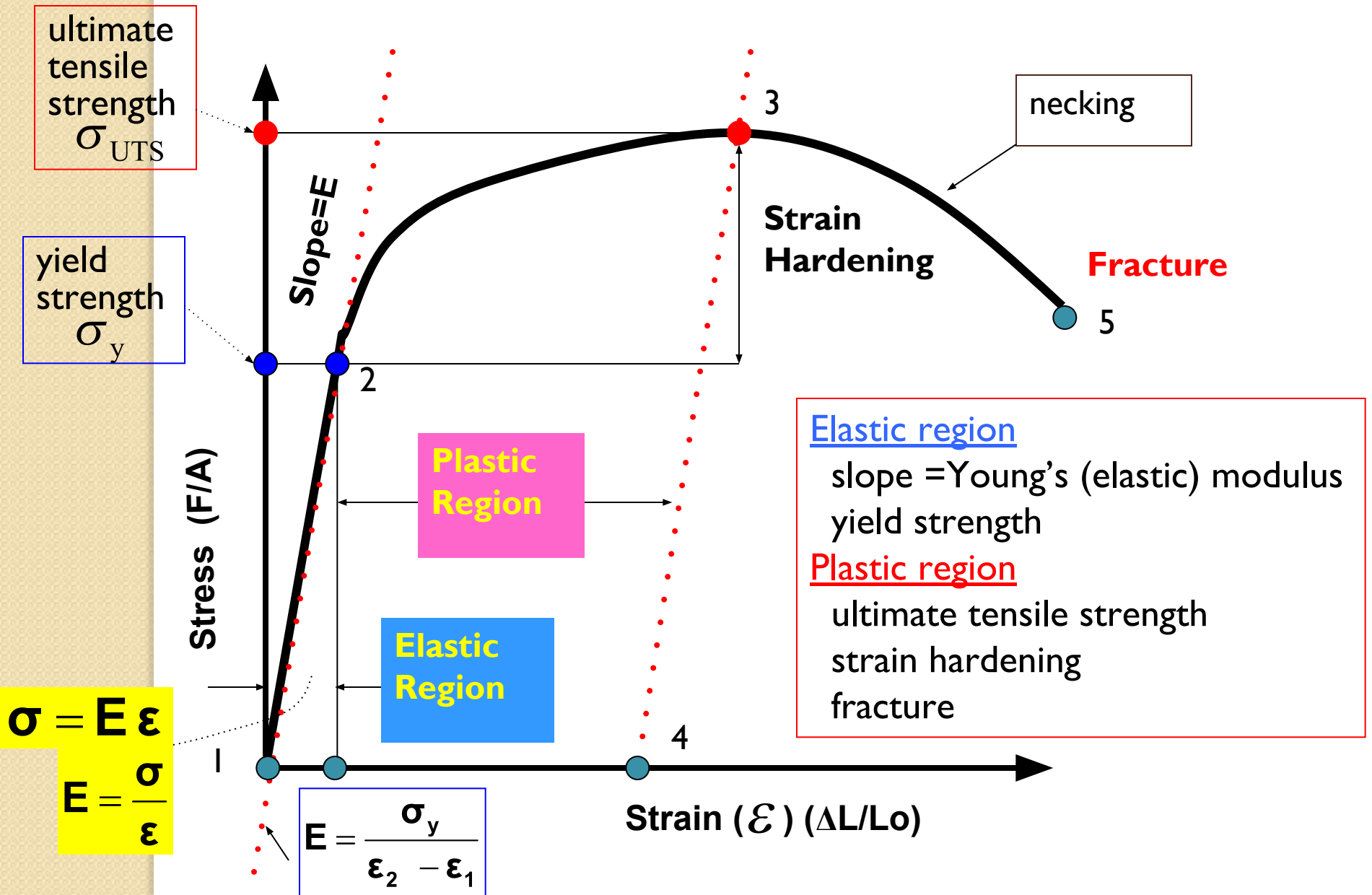
## External Work and Strain Energy

Energy methods are based on the conservation of energy principle, which states that the work done by all the external forces acting on a structure, is transformed into internal work or strain energy, which is developed when the structure deforms. If the material's elastic limit is not exceeded, the elastic strain energy will return the structure to its un deformed state when the loads are removed. The conservation of energy principle can be stated mathematically as

$$Ue = Ui$$

Before developing any of the energy methods based on this principle, however, we will first determine the external work and strain energy caused by a force and a moment. The formulations to be presented will provide a basis for understanding the work and energy methods that follow

# Stress-Strain Diagram




Elastic region
- slope =Young's (elastic) modulus
- yield strength

Plastic region
- ultimate tensile strength
- strain hardening
- fracture

# Structural Analysis & Design I

**External Work—Force.** When a force **F** undergoes a displacement $dx$ in the *same direction* as the force, the work done is $dU_e = F\,dx$. If the total displacement is $x$, the work becomes

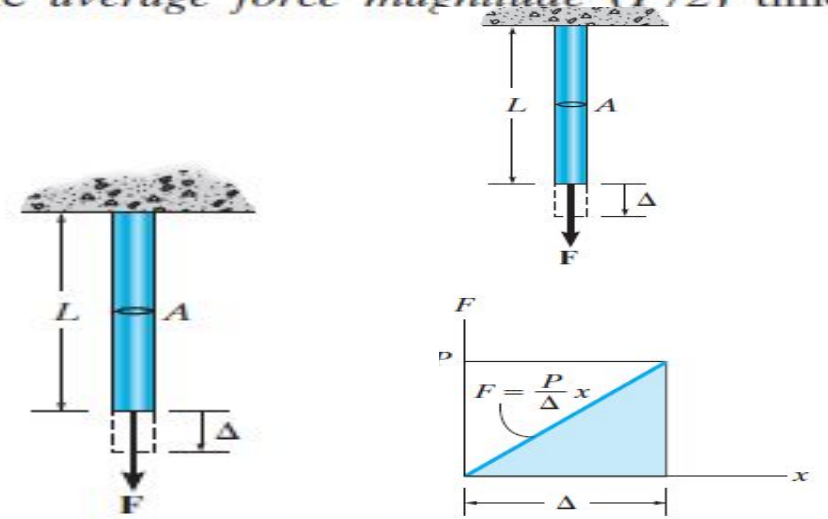
$$U_e = \int_0^x F\,dx \tag{9–2}$$

Consider now the effect caused by an axial force applied to the end of a bar as shown in Fig. 9–1*a*. As the magnitude of **F** is *gradually* increased from zero to some limiting value $F = P$, the final elongation of the bar becomes $\Delta$. If the material has a linear elastic response, then $F = (P/\Delta)x$. Substituting into Eq. 9–2, and integrating from 0 to $\Delta$, we get

$$U_e = \tfrac{1}{2}P\Delta \tag{9–3}$$

which represents the shaded *triangular area* in Fig. 9–1*a*.

We may also conclude from this that as a force is gradually applied to the bar, and its magnitude builds linearly from zero to some value $P$, the work done is equal to the *average force magnitude* $(P/2)$ times the displacement $(\Delta)$.



# Structural Analysis & Design I

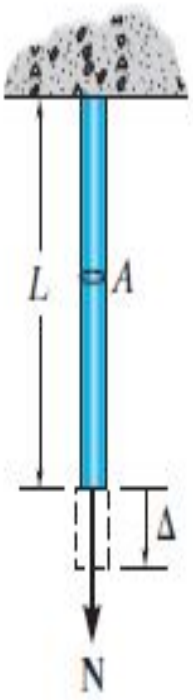



Fig. 9–3

**Strain Energy—Axial Force.** When an axial force **N** is applied gradually to the bar in Fig. 9–3, it will strain the material such that the *external work* done by **N** will be converted into *strain energy*, which is stored in the bar (Eq. 9–1). Provided the material is *linearly elastic*, Hooke's law is valid, $\sigma = E\epsilon$, and if the bar has a constant cross-sectional area $A$ and length $L$, the normal stress is $\sigma = N/A$ and the final strain is $\epsilon = \Delta/L$. Consequently, $N/A = E(\Delta/L)$, and the final deflection is

$$\Delta = \frac{NL}{AE} \tag{9–8}$$

Substituting into Eq. 9–3, with $P = N$, the strain energy in the bar is therefore

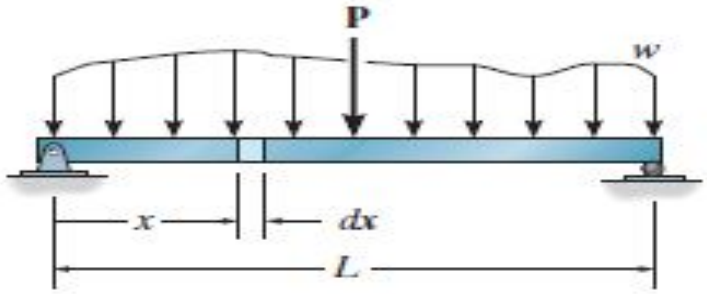
$$U_i = \frac{N^2L}{2AE} \tag{9–9}$$

**Strain Energy—Bending.** Consider the beam shown in Fig. 9–4*a*, which is distorted by the *gradually* applied loading **P** and $w$. These loads create an internal moment **M** in the beam at a section located a distance $x$ from the left support. The resulting rotation of the differential element $dx$, Fig. 9–4*b*, can be found from Eq. 8–2, that is, $d\theta = (M/EI)\,dx$. Consequently, the strain energy, or work stored in the element, is determined from Eq. 9–6 since the internal moment is gradually developed. Hence,

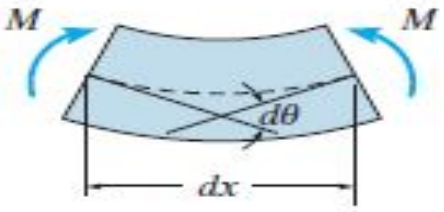
$$dU_i = \frac{M^2\,dx}{2EI} \tag{9–10}$$

The strain energy for the beam is determined by integrating this result over the beam's entire length $L$. The result is

$$U_i = \int_0^L \frac{M^2\,dx}{2EI} \tag{9–11}$$




(a)




(b)

## 9.2 Principle of Work and Energy

Now that the work and strain energy for a force and a moment have been formulated, we will illustrate how the conservation of energy or the principle of work and energy can be applied to determine the displacement at a point on a structure. To do this, consider finding the displacement $\Delta$ at the point where the force **P** is applied to the cantilever beam in Fig. 9–5. From Eq. 9–3, the external work is $U_e = \frac{1}{2}P\Delta$. To obtain the resulting strain energy, we must first determine the internal moment as a function of position $x$ in the beam and then apply Eq. 9–11. In this case $M = -Px$, so that

$$U_i = \int_0^L \frac{M^2\,dx}{2EI} = \int_0^L \frac{(-Px)^2\,dx}{2EI} = \frac{1}{6}\frac{P^2L^3}{EI}$$

Equating the external work to internal strain energy and solving for the unknown displacement $\Delta$, we have

$$U_e = U_i$$

$$\frac{1}{2}P\Delta = \frac{1}{6}\frac{P^2L^3}{EI}$$

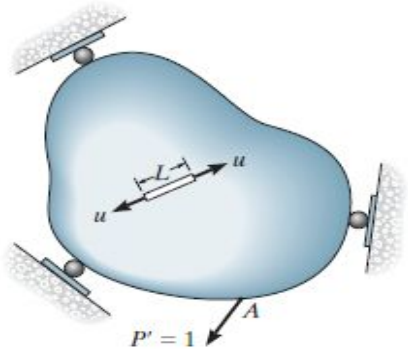
$$\Delta = \frac{PL^3}{3EI}$$

# Structural Analysis & Design I

## Virtual work

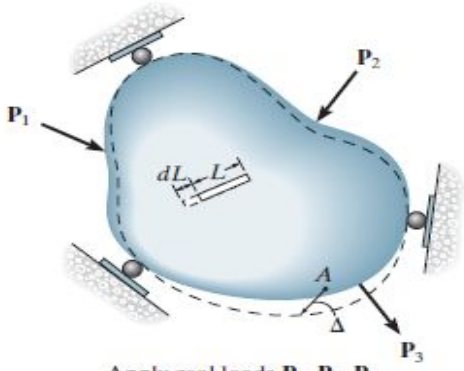
The principle of virtual work was developed by John Bernoulli in 1717 and is sometimes referred to as the unit-load method. It provides a general means of obtaining the displacement and slope at a specific point on a structure, be it a beam, frame, or truss.

Deformable structure of any shape or size and apply a series of *external loads* ***P to*** it, it will cause *internal loads* ***u at points throughout the structure. It is*** *necessary that the external and internal loads be related by the equations of equilibrium. As a consequence of these loadings, external displacements* will occur at the **P loads and internal displacements will occur at** each point of internal load **u**. **In general, *these displacements do not have*** *to be elastic, and they may not be related to the loads; however, the external and internal displacements must be related by the compatibility of the displacements.*




Apply virtual load $P' = 1$

(a)

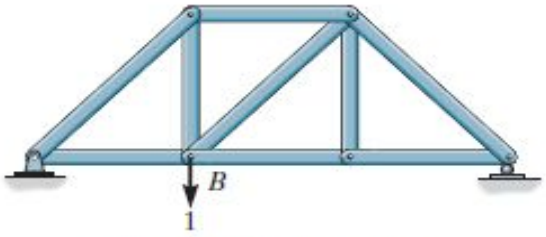



Apply real loads $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$

(b)

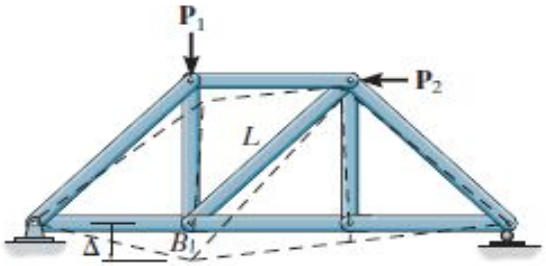
$$\underset{\text{Work of External Loads}}{\Sigma P\Delta} = \underset{\text{Work of Internal Loads}}{\Sigma u\delta} \qquad (9\text{–}12)$$

Based on this concept, the principle of virtual work will now be developed. To do this, we will consider the structure (or body) to be of arbitrary shape as shown in Fig. 9–6*b*.* Suppose it is necessary to determine the displacement $\Delta$ of point $A$ on the body caused by the "real loads" $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$. It is to be understood that these loads cause no movement of the supports; in general, however, they can strain the material *beyond the elastic limit*. Since no external load acts on the body at $A$ and in the direction of $\Delta$, the displacement $\Delta$ can be determined by *first* placing on the body a *"virtual" load* such that this force $\mathbf{P}'$ acts in the *same direction* as $\Delta$, Fig. 9–6*a*. For convenience, which will be apparent later, we will choose $\mathbf{P}'$ to have a "unit" magnitude, that is, $P' = 1$. The term "virtual" is used to describe the load, since *it is imaginary and does not actually exist as part of the real loading*. The unit load ($\mathbf{P}'$) does, however, create an internal virtual load $\mathbf{u}$ in a representative element or fiber of the body, as shown in Fig. 9–6*a*. Here it is required that $\mathbf{P}'$ and $\mathbf{u}$ be related by the equations of equilibrium.†




Apply virtual unit load to $B$

(a)




Apply real loads $\mathbf{P}_1$, $\mathbf{P}_2$

(b)

Fig. 9–7

## 9.4 Method of Virtual Work: Trusses

We can use the method of virtual work to determine the displacement of a truss joint when the truss is subjected to an external loading, temperature change, or fabrication errors. Each of these situations will now be discussed.

