c' bc$$

$$T = A_s f_s$$

FLEXURAL STRENGTH

- **For failure initiated by yielding of the tension steel:**
- Distance to the neutral axis:

$$C = T \Rightarrow \alpha c f_c' b = A_s f_y \therefore c = \frac{A_s f_y}{\alpha f_c' b}$$



- Total compression force on concrete:

$$C = \alpha f_c' b c$$

$$c = \frac{A_s f_y}{\alpha f_c' b} = \frac{\rho b d f_y}{\alpha f_c' b} = \frac{\rho d f_y}{\alpha f_c'}$$

- The nominal moment:

$$M_n = Cz = \alpha f_c' b c (d - \beta c) = \alpha f_c' b \frac{\rho f_y d}{\alpha f_c'} \left(d - \beta \frac{\rho f_y d}{\alpha f_c'} \right) = \rho f_y b d^2 \left(1 - \frac{\beta \rho f_y}{\alpha f_c'} \right)$$

$$M_n = \rho f_y b d^2 \left(1 - 0.59 \frac{\rho f_y}{f_c'} \right)$$

FLEXURAL STRENGTH

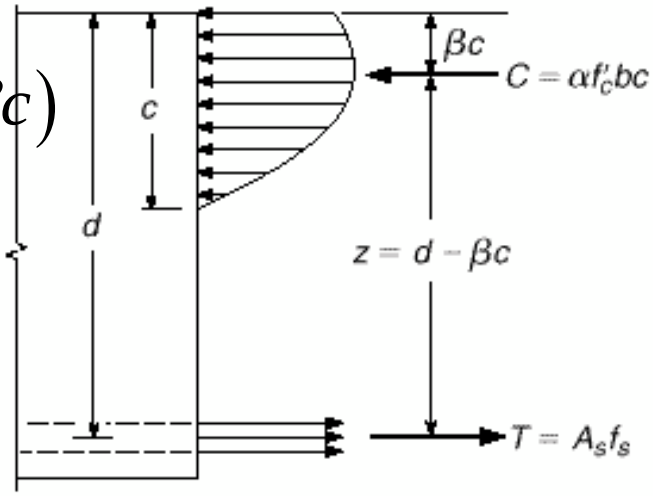
- **For failure by compression of concrete:**
- Distance to the neutral axis:

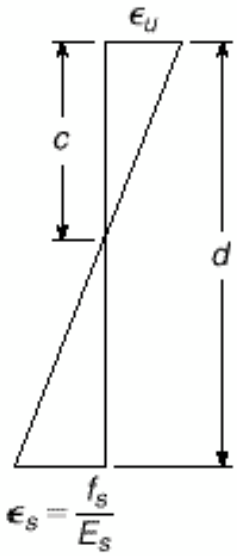
$$C = T \Rightarrow \alpha c f_c' b = A_s f_s \therefore c = \frac{A_s f_s}{\alpha f_c' b}$$

- The nominal moment:

$$M_n = Cz = \alpha f_c' bcz = \alpha f_c' bc(d - \beta c)$$

$$M_n = Tz = A_s f_s z = A_s f_s (d - \beta c)$$





- Steel stress:

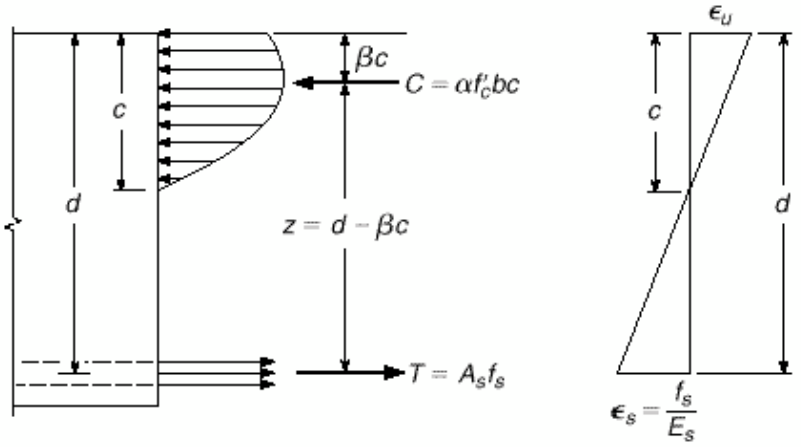
$$f_s = \varepsilon_s E_s = \varepsilon_u \frac{d - c}{c} E_s$$

- Hence, $$\alpha c f_c' b = A_s \varepsilon_u \frac{d - c}{c} E_s$$
- Solve this quadratic equation for c.

FLEXURAL STRENGTH

Balanced Steel Ratio, ρ_b

- The balanced steel ratio , representing that amount of reinforcement necessary for beam to fail by crashing of the concrete at the same load that causes the steel to yield.
- This means concrete reaches its compressive strain, ε_u and for steel, $f_s = f_y$.
- For this case, $$c = \frac{\varepsilon_u}{\varepsilon_u + \varepsilon_y} d$$



Whether or not the steel has yielded can be determined by comparing the actual reinforcement ratio with the *balanced reinforcement ratio*, ρ_b.

$$\rho_b = \frac{\alpha f_c'}{f_y} \frac{\varepsilon_u}{\varepsilon_u + \varepsilon_y}$$

EXAMPLE 2.3

- A rectangular beam has the dimensions b = 10 in., h = 25 in., and d = 23 in. and is reinforced with three No. 8 (No. 25) bars so that A_s = 2.37 in^2.
- The concrete cylinder strength is f_c' is 4,000 psi. The yield point of the steel f_y is 60,000 psi. Given, E_s = 29,000 ksi and E_c = 3,600 ksi. Also ε_u = 0.003 and ε_y = 0.002
- Determine the nominal moment M_n at which the beam will fail.

EXAMPLE 2.3 (SOLUTION)

- Given: b = 10 in., h = 25 in., and d = 23 in. and A_s = 2.37 in^2.
- f_c' = 4,000 psi, f_y = 60,000 psi.
- Actual reinforcement ratio:

$$\rho = \frac{A_s}{bd} = \frac{2.37}{10 \times 23} = 0.0103$$

- Balanced reinforcement ratio:

$$\rho_b = \frac{\alpha f_c'}{f_y} \frac{\varepsilon_u}{\varepsilon_u + \varepsilon_y} \Rightarrow \rho_b = \frac{0.72 \times 4,000}{60,000} \frac{0.003}{0.003 + 0.002} \therefore \rho_b = 0.0284$$

- Since, $\rho < \rho_b$, the beam will fail in tension by yielding of the steel.
- Nominal moment: $M_n = \rho f_y b d^2 \left(1 - 0.59 \frac{\rho f_y}{f_c'}\right)$

$$= 0.103 \times 60,000 \times 10 \times 23^2 \left(1 - 0.59 \frac{0.0103 \times 60,000}{4,000}\right)$$

$$= 2,971,214 \text{ in-lb} = 248 \text{ ft-kips}$$