**External Loading.** For the purpose of explanation let us consider the vertical displacement $\Delta$ of joint $B$ of the truss in Fig. 9–7*a*. Here a typical element of the truss would be one of its *members* having a length $L$, Fig. 9–7*b*. If the applied loadings $\mathbf{P}_1$ and $\mathbf{P}_2$ cause a *linear elastic material response*, then this element deforms an amount $\Delta L = NL/AE$, where $N$ is the normal or axial force in the member, caused by the loads. Applying Eq. 9–13, the virtual-work equation for the truss is therefore

$$1 \cdot \Delta = \sum \frac{nNL}{AE} \tag{9–15}$$

where

$1$ = external virtual unit load acting on the truss joint in the stated direction of $\Delta$.

$n$ = internal virtual normal force in a truss member caused by the external virtual unit load.

$\Delta$ = external joint displacement caused by the real loads on the truss.

$N$ = internal normal force in a truss member caused by the real loads.

$L$ = length of a member.

$A$ = cross-sectional area of a member.

$E$ = modulus of elasticity of a member.

## Procedure for Analysis

The following procedure may be used to determine a specific displacement of any joint on a truss using the method of virtual work.

### Virtual Forces n

- Place the unit load on the truss at the joint where the desired displacement is to be determined. The load should be in the same direction as the specified displacement, e.g., horizontal or vertical.
- With the unit load so placed, and all the real loads *removed* from the truss, use the method of joints or the method of sections and calculate the internal **n** force in each truss member. Assume that tensile forces are positive and compressive forces are negative.

### Real Forces N

- Use the method of sections or the method of joints to determine the **N** force in each member. These forces are caused only by the real loads acting on the truss. Again, assume tensile forces are positive and compressive forces are negative.

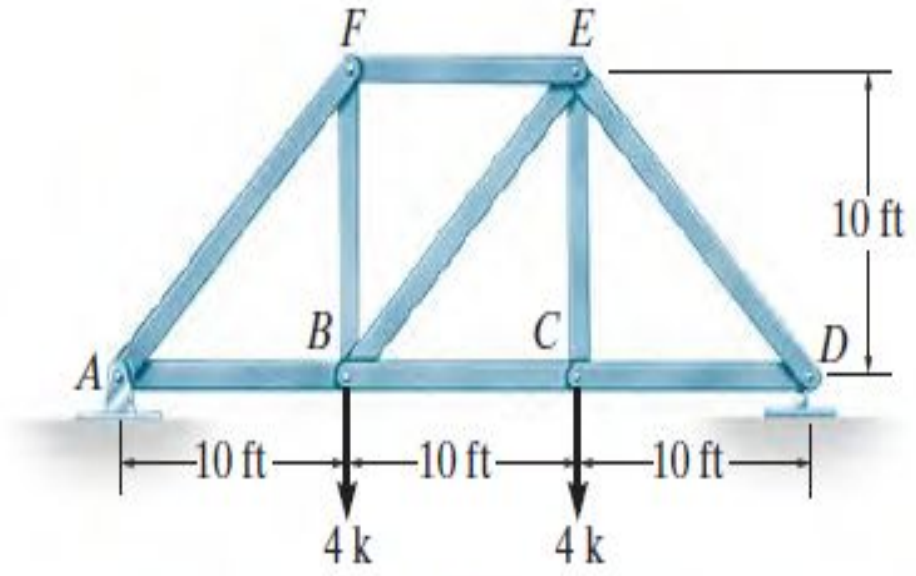
# Structural Analysis & Design I

## Virtual-Work Equation

- Apply the equation of virtual work, to determine the desired displacement. It is important to retain the algebraic sign for each of the corresponding **n** and **N** forces when substituting these terms into the equation.
- If the resultant sum $\Sigma nNL/AE$ is positive, the displacement $\Delta$ is in the same direction as the unit load. If a negative value results, $\Delta$ is opposite to the unit load.
- When applying $1 \cdot \Delta = \Sigma n\alpha\, \Delta T L$, realize that if any of the members undergoes an *increase in temperature*, $\Delta T$ will be *positive*, whereas a *decrease in temperature* results in a *negative* value for $\Delta T$.
- For $1 \cdot \Delta = \Sigma n\, \Delta L$, when a fabrication error *increases the length* of a member, $\Delta L$ is *positive*, whereas a *decrease in length* is *negative*.
- When applying any formula, attention should be paid to the units of each numerical quantity. In particular, the virtual unit load can be assigned any arbitrary unit (lb, kip, N, etc.), since the **n** forces will have these *same units*, and as a result the units for both the virtual unit load and the **n** forces will cancel from both sides of the equation.

# Structural Analysis & Design I

Determine the vertical displacement of joint $C$ of the steel truss shown in Fig. . The cross-sectional area of each member is $A = 0.5\ \text{in}^2$ and $E = 29(10^3)$ ksi.

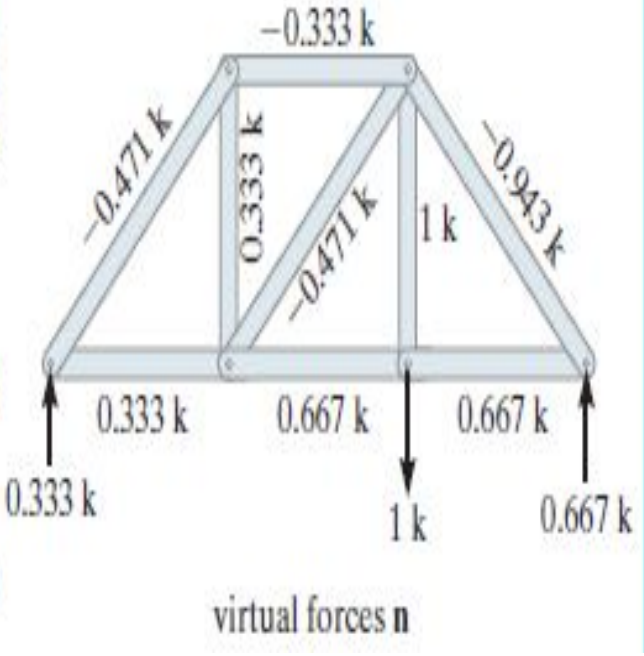


## SOLUTION

**Virtual Forces n.** Only a vertical 1-k load is placed at joint $C$, and the force in each member is calculated using the method of joints. The results are shown in Fig. 9–8*b*. Positive numbers indicate tensile forces and negative numbers indicate compressive forces.

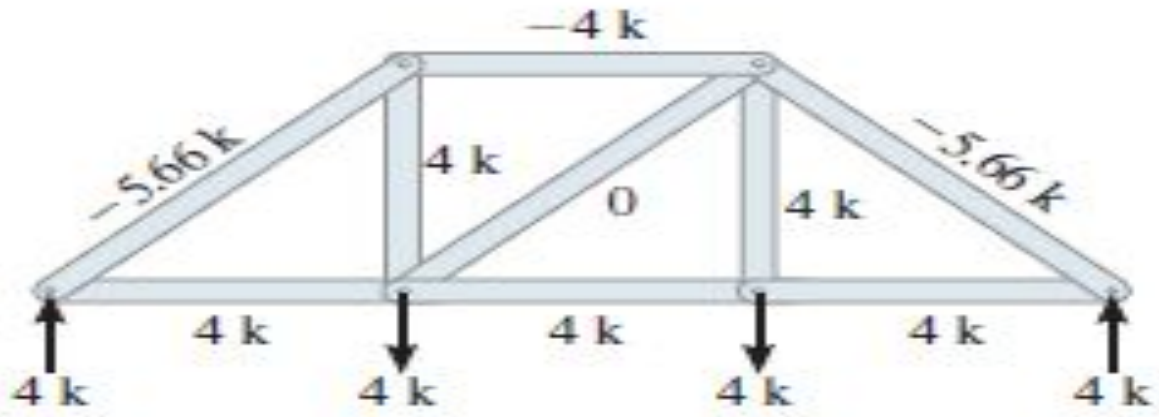
**Real Forces N.** The real forces in the members are calculated using the method of joints. The results are shown in Fig. 9–8*c*.

**Virtual-Work Equation.** Arranging the data in tabular form, we have




virtual forces **n**

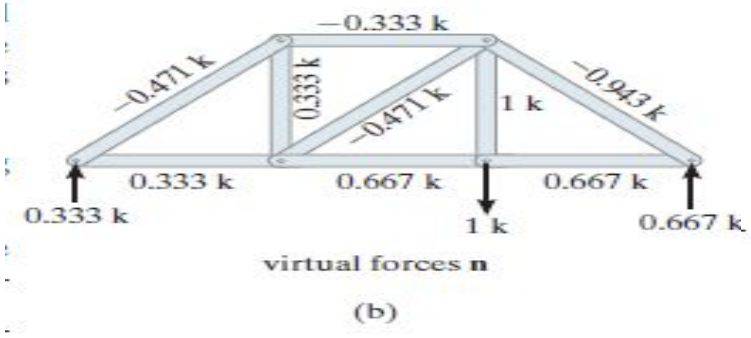
# Structural Analysis & Design I




real forces N

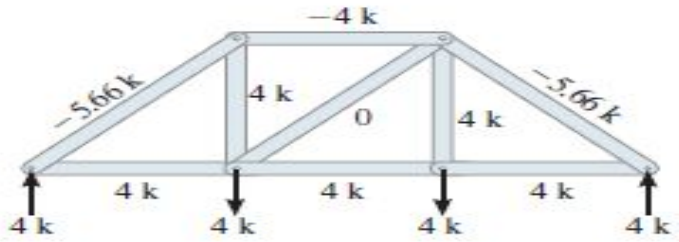
(c)

# Structural Analysis & Design I




virtual forces **n**

(b)




real forces **N**

(c)

| Member | $n$ (k) | $N$ (k) | $L$ (ft) | $nNL$ (k$^2$ · ft) |
|---|---|---|---|---|
| $AB$ | 0.333 | 4 | 10 | 13.33 |
| $BC$ | 0.667 | 4 | 10 | 26.67 |
| $CD$ | 0.667 | 4 | 10 | 26.67 |
| $DE$ | −0.943 | −5.66 | 14.14 | 75.42 |
| $FE$ | −0.333 | −4 | 10 | 13.33 |
| $EB$ | −0.471 | 0 | 14.14 | 0 |
| $BF$ | 0.333 | 4 | 10 | 13.33 |
| $AF$ | −0.471 | −5.66 | 14.14 | 37.71 |
| $CE$ | 1 | 4 | 10 | 40 |
| | | | | $\Sigma$246.47 |

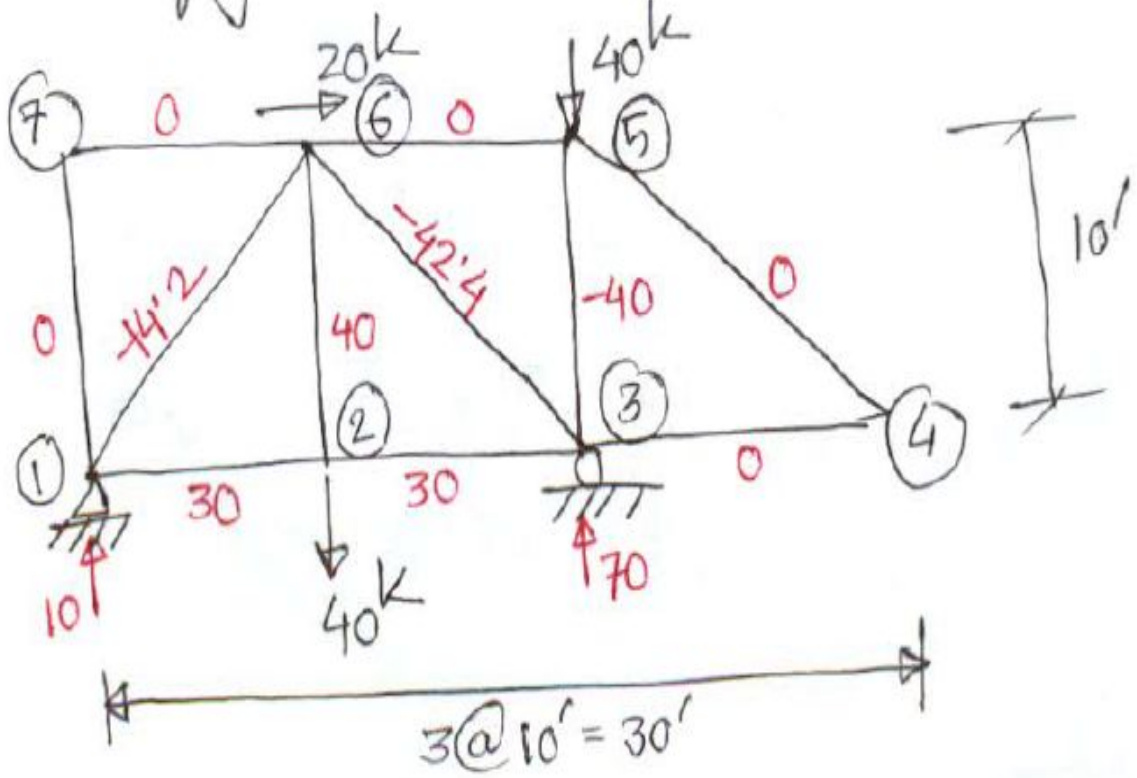
Thus $$1\ \text{k} \cdot \Delta_{C_v} = \sum \frac{nNL}{AE} = \frac{246.47\ \text{k}^2 \cdot \text{ft}}{AE}$$

Converting the units of member length to inches and substituting the numerical values for $A$ and $E$, we have

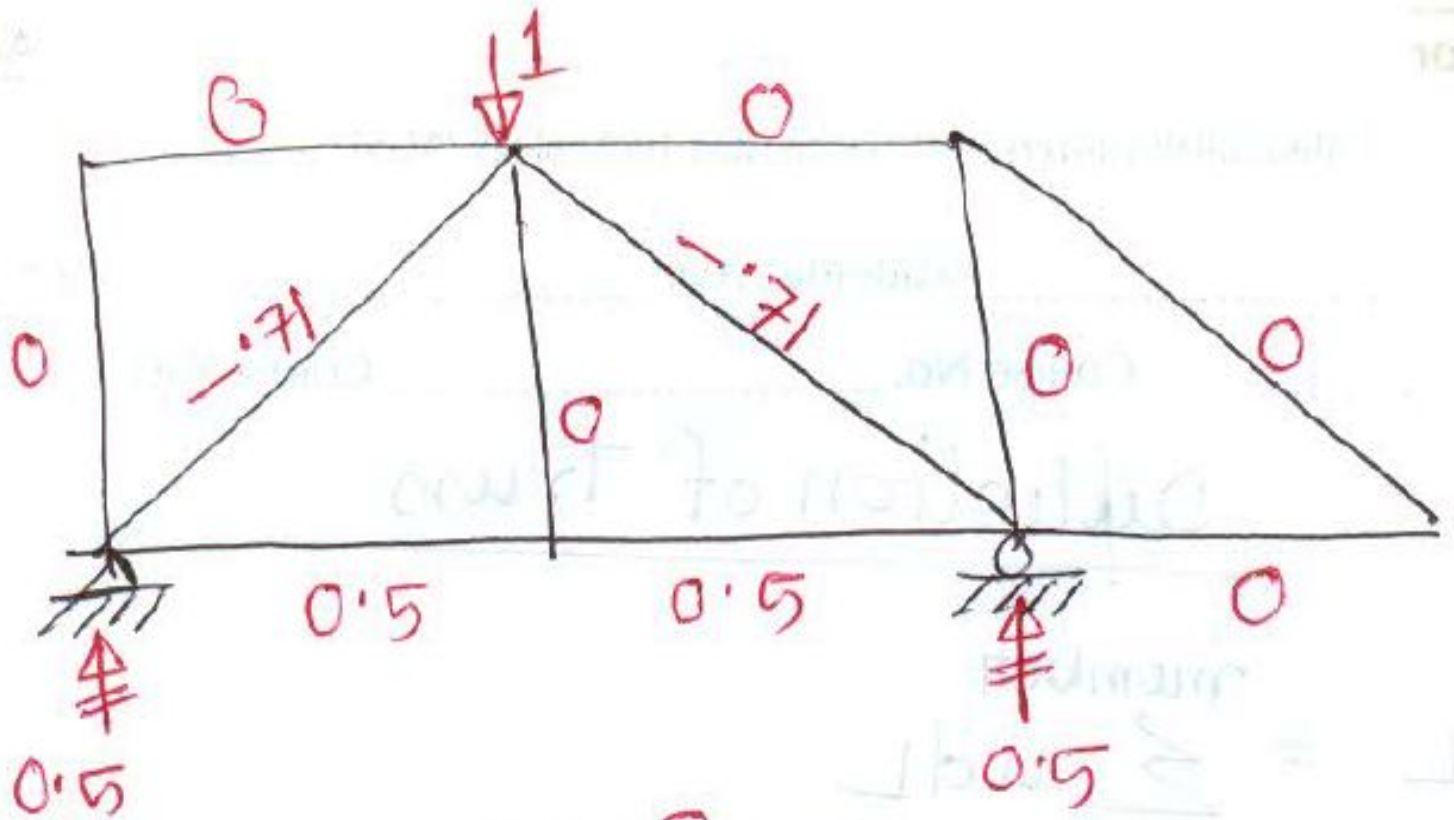
$$1\ \text{k} \cdot \Delta_{C_v} = \frac{(246.47\ \text{k}^2 \cdot \text{ft})(12\ \text{in./ft})}{(0.5\ \text{in}^2)(29(10^3)\ \text{k/in}^2)}$$

$$\Delta_{C_v} = 0.204\ \text{in.}$$ *Ans.*

compute the vertical dfluction of joint #6 of the trus
shown in fig. $A = 2\,in^2$, $E = 30,000$ ksi



# Structural Analysis & Design I



# Structural Analysis & Design I

| member | L (ft) | $n_0$ (k) | $n_1$ | $n_0 n_1 L$ |
|---|---|---|---|---|
| 12 | 10×12 | 30 | 0·5 | 1800 |
| 23 | 10×12 | 30 | 0·5 | 1800 |
| 16 | 10√2×12 | −14·2 | −·71 | 1710·71 |
| 36 | 10√2×12 | −42·4 | −·71 | 5108·05 |

$\Sigma$ 10418·76

$$\Delta_{6V} = \sum \frac{n_1 n_0 L}{AE} = 0.174 \ (\downarrow)$$

# Structural Analysis & Design I

In **1879** Alberto Castigliano, an Italian railroad engineer, published a book in which he outlined a method for **determining the deflection** or slope at a point in a structure, be it a truss, beam, or frame.

This method, which is referred to as ***Castigliano's second theorem****, or the method of least work, applies only to* structures that have constant temperature, unyielding supports, and ***linear elastic material*** *response.* The **displacement of a point** is to be determined, the theorem states that it is equal to **the first partial derivative of the strain energy** in the structure with respect to a force acting at the point and in the direction of displacement.

# Structural Analysis & Design I

To derive Castigliano's second theorem, consider a body (structure) of any arbitrary shape which is subjected to a series of *n forces.* Since the **external work done** by these loads is equal to the **internal strain energy stored** in the body, we can write.

***Ui = Ue***

Now, if any one of the forces, say *Pi* is increased by a differential amount *dPi* the internal work is also increased such that the new strain energy becomes

$$U_i + dU_i = U_i + \frac{\partial U_i}{\partial P_i} dP_i \qquad (9\text{–}18)$$

This value, however, should not depend on the sequence in which the $n$ forces are applied to the body. For example, if we apply $dP_i$ to the body *first*, then this will cause the body to be displaced a differential amount $d\Delta_i$ in the direction of $dP_i$. By Eq. 9–3 $\left(U_e = \frac{1}{2}P\Delta\right)$, the increment of strain energy would be $\frac{1}{2}dP_i\,d\Delta_i$. This quantity, however, is a second-order differential and may be neglected. Further application of the loads $P_1$, $P_2$, $\ldots$, $P_n$, which displace the body $\Delta_1$, $\Delta_2, \ldots, \Delta_n$, yields the strain energy.

$$U_i + dU_i = U_i + dP_i\Delta_i \tag{9–19}$$

Here, as before, $U_i$ is the internal strain energy in the body, caused by the loads $P_1, P_2, \ldots, P_n$, and $dU_i = dP_i\Delta_i$ is the *additional* strain energy caused by $dP_i$. (Eq. 9–4, $U_e = P\Delta'$.)

In summary, then, Eq. 9–18 represents the strain energy in the body determined by first applying the loads $P_1$, $P_2$, $\ldots$, $P_n$, *then* $dP_i$, and Eq. 9–19 represents the strain energy determined by first applying $dP_i$ and *then* the loads $P_1$, $P_2$, $\ldots$, $P_n$. Since these two equations must be equal, we require

$$\Delta_i = \frac{\partial U_i}{\partial P_i} \tag{9–20}$$

## Castigliano's Theorem for Trusses

The strain energy for a member of a truss is given by Eq. 9–9, $U_i = N^2L/2AE$. Substituting this equation into Eq. 9–20 and omitting the subscript $i$, we have

$$\Delta = \frac{\partial}{\partial P}\sum \frac{N^2L}{2AE}$$

It is generally easier to perform the differentiation prior to summation. In the general case $L$, $A$, and $E$ are constant for a given member, and therefore we may write

$$\Delta = \sum N\left(\frac{\partial N}{\partial P}\right)\frac{L}{AE} \qquad (9\text{–}21)$$

where

$\Delta$ = external joint displacement of the truss.

$P$ = external force applied to the truss joint in the direction of $\Delta$.

$N$ = internal force in a member caused by *both* the force $P$ and the loads on the truss.

$L$ = length of a member.

$A$ = cross-sectional area of a member.

$E$ = modulus of elasticity of a member.

# Structural Analysis & Design I

## Procedure for Analysis

The following procedure provides a method that may be used to determine the displacement of any joint of a truss using Castigliano's theorem.

### External Force *P*

- Place a force $P$ on the truss at the joint where the desired displacement is to be determined. This force is assumed to have a *variable magnitude* in order to obtain the change $\partial N/\partial P$. Be sure **P** is directed along the line of action of the displacement.

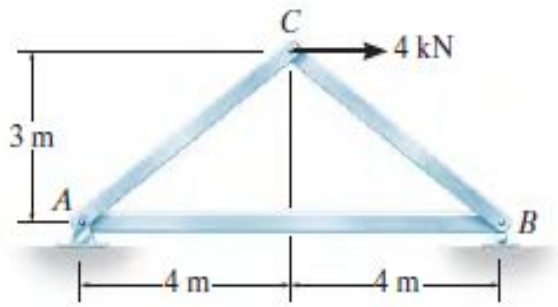
### Internal Forces *N*

- Determine the force $N$ in each member caused by both the real (numerical) loads and the variable force $P$. Assume tensile forces are positive and compressive forces are negative.
- Compute the respective partial derivative $\partial N/\partial P$ for each member.
- After $N$ and $\partial N/\partial P$ have been determined, assign $P$ its numerical value if it has replaced a real force on the truss. Otherwise, set $P$ equal to zero.

### Castigliano's Theorem

- Apply Castigliano's theorem to determine the desired displacement $\Delta$. It is important to retain the algebraic signs for corresponding values of $N$ and $\partial N/\partial P$ when substituting these terms into the equation.
- If the resultant sum $\Sigma N(\partial N/\partial P)L/AE$ is positive, $\Delta$ is in the same direction as $P$. If a negative value results, $\Delta$ is opposite to $P$.

# Structural Analysis & Design I



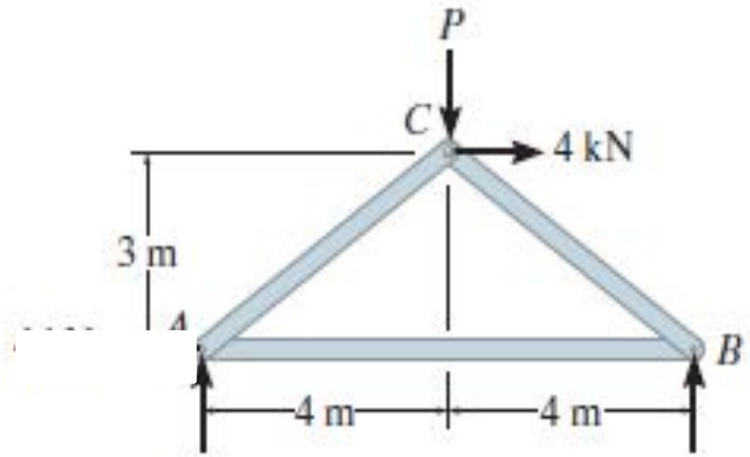
(a)

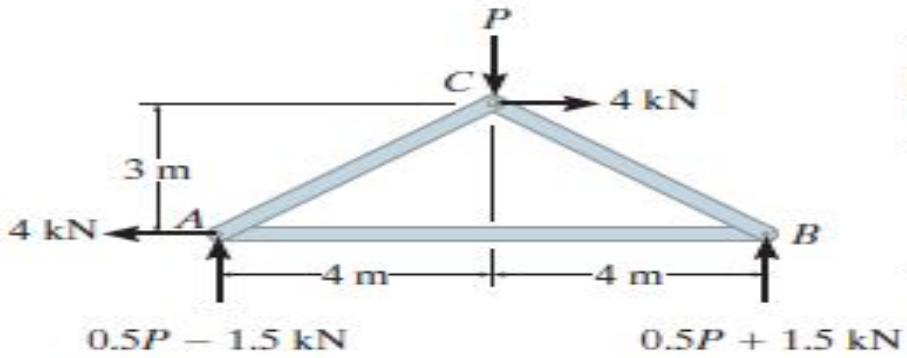
Determine the vertical displacement of joint $C$ of the truss shown in Fig. 9–11*a*. The cross-sectional area of each member is $A = 400\ \text{mm}^2$ and $E = 200\ \text{GPa}$.

## SOLUTION

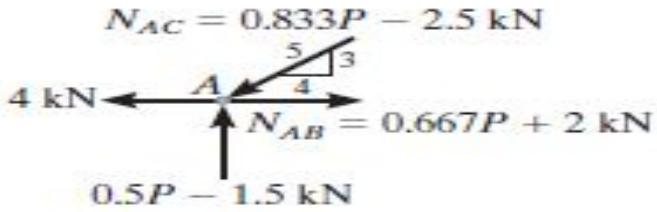
**External Force *P*.** A vertical force **P** is applied to the truss at joint $C$, since this is where the vertical displacement is to be determined, Fig. 9–11*b*.

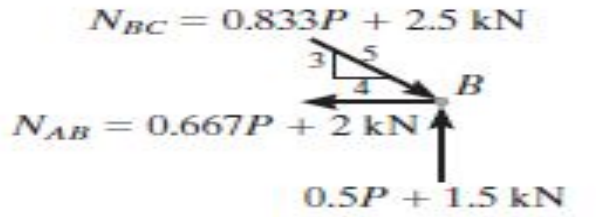
**Internal Forces *N*.** The reactions at the truss supports at $A$ and $B$ are determined and the results are shown in Fig. 9–11*b*. Using the method of joints, the $N$ forces in each member are determined, Fig. 9–11*c*.* For convenience, these results along with the partial derivatives $\partial N/\partial P$ are listed in tabular form as follows:



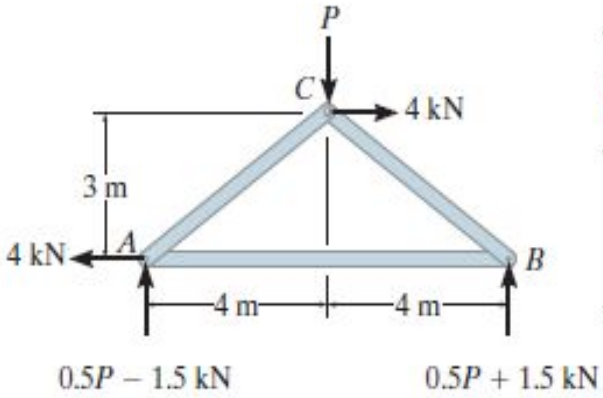

P
C
4 kN
3 m
4 kN
A
B
4 m
4 m
0.5P − 1.5 kN
0.5P + 1.5 kN

(b)


$N_{AC} = 0.833P - 2.5$ kN
5
3
4
4 kN
A
$N_{AB} = 0.667P + 2$ kN
$0.5P - 1.5$ kN


$N_{BC} = 0.833P + 2.5$ kN
3
5
4
B
$N_{AB} = 0.667P + 2$ kN
$0.5P + 1.5$ kN

(c)




(b)

| Member | $N$ | $\frac{\partial N}{\partial P}$ | $N\,(P=0)$ | $L$ | $N\left(\frac{\partial N}{\partial P}\right)L$ |
|---|---|---|---|---|---|
| $AB$ | $0.667P + 2$ | 0.667 | 2 | 8 | 10.67 |
| $AC$ | $-(0.833P - 2.5)$ | −0.833 | 2.5 | 5 | −10.42 |
| $BC$ | $-(0.833P + 2.5)$ | −0.833 | −2.5 | 5 | 10.42 |

$\Sigma = 10.67\ \text{kN}\cdot\text{m}$

Since $P$ does not actually exist as a real load on the truss, we require $P = 0$ in the table above.

**Castigliano's Theorem.** Applying Eq. 9–21, we have

$$\Delta_{C_v} = \sum N\left(\frac{\partial N}{\partial P}\right)\frac{L}{AE} = \frac{10.67\ \text{kN}\cdot\text{m}}{AE}$$

Substituting $A = 400\ \text{mm}^2 = 400(10^{-6})\ \text{m}^2$, $E = 200\ \text{GPa} = 200(10^9)\ \text{Pa}$, and converting the units of $N$ from kN to N, we have

$$\Delta_{C_v} = \frac{10.67(10^3)\ \text{N}\cdot\text{m}}{400(10^{-6})\ \text{m}^2(200(10^9)\ \text{N/m}^2)} = 0.000133\ \text{m} = 0.133\ \text{mm}$$

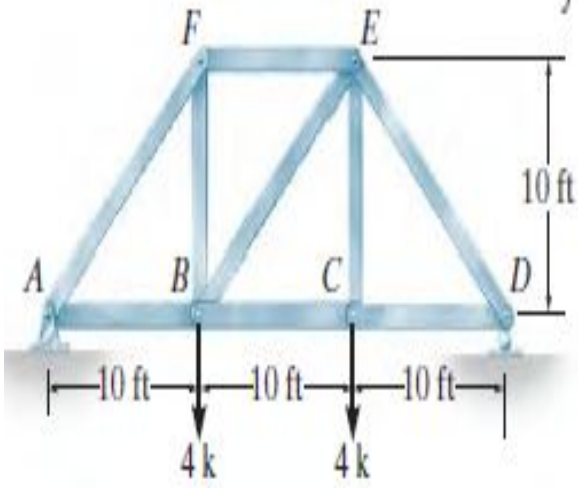
*Ans.*

# Structural Analysis & Design I

Determine the vertical displacement of joint $C$ of the truss shown in Fig. 9–13*a*. Assume that $A = 0.5\ \text{in}^2$ and $E = 29(10^3)$ ksi.

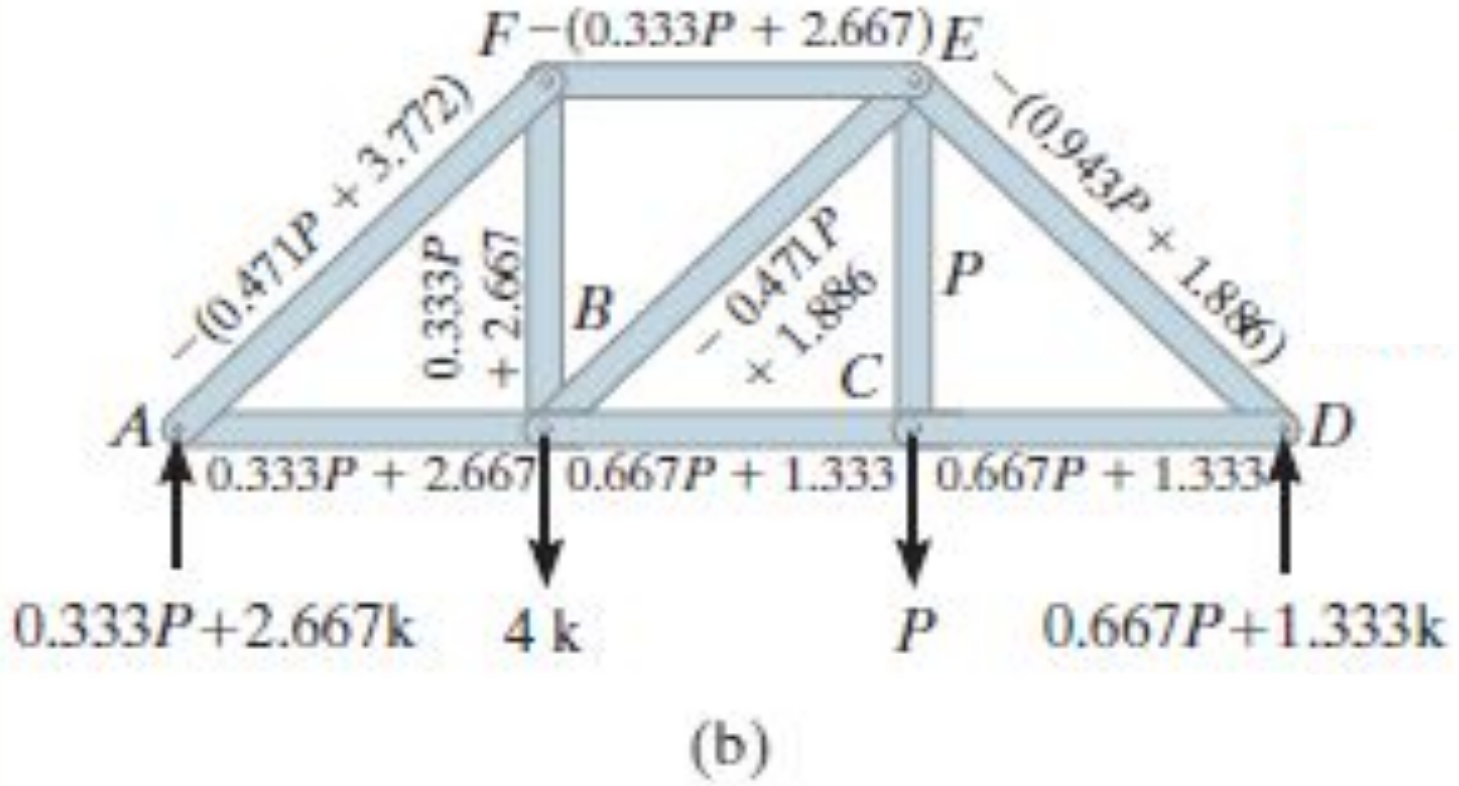
SOLUTION

**External Force P.** The 4-k force at $C$ is replaced with a *variable force* $P$ at joint $C$, Fig. 9–13*b*.




**Internal Forces N.** The method of joints is used to determine the force $N$ in each member of the truss. The results are summarized in Fig. 9–13*b*. Here $P = 4$ k when we apply Eq. 9–21. The required data can be arranged in tabulated form as follows:

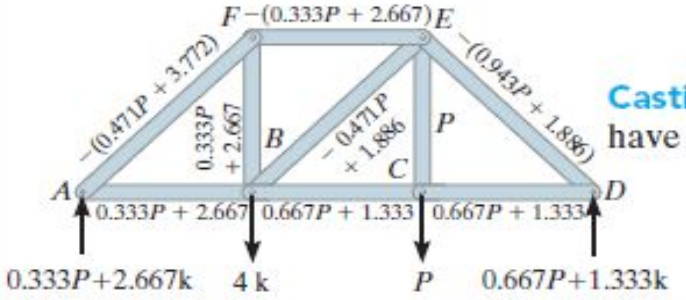
# Structural Analysis & Design I




(b)

| Member | $N$ | $\frac{\partial N}{\partial P}$ | $N\,(P = 4\text{ k})$ | $L$ | $N\left(\frac{\partial N}{\partial P}\right)L$ |
|---|---|---|---|---|---|
| $AB$ | $0.333P + 2.667$ | 0.333 | 4 | 10 | 13.33 |
| $BC$ | $0.667P + 1.333$ | 0.667 | 4 | 10 | 26.67 |
| $CD$ | $0.667P + 1.333$ | 0.667 | 4 | 10 | 26.67 |
| $DE$ | $-(0.943P + 1.886)$ | −0.943 | −5.66 | 14.14 | 75.42 |
| $EF$ | $-(0.333P + 2.667)$ | −0.333 | −4 | 10 | 13.33 |
| $FA$ | $-(0.471P + 3.771)$ | −0.471 | −5.66 | 14.14 | 37.71 |
| $BF$ | $0.333P + 2.667$ | 0.333 | 4 | 10 | 13.33 |
| $BE$ | $-0.471P + 1.886$ | −0.471 | 0 | 14.14 | 0 |
| $CE$ | $P$ | 1 | 4 | 10 | 40 |

$\Sigma = 246.47\text{ k}\cdot\text{ft}$




(b)

Fig. 9–13

**Castigliano's Theorem.** Substituting the data into Eq. 9–21, we have

$$\Delta_{C_v} = \sum N\left(\frac{\partial N}{\partial P}\right)\frac{L}{AE} = \frac{246.47\text{ k}\cdot\text{ft}}{AE}$$

Converting the units of member length to inches and substituting the numerical value for $AE$, we have

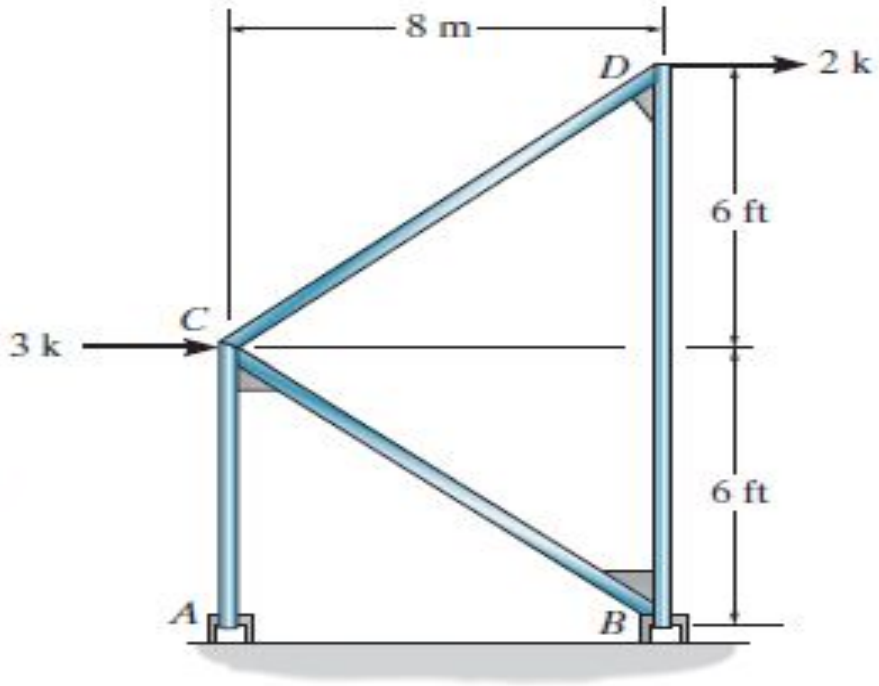
$$\Delta_{C_v} = \frac{(246.47\text{ k}\cdot\text{ft})(12\text{ in./ft})}{(0.5\text{ in}^2)(29(10^3)\text{ k/in}^2)} = 0.204\text{ in.} \qquad \textit{Ans.}$$

The similarity between this solution and that of the virtual-work method, Example 9–1, should be noted.

# Structural Analysis & Design I

## Assignment: 2

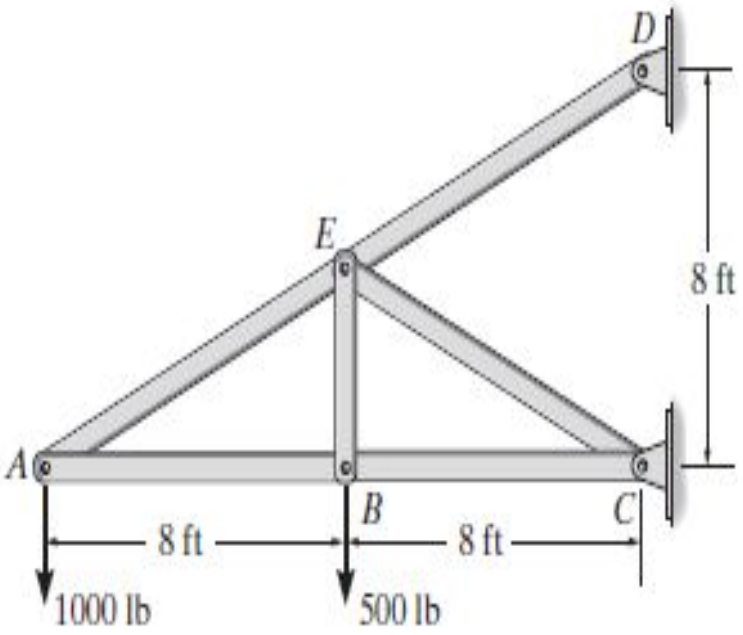
Determine the horizontal displacement of joint $D$. Assume the members are pin connected at their end points. $AE$ is constant. Use the method of virtual work. and Castigliano



# Structural Analysis & Design I

## Assignment: 3

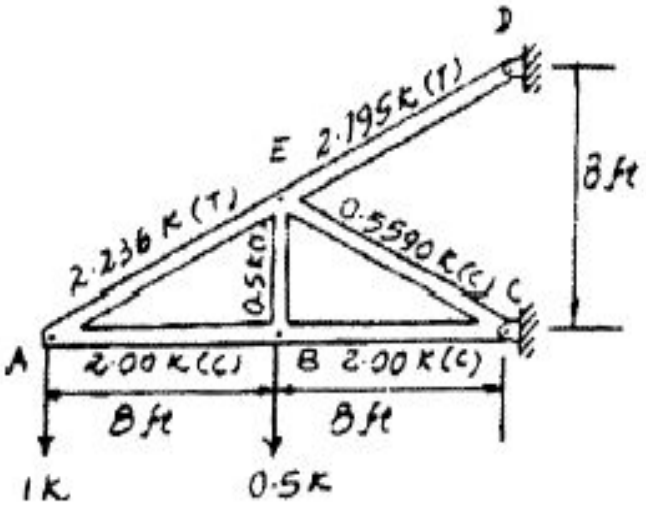
Determine the vertical displacement of joint $A$. Assume the members are pin connected at their end points. Take $A = 2\ \text{in}^2$ and $E = 29\,(10^3)$ for each member. Use the method of virtual work. and Castigliano's theory.

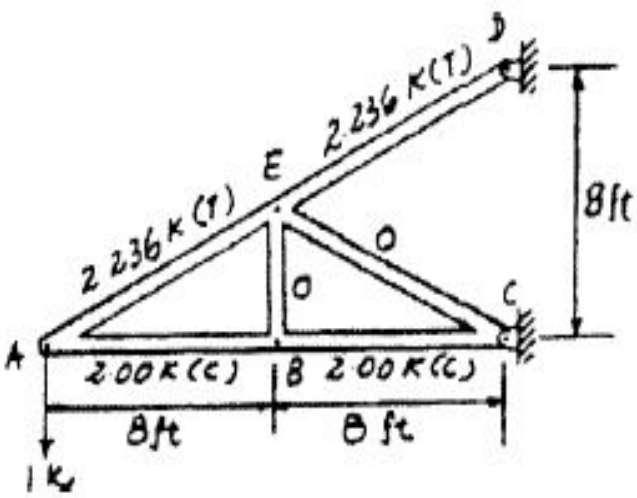


# Structural Analysis & Design I

$$\Delta_{A_v} = \sum \frac{nNL}{AE} = \frac{1}{AE}[2(-2.00)(-2.00)(8) + (2.236)(2.236)(8.944) + (2.236)(2.795)(8.944)]$$

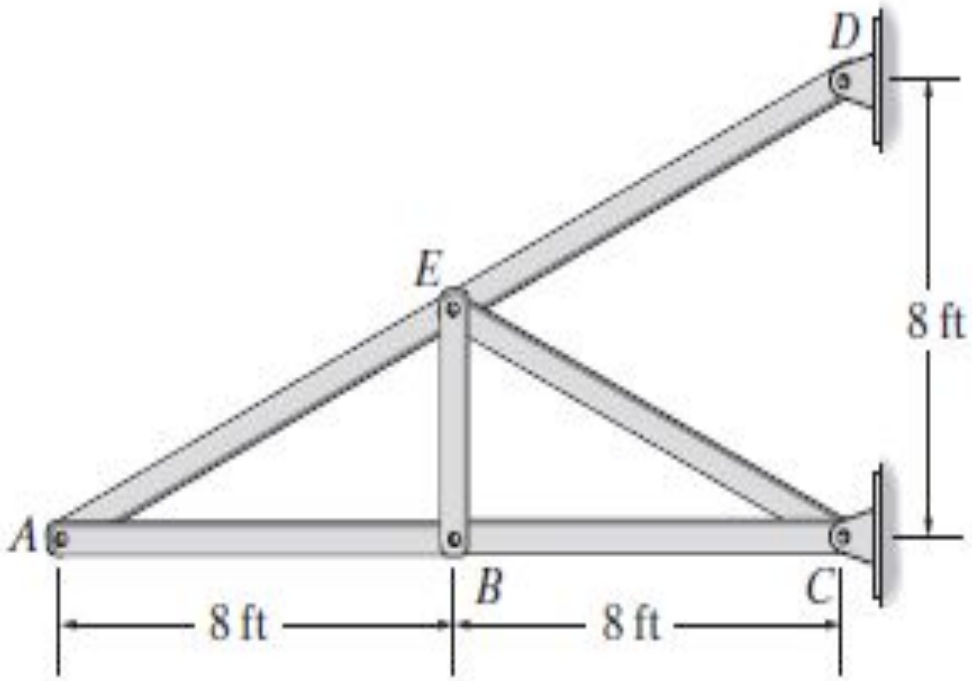
$$= \frac{164.62(12)}{(2)(29)(10^3)} = 0.0341 \text{ in.} \downarrow \qquad \textbf{Ans.}$$



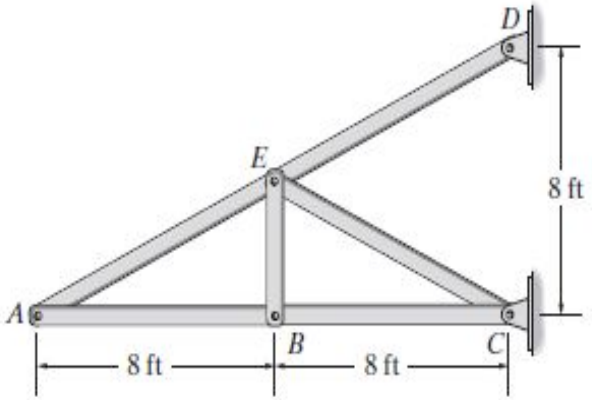
# Structural Analysis & Design I

Determine the vertical displacement of joint $A$ if members $AB$ and $BC$ experience a temperature increase of $\Delta T = 200°\text{F}$. Take $A = 2\ \text{in}^2$ and $E = 29(10^3)$ ksi. Also, $\alpha = 6.60\,(10^{-6})/°\text{F}$.



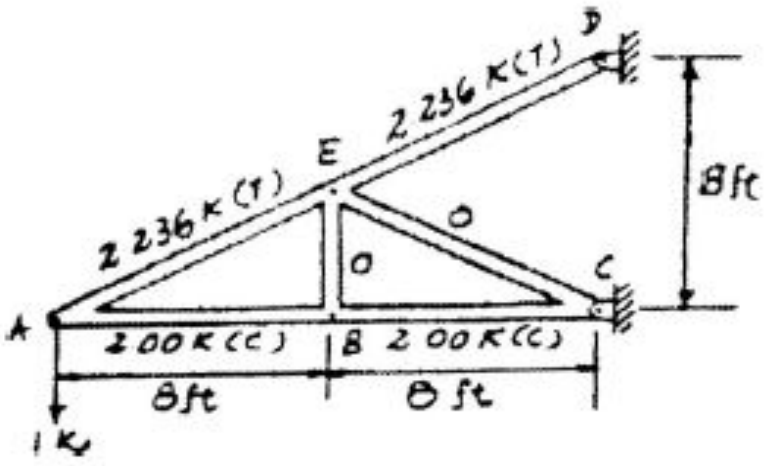
# Structural Analysis & Design I

Notice that only the member undergoing increase in length due to temperature change is considered during calculation



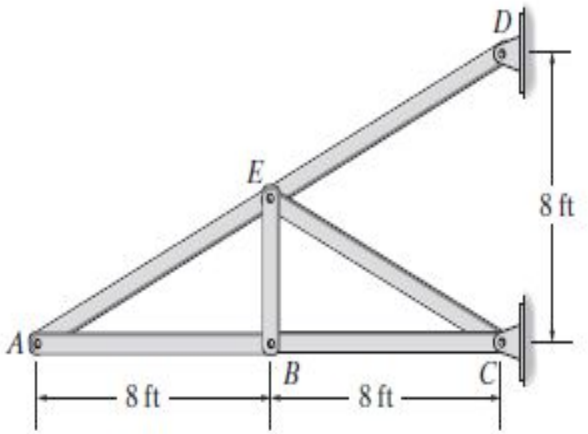

$$\Delta_{A_v} = \sum n\alpha\Delta TL = (-2)(6.60)(10^{-6})(200)(8)(12) + (-2)(6.60)(10^{-6})(200)(8)(12)$$

$$= -0.507 \text{ in.} = 0.507 \text{ in.} \uparrow \qquad \textbf{Ans.}$$



# Structural Analysis & Design I

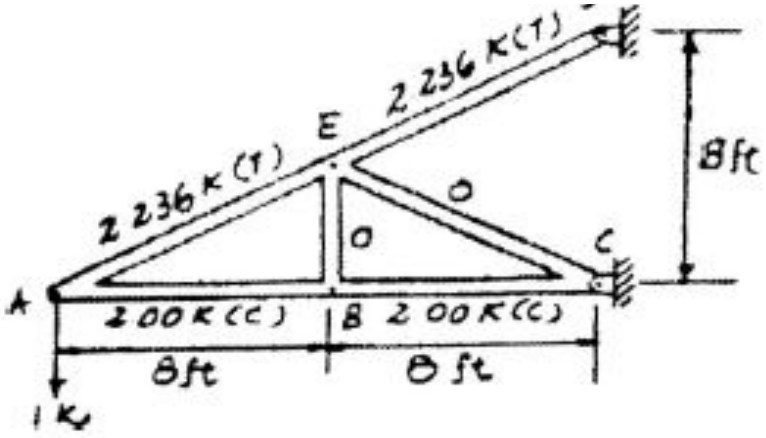
Determine the vertical displacement of joint $A$ if member $AE$ is fabricated 0.5 in. too short.




$$\Delta_{A_v} = \sum n\Delta L = (2.236)(-0.5)$$

$$= -1.12 \text{ in} = 1.12 \text{ in.} \uparrow$$

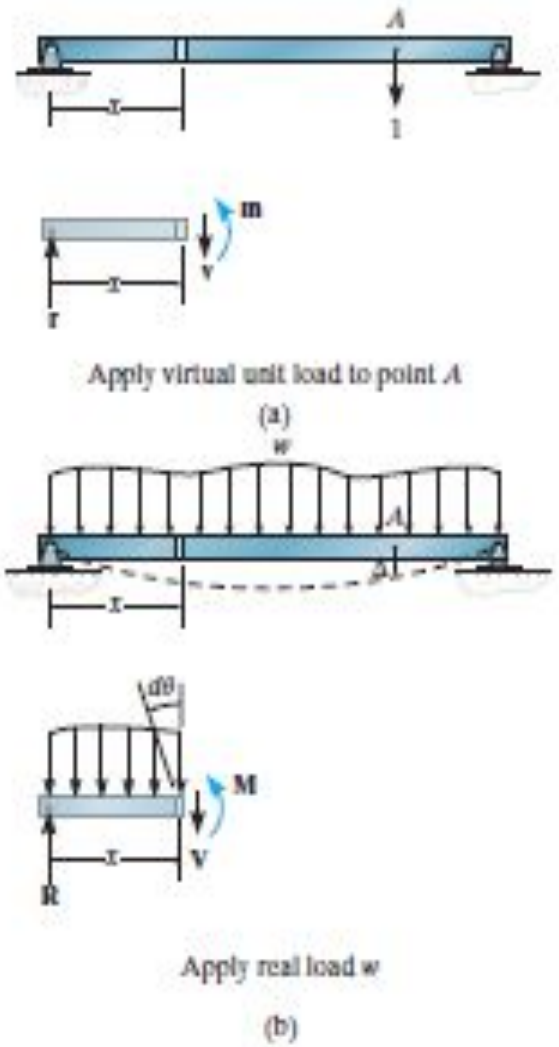
**Ans.**



# Structural Analysis & Design I

- **Assignment 4 (9.3, Hibbeler)**

# Structural Analysis & Design I




## Method of Virtual Work: Beams and Frames

The method of virtual work can also be applied to deflection problems involving beams and frames. Since strains due to *bending* are the *primary cause* of beam or frame deflections, we will discuss their effects first. Deflections due to shear, axial and torsional loadings, and temperature will be considered in Sec. 9–8.

The principle of virtual work, or more exactly, the method of virtual force, may be formulated for beam and frame deflections by considering the beam shown in Fig. 9–14*b*. Here the displacement $\Delta$ of point $A$ is to be determined. To compute $\Delta$ a virtual unit load acting in the direction of $\Delta$ is placed on the beam at $A$, and the *internal virtual moment* $\mathbf{m}$ is determined by the method of sections at an arbitrary location $x$ from the left support, Fig. 9–14*a*. When the real loads act on the beam, Fig. 9–14*b*, point $A$ is displaced $\Delta$. Provided these loads cause *linear elastic material response*, then from Eq. 8–2, the element $dx$ deforms or rotates $d\theta = (M/EI)\,dx$.* Here $M$ is the internal moment at $x$ caused by the real loads. Consequently, the *external virtual work* done by the unit load is $1 \cdot \Delta$, and the *internal virtual work* done by the moment $\mathbf{m}$ is $m\,d\theta = m(M/EI)\,dx$. Summing the effects on all the elements $dx$ along the beam requires an integration, and therefore Eq. 9–13 becomes

$$1 \cdot \Delta = \int_0^L \frac{mM}{EI}\,dx \qquad (9\text{–}22)$$

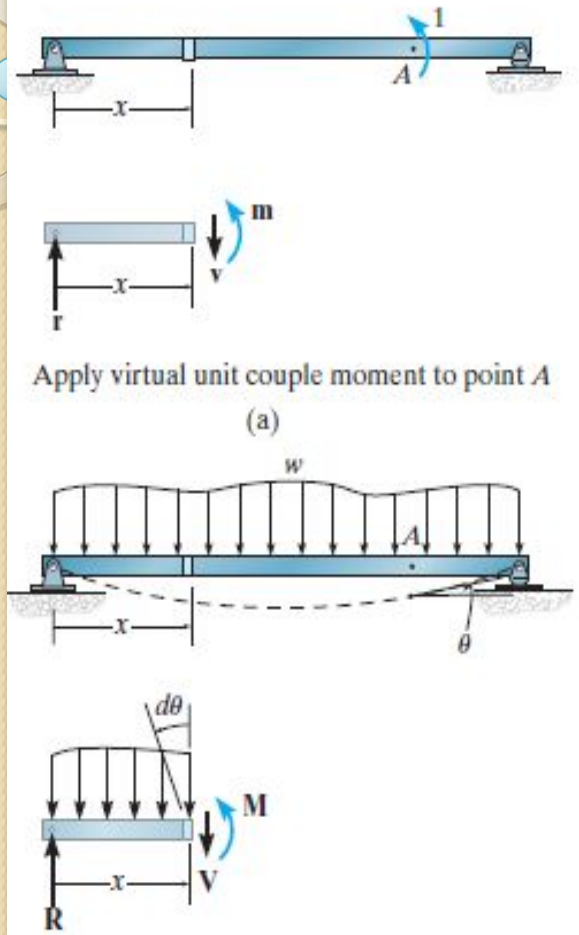



Fig. 9–15

(9–22)

where

$1$ = external virtual unit load acting on the beam or frame in the direction of $\Delta$.

$m$ = internal virtual moment in the beam or frame, expressed as a function of $x$ and caused by the external virtual unit load.

$\Delta$ = external displacement of the point caused by the real loads acting on the beam or frame.

$M$ = internal moment in the beam or frame, expressed as a function of $x$ and caused by the real loads.

$E$ = modulus of elasticity of the material.

$I$ = moment of inertia of cross-sectional area, computed about the neutral axis.

In a similar manner, if the tangent rotation or slope angle $\theta$ at a point $A$ on the beam's elastic curve is to be determined, Fig. 9–15, a unit couple moment is first applied at the point, and the corresponding internal moments $m_\theta$ have to be determined. Since the work of the unit couple is $1 \cdot \theta$, then

$$1 \cdot \theta = \int_0^L \frac{m_\theta M}{EI} dx \qquad (9\text{–}23)$$

*Recall that if the material is strained beyond its elastic limit, the principle of virtual work can still be applied, although in this case a nonlinear or plastic analysis must be used.

# Structural Analysis & Design I

**Sign Convention.** Before presenting a method for finding the internal normal force, shear force, and bending moment, we will need to establish a sign convention to define their "positive" and "negative" values.* Although the choice is arbitrary, the sign convention to be adopted here has been widely accepted in structural engineering practice, and is illustrated in Fig. 4–1*a*. On the *left-hand face* of the cut member the normal force **N** acts to the right, the internal shear force **V** acts downward, and the moment **M** acts counterclockwise. In accordance with Newton's third law, an equal but opposite normal force, shear force, and bending moment must act on the right-hand face of the member at the section. Perhaps an easy way to remember this sign convention is to isolate a small segment of the member and note that *positive normal force tends to elongate the segment,* Fig. 4–1*b*; *positive shear tends to rotate the segment clockwise,* Fig. 4–1*c*; and *positive bending moment tends to bend the segment concave upward,* so as to "hold water," Fig. 4–1*d*.

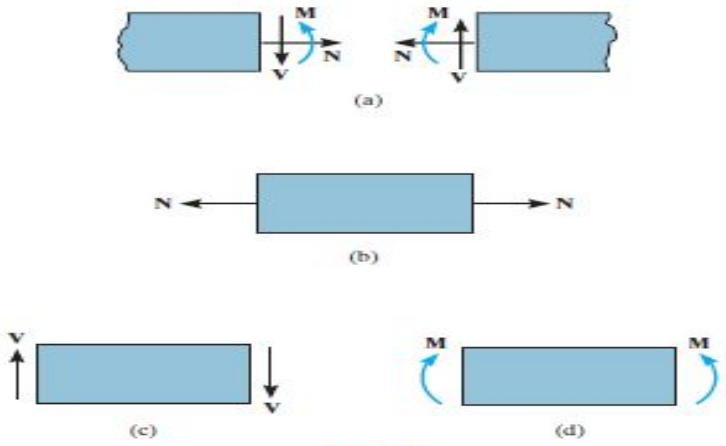


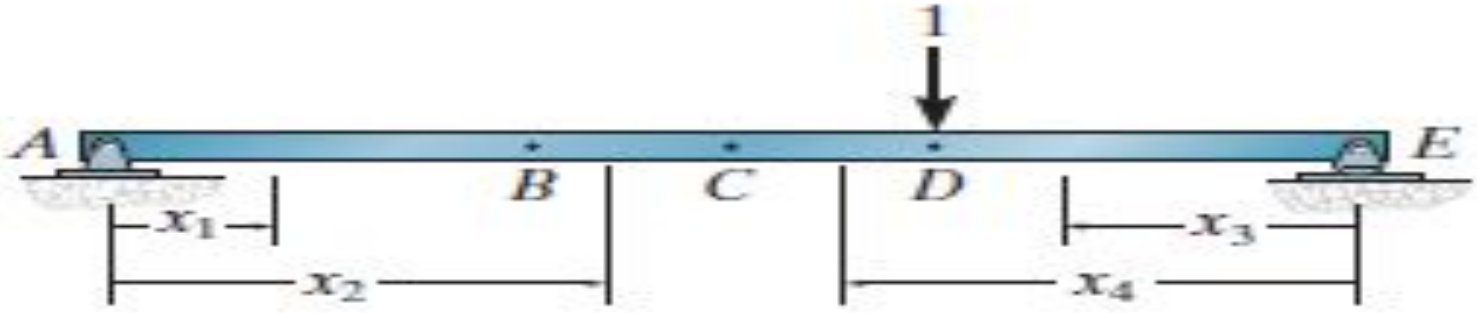

Fig. 4–1

# Structural Analysis & Design I

## Table for Evaluating $\int_0^L m\, m'\, dx$

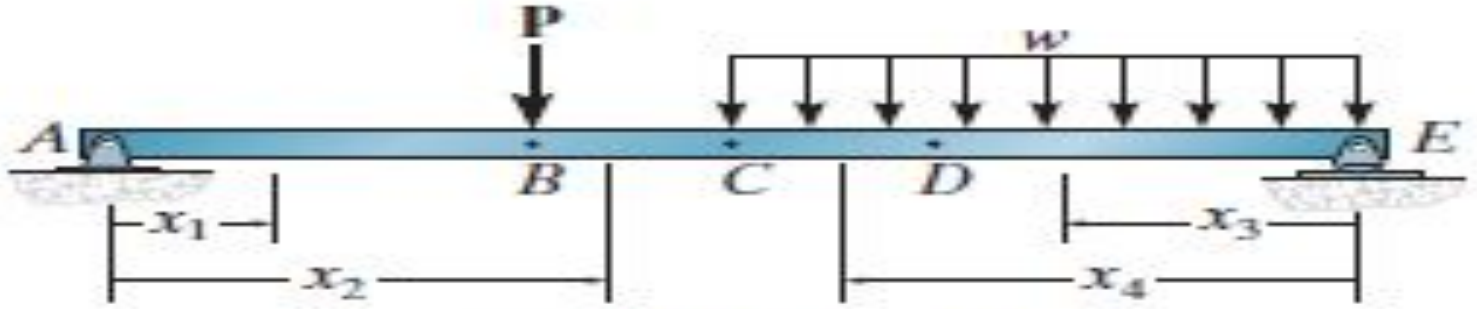
| $\int_0^L m\, m'\, dx$ | $m'$ (rectangle, $L$) | $m'$ (triangle, $L$) | $m'_1$, $m'_2$ (trapezoid, $L$) | parabola, $m'$ ($L$) |
|---|---|---|---|---|
| $m$ (rectangle, $L$) | $mm'L$ | $\frac{1}{2}mm'L$ | $\frac{1}{2}m(m'_1 + m'_2)L$ | $\frac{2}{3}mm'L$ |
| $m$ (triangle, $L$) | $\frac{1}{2}mm'L$ | $\frac{1}{3}mm'L$ | $\frac{1}{6}m(m'_1 + 2m'_2)L$ | $\frac{5}{12}mm'L$ |
| $m_1$, $m_2$ (trapezoid, $L$) | $\frac{1}{2}m'(m_1 + m_2)L$ | $\frac{1}{6}m'(m_1 + 2m_2)L$ | $\frac{1}{6}[m'_1(2m_1 + m_2) + m'_2(m_1 + 2m_2)]L$ | $\frac{1}{12}[m'(3m_1 + 5m_2)]L$ |
| $m$ (triangle, $a$, $b$, $L$) | $\frac{1}{2}mm'L$ | $\frac{1}{6}mm'(L + a)$ | $\frac{1}{6}m[m'_1(L + b) + m_2(L + a)]$ | $\frac{1}{12}mm'\left(3 + \frac{3a}{L} - \frac{a^2}{L^2}\right)L$ |
| $m$ (triangle, $L$) | $\frac{1}{2}mm'L$ | $\frac{1}{6}mm'L$ | $\frac{1}{6}m(2m'_1 + m'_2)L$ | $\frac{1}{4}mm'L$ |

# Structural Analysis & Design I




Apply virtual unit load

(a)




Apply real loads

(b)

## Virtual Moments m or mu

- Place a *unit load on the beam or frame at the point and in the* direction of the desired *displacement.*
- If the *slope is to be determined, place a unit couple moment at the* point.
- Establish appropriate *x coordinates that are valid within regions* of the beam or frame where there is no discontinuity of real or virtual load.
- With the virtual load in place, and all the real loads *removed from* the beam or frame, calculate the internal moment *m or as a* function of each *x coordinate.*
- Assume *m or acts in the conventional positive direction as* indicated in Fig. 4–1.

# Structural Analysis & Design I

## Real Moments

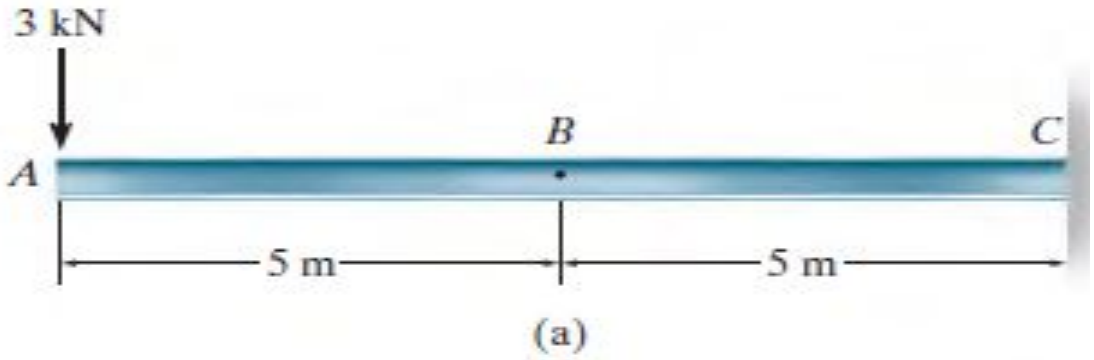
- Using the *same* $x$ coordinates as those established for $m$ or $m_\theta$, determine the internal moments $M$ caused only by the real loads.
- Since $m$ or $m_\theta$ was assumed to act in the conventional positive direction, *it is important that positive* $M$ acts in this *same* direction. This is necessary since positive or negative internal work depends upon the directional sense of load (defined by $\pm m$ or $\pm m_\theta$) and displacement (defined by $\pm M\,dx/EI$).

## Virtual-Work Equation

- Apply the equation of virtual work to determine the desired displacement $\Delta$ or rotation $\theta$. It is important to retain the algebraic sign of each integral calculated within its specified region.
- If the algebraic sum of all the integrals for the entire beam or frame is positive, $\Delta$ or $\theta$ is in the same direction as the virtual unit load or unit couple moment, respectively. If a negative value results, the direction of $\Delta$ or $\theta$ is opposite to that of the unit load or unit couple moment.

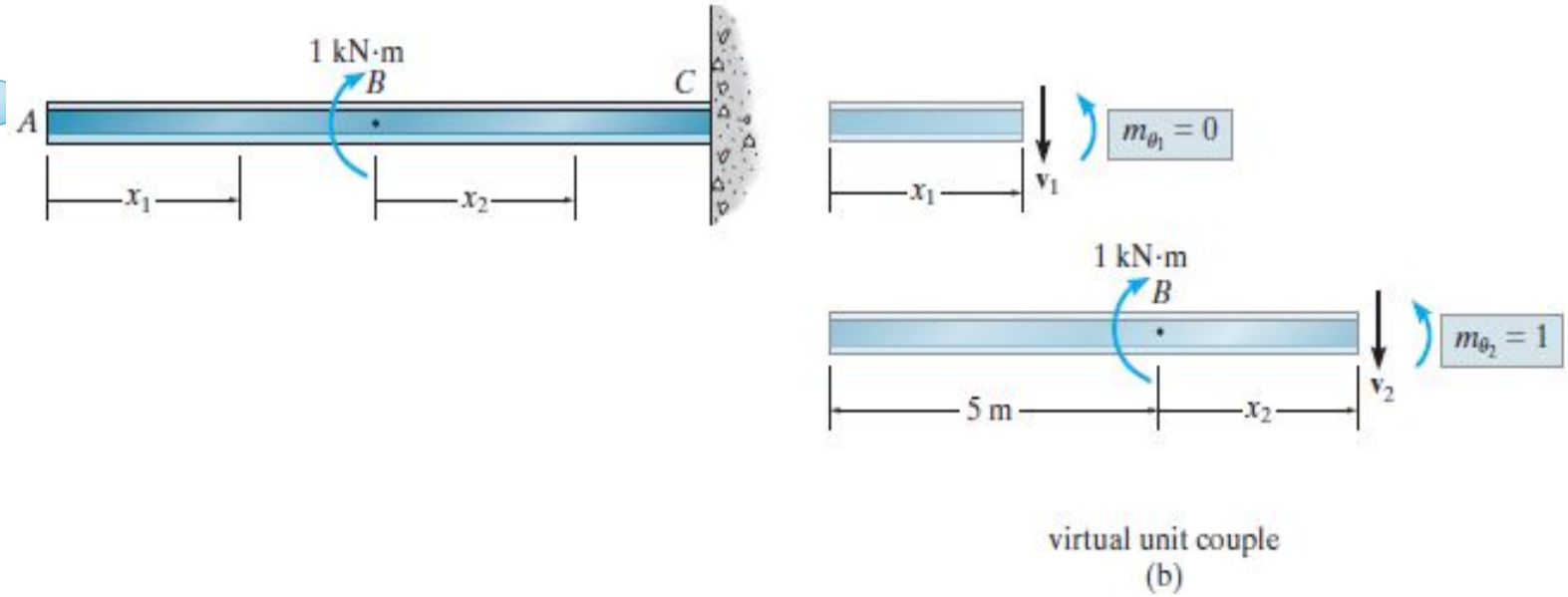
# Structural Analysis & Design I

Determine the slope $\theta$ at point $B$ of the steel beam shown in Fig. 9–18*a*. Take $E = 200\ \text{GPa}$, $I = 60(10^6)\ \text{mm}^4$.




(a)

# Structural Analysis & Design I

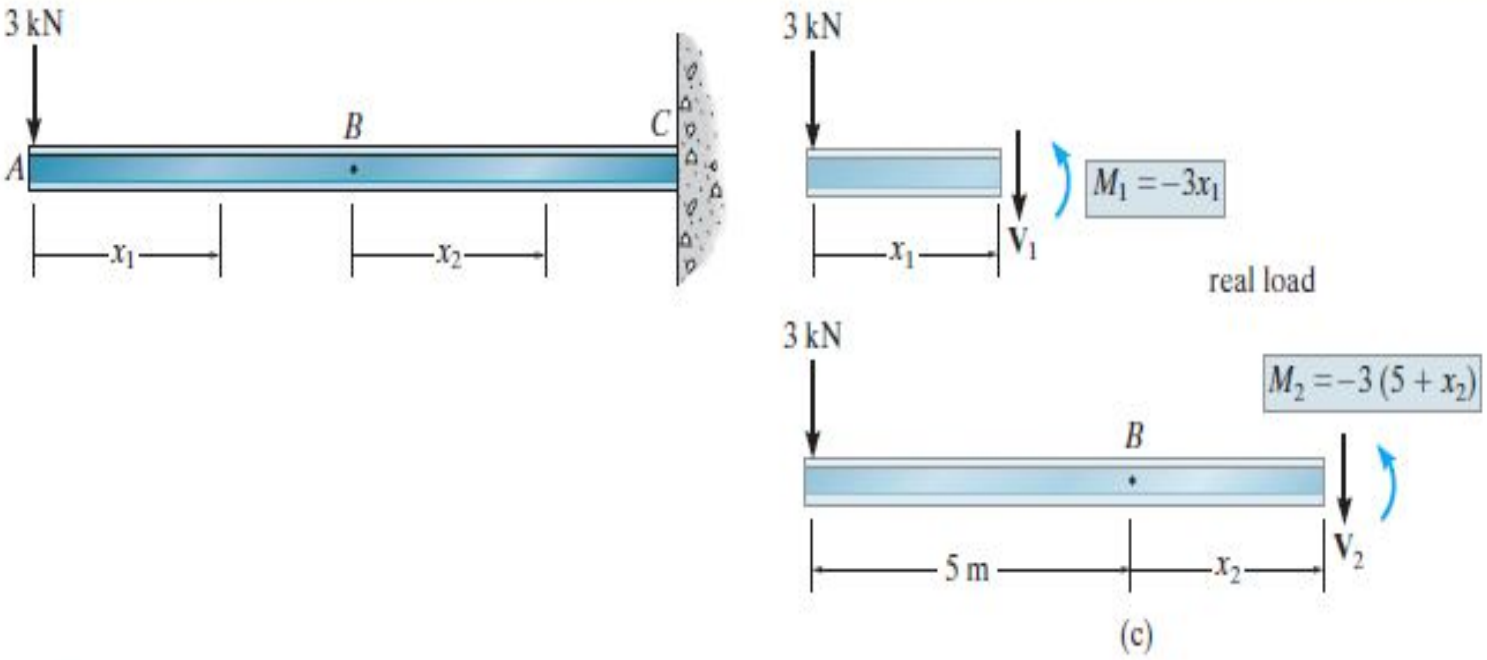



virtual unit couple
(b)

## Virtual Moment (m)

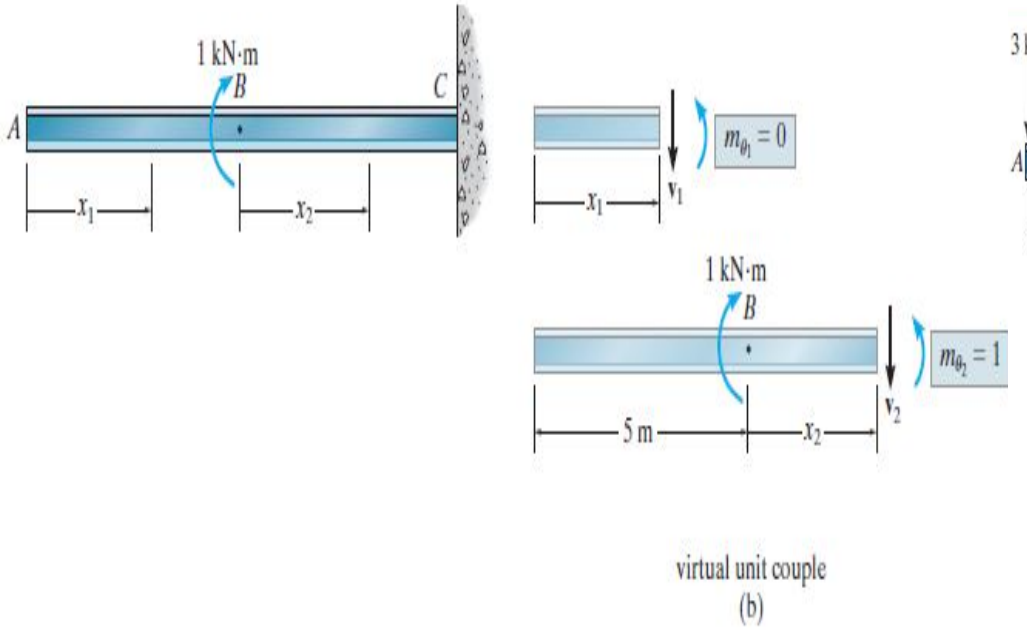
**The slope at *B is determined by placing a virtual* unit couple moment of at *B, Fig. 9–18b. Here two x coordinates* must be selected in order to determine the total virtual strain energy in the beam. Coordinate accounts for the strain energy within segment *AB and coordinate accounts for that in segment BC. The internal* moments within each of these segments are computed using the method of sections as shown in Fig. 9–18*b.***

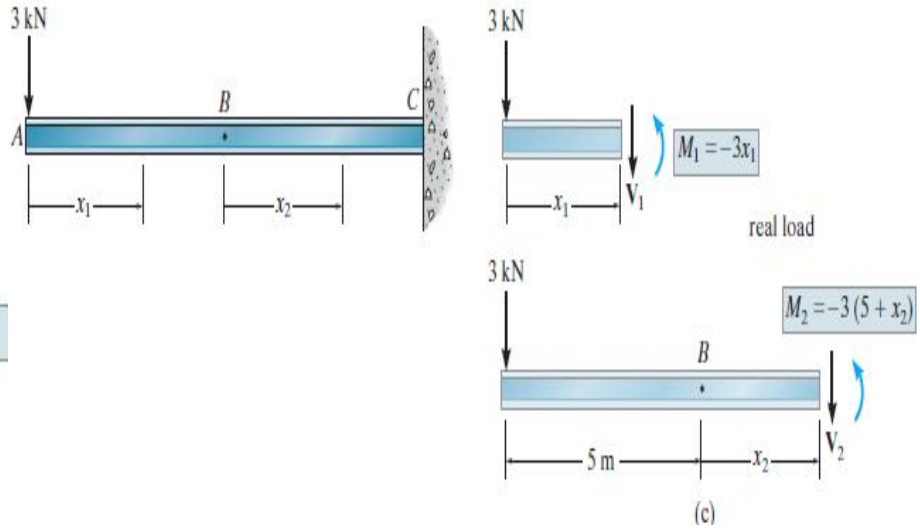
# Structural Analysis & Design I




**Real Moments *M*.** Using the *same* coordinates $x_1$ and $x_2$, the internal moments $M$ are computed as shown in Fig. 9–18*c*.

# Structural Analysis & Design I




**Virtual-Work Equation.** The slope at $B$ is thus given by

$$1 \cdot \theta_B = \int_0^L \frac{m_\theta M}{EI} dx$$

$$= \int_0^5 \frac{(0)(-3x_1)\, dx_1}{EI} + \int_0^5 \frac{(1)[-3(5 + x_2)]\, dx_2}{EI}$$

$$\theta_B = \frac{-112.5\ \text{kN} \cdot \text{m}^2}{EI} \qquad (1)$$

(1)

We can also evaluate the integrals $\int m_\theta M\, dx$ graphically, using the table given on the inside front cover of the book. To do so it is first necessary to draw the moment diagrams for the beams in Figs. 9–18*b* and 9–18*c*. These are shown in Figs. 9–18*d* and 9–18*e*, respectively. Since there is no moment $m$ for $0 \le x < 5$ m, we use only the shaded rectangular and trapezoidal areas to evaluate the integral. Finding these shapes in the appropriate row and column of the table, we have

$$\int_5^{10} m_\theta M\, dx = \tfrac{1}{2} m_\theta (M_1 + M_2) L = \tfrac{1}{2}(1)(-15 - 30)5$$

$$= -112.5\ \text{kN}^2 \cdot \text{m}^3$$

This is the same value as that determined in Eq. 1. Thus,

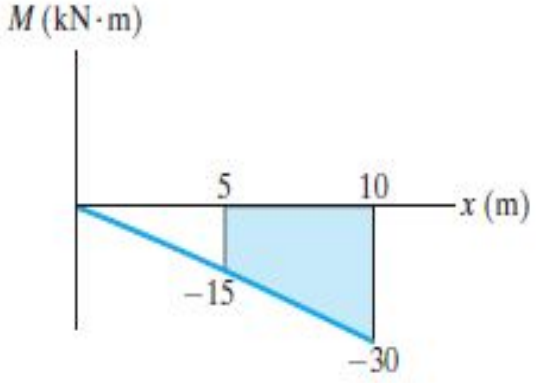
$$(1\ \text{kN} \cdot \text{m}) \cdot \theta_B = \frac{-112.5\ \text{kN}^2 \cdot \text{m}^3}{200(10^6)\ \text{kN/m}^2 [60(10^6)\ \text{mm}^4](10^{-12}\ \text{m}^4/\text{mm}^4)}$$

$$\theta_B = -0.00938\ \text{rad} \qquad \textit{Ans.}$$

The *negative sign* indicates $\theta_B$ is *opposite* to the direction of the virtual couple moment shown in Fig. 9–18*b*.



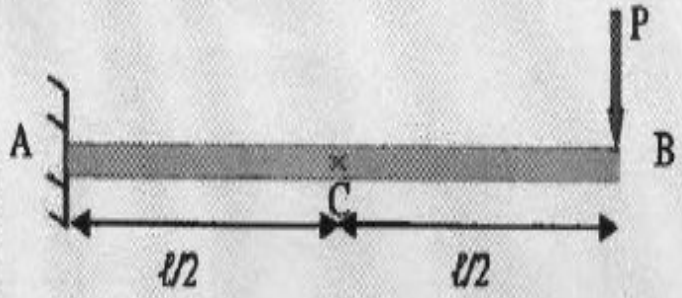

(d)



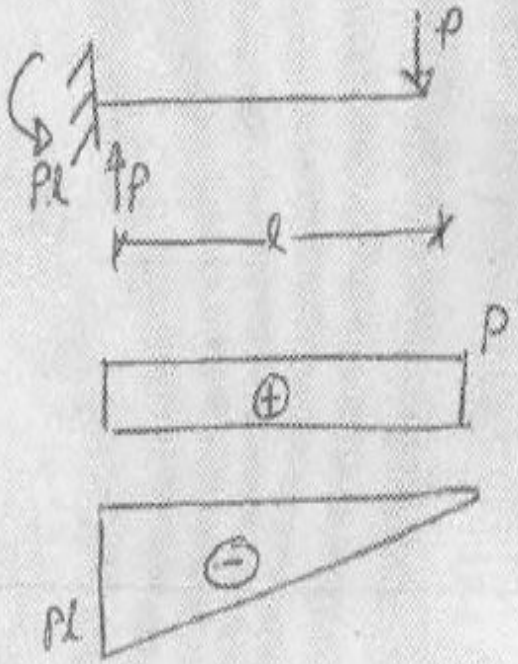

(e)

## 5.8 Examples using Virtual Work Method

**Example 3:** Calculate the rotation at B and the deflection at C. EI is constant

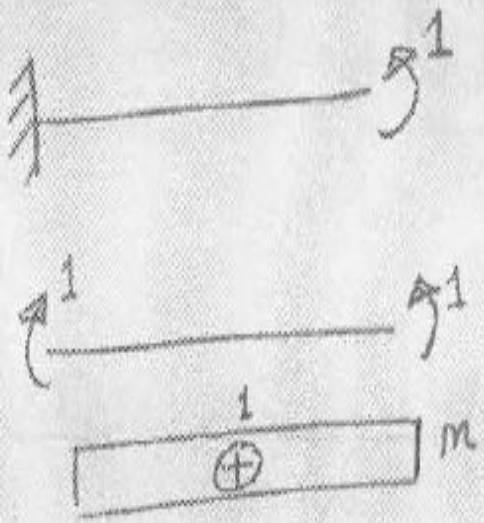



Real structure



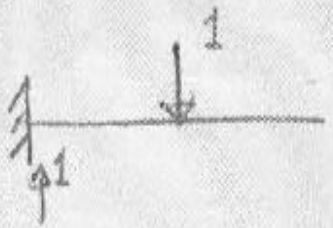

Rotation at B, $\theta_B$

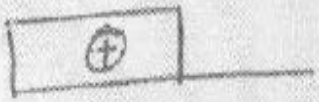
Virtual structure



# Structural Analysis & Design I

Deflection at C



$$\theta_{B3} = \int_0^{\ell} \frac{mM}{EI}\,dx = \frac{1}{EI}\left[\text{(1)} \quad \text{(Pℓ)}\right]$$

$$= \frac{1}{EI}\left[(1\times\ell)\left(\tfrac{1}{2}\times\ell\times -P\ell\right)\right]$$

(1xlx(-pl))/2

$$\theta_{B3} = -\frac{P\ell^2}{2EI}$$

m

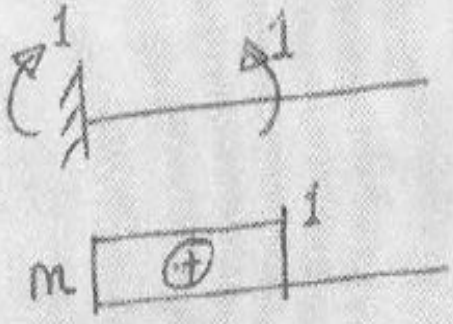



$$\Delta_C = \int_0^{\ell} \frac{mM}{EI}\,dx = \int_0^{\ell/2} \frac{}{EI} = \int_0^{\ell/2} \frac{-\frac{Px}{2}}{EI} + \int_0^{\ell/2} \frac{0}{EI}$$

$$= \frac{1}{EI}\left[\frac{1}{6}(-\ell/2)\left(-\frac{P\ell}{2} + (-2P\ell)\frac{\ell}{2}\right)\right] = \frac{5P\ell^3}{48EI} \quad (+ve)$$

# Structural Analysis & Design I

Rotation at C, $\theta_C$




$$\theta_C = \int_0^L \frac{m \cdot M}{EI} = \int_0^{L/2} \frac{1 \cdot (-Pt)}{EI} + \int_0^{L/2} \frac{0 \cdot (-PL/2)}{EI}$$

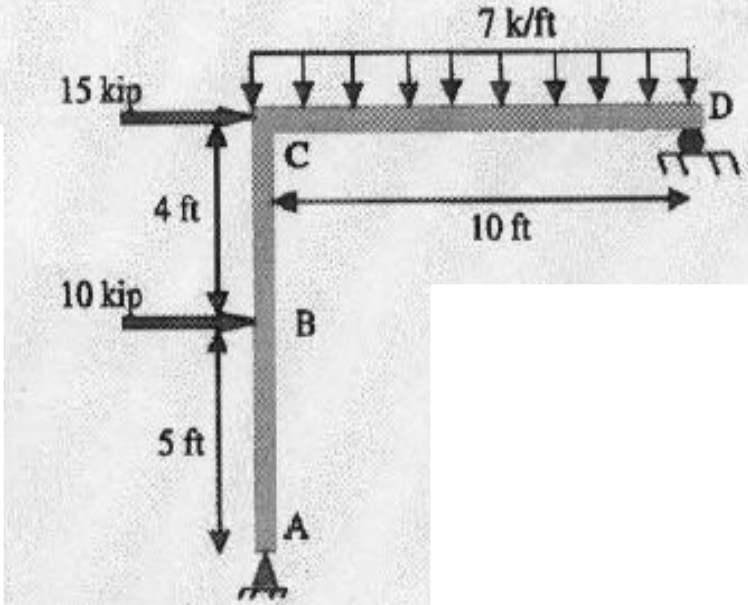
$$= \frac{1}{EI}\left[\frac{1}{2}(1)\left(-\frac{PL}{2} - PL\right)\left(\frac{L}{2}\right)\right] = -\frac{3PL^2}{8EI}$$

# Structural Analysis & Design I

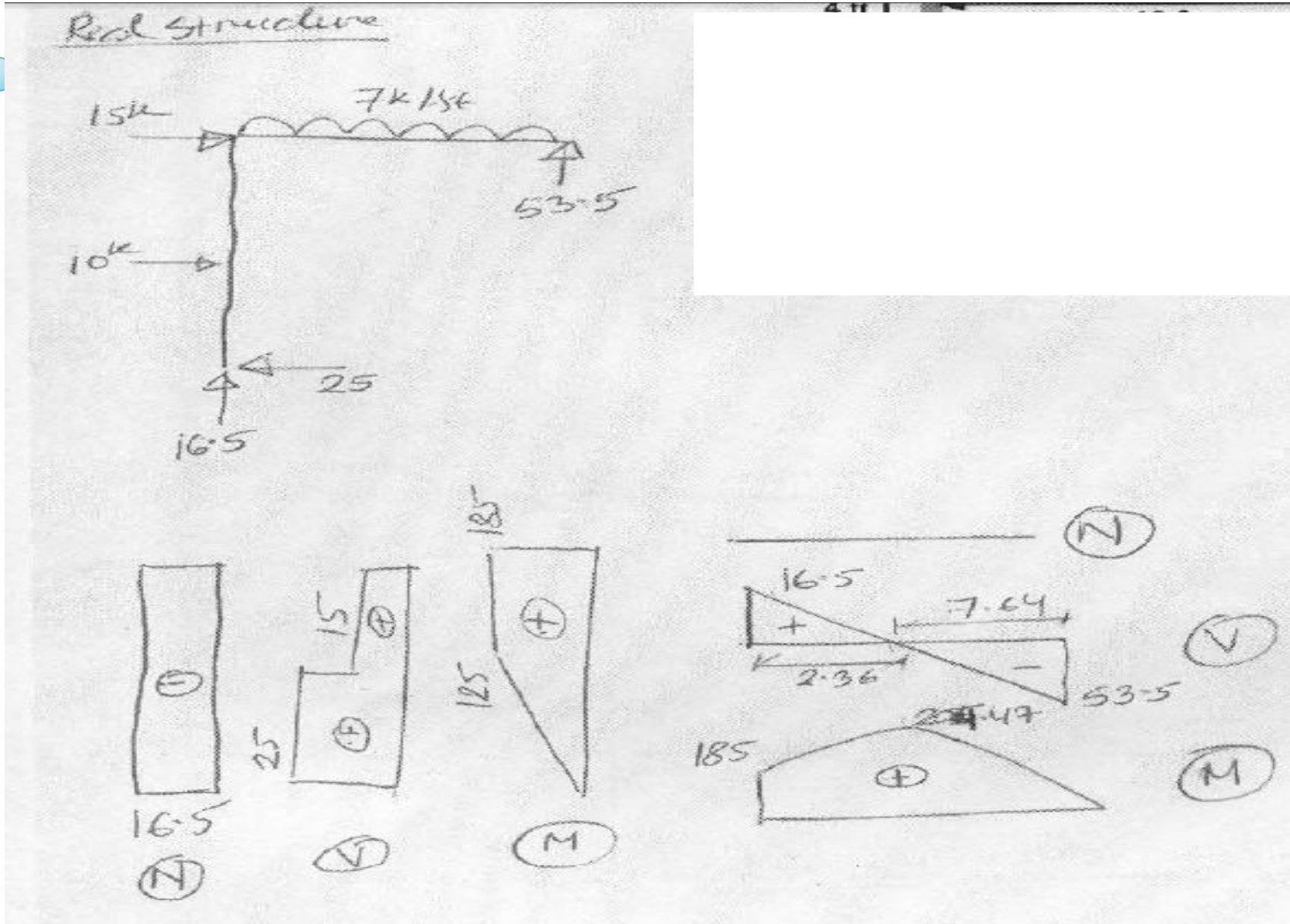
**Example 7:** The frame below has the following material and geometrical properties: A= 13 $in^2$, I= 600 $in^4$ and E= 29000 ksi.

a- Determine the horizontal displacement of point C due to the axial and flexural forces

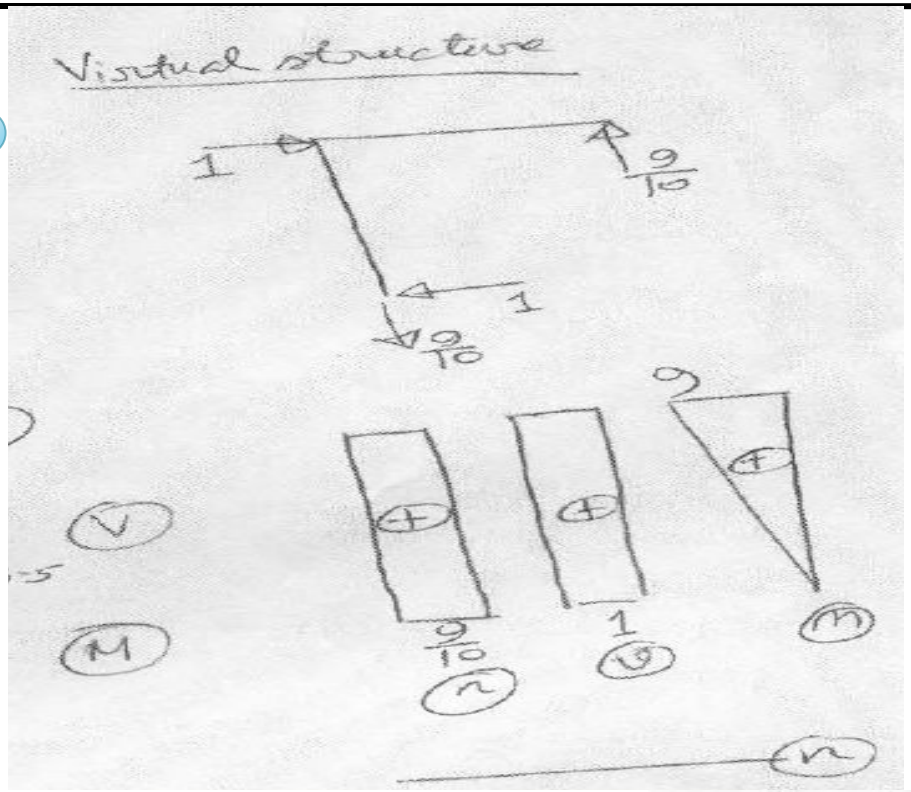
b- Calculate the slope at point D due to flexural forces only

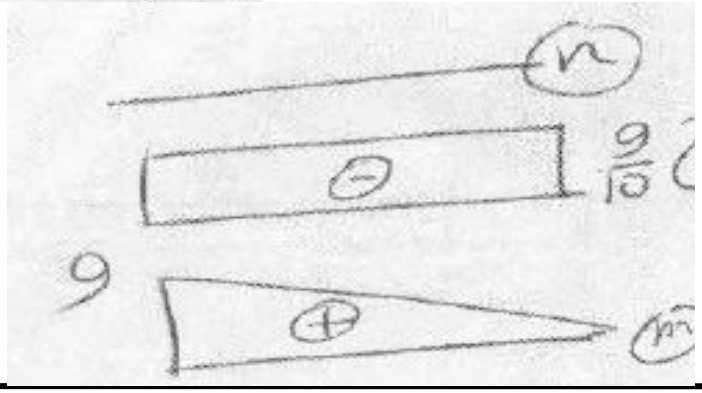


# Structural Analysis & Design I



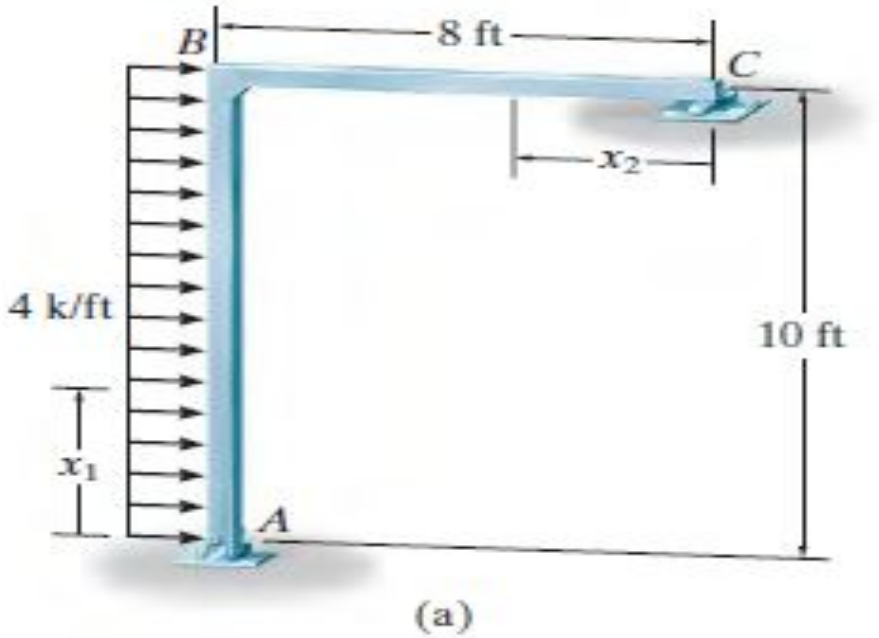
# Structural Analysis & Design I



# Structural Analysis & Design I

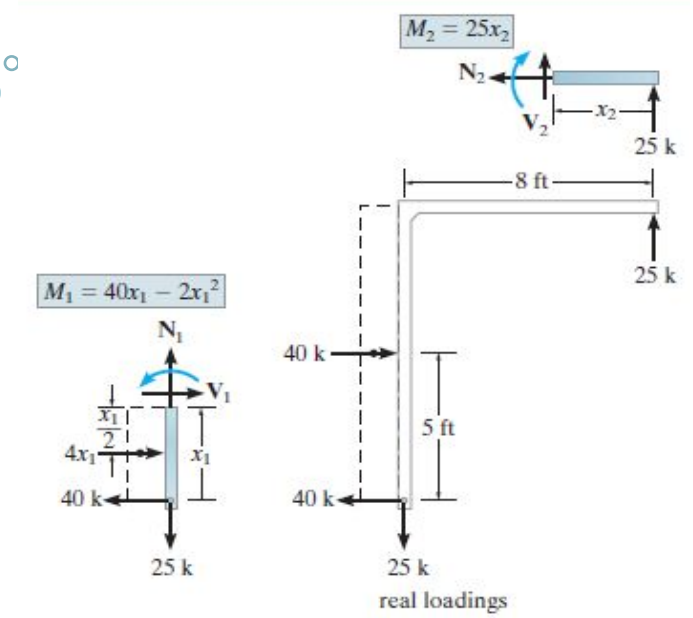
Determine the horizontal displacement of point $C$ on the frame shown in Fig. 9–20*a*. Take $E = 29(10^3)$ ksi and $I = 600\ \text{in}^4$ for both members.

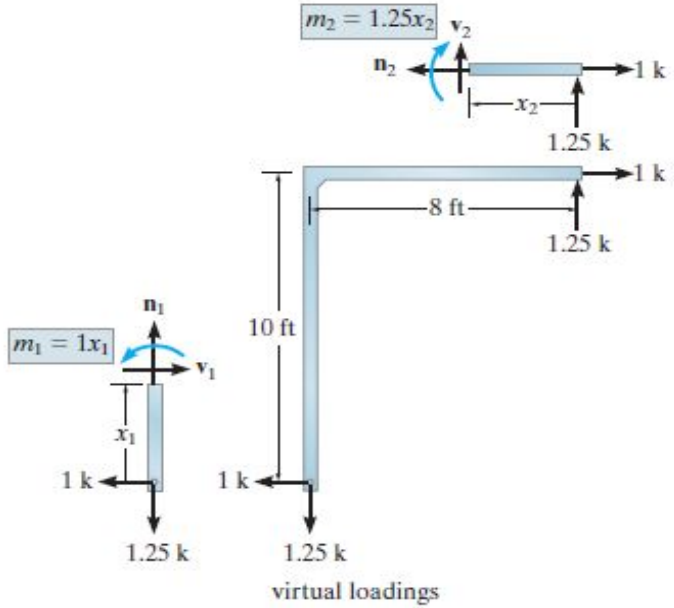


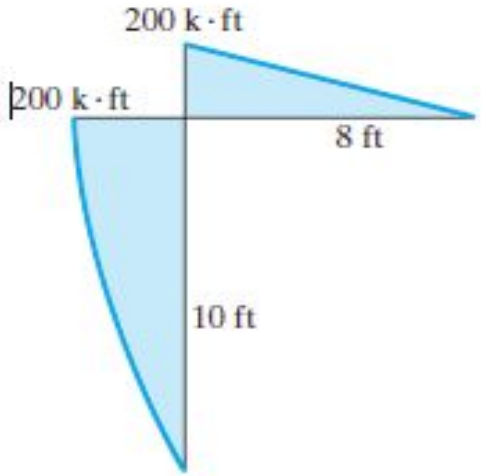

(a)

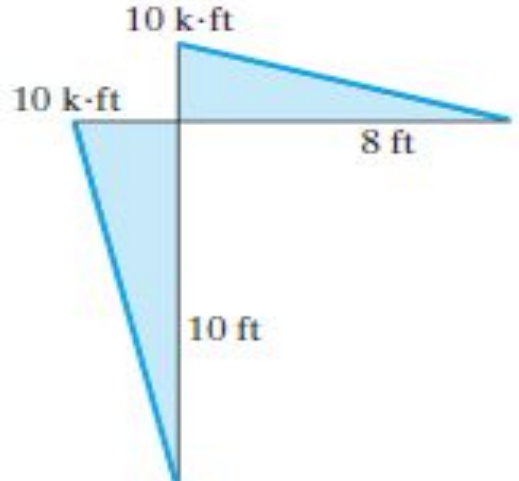
Fig. 9–20

# Structural Analysis & Design I



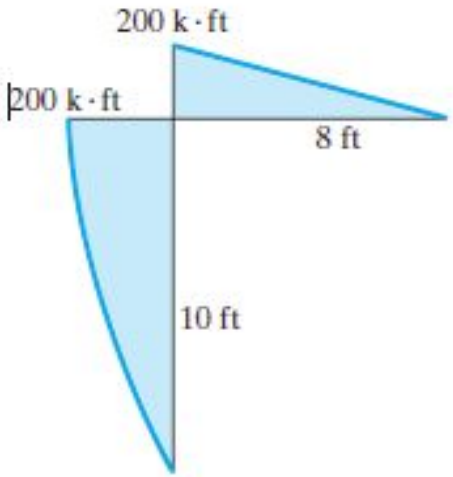
$$\int mM\,dx = \tfrac{5}{12}(10)(200)(10) + \tfrac{1}{3}(10)(200)(8)$$

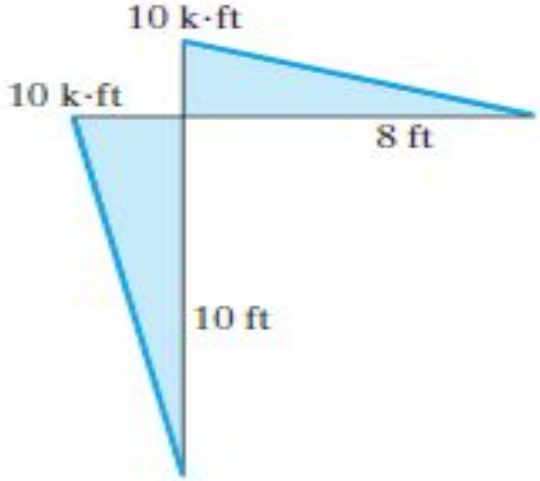
$$= 8333.3 + 5333.3 = 13\,666.7\ \text{k}^2 \cdot \text{ft}^3$$

This is the same as that calculated in Eq. 1. Thus

$$\Delta_{C_h} = \frac{13\,666.7\ \text{k} \cdot \text{ft}^3}{[29(10^3)\ \text{k/in}^2((12)^2\ \text{in}^2/\text{ft}^2)][600\ \text{in}^4(\text{ft}^4/(12)^4\ \text{in}^4)]}$$

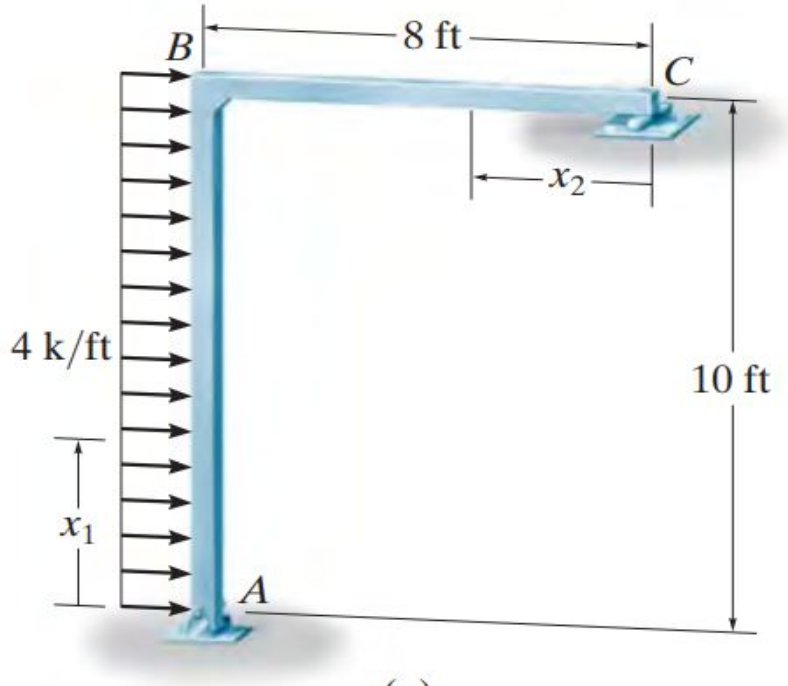
$$= 0.113\ \text{ft} = 1.36\ \text{in.}$$ *Ans.*



# Structural Analysis & Design I

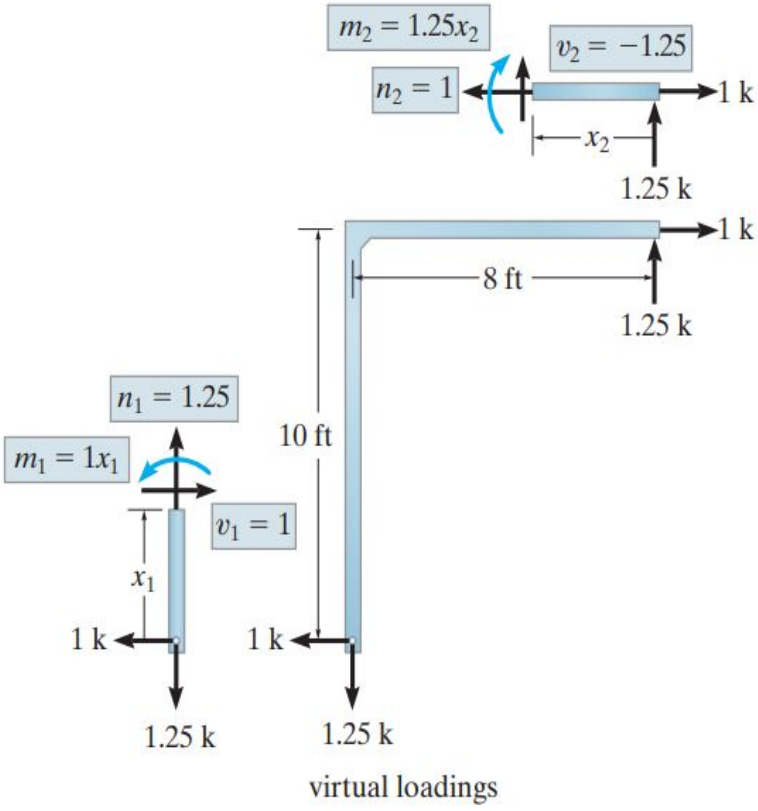
Determine the horizontal displacement of point $C$ on the frame shown in Fig. 9–25*a*. Take $E = 29(10^3)$ ksi, $G = 12(10^3)$ ksi, $I = 600\ \text{in}^4$, and $A = 80\ \text{in}^2$ for both members. The cross-sectional area is rectangular. Include the internal strain energy due to axial load and shear.

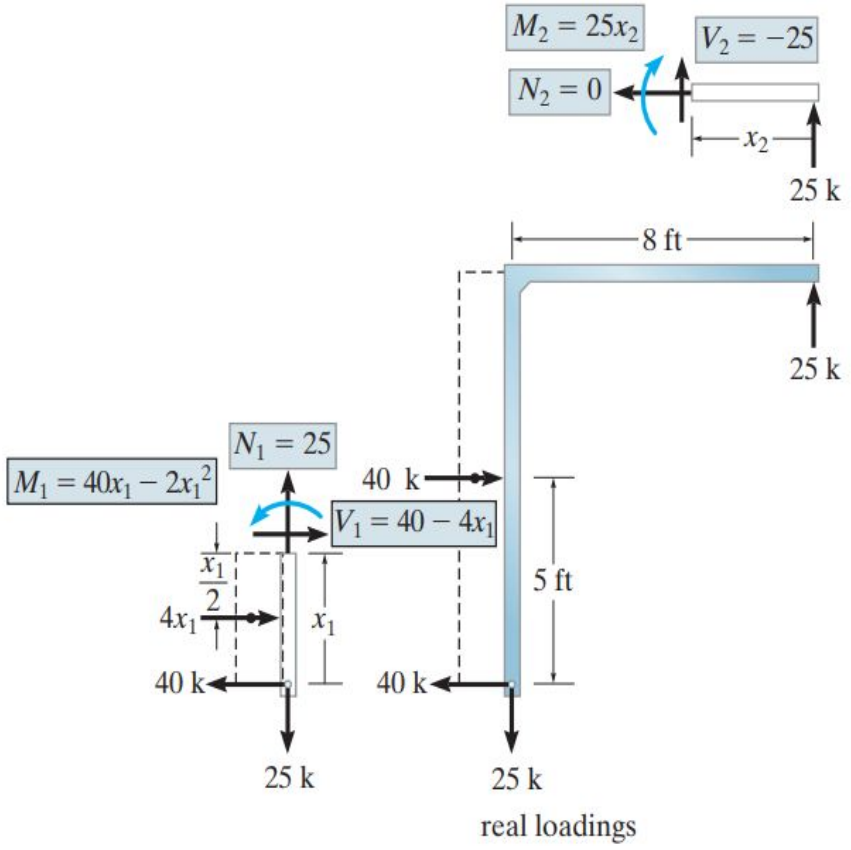



(a)

SOLUTION

Here we must apply a horizontal unit load at $C$. The necessary free-body diagrams for the real and virtual loadings are shown in Figs. 9–25*b* and 9–25*c*.



**Bending.** The virtual strain energy due to bending has been determined in Example 9–10. There it was shown that

$$U_b = \int_0^L \frac{mM\,dx}{EI} = \frac{13\,666.7\ \text{k}^2\cdot\text{ft}^3}{EI} = \frac{13\,666.7\ \text{k}^2\cdot\text{ft}^3\,(12^3\ \text{in}^3/1\ \text{ft}^3)}{[29(10^3)\ \text{k/in}^2](600\ \text{in}^4)} = 1.357\ \text{in.}\cdot\text{k}$$

**Axial load.** From the data in Fig. 9–25*b* and 9–25*c*, we have

$$U_a = \sum \frac{nNL}{AE}$$

$$= \frac{1.25\ \text{k}(25\ \text{k})(120\ \text{in.})}{80\ \text{in}^2[29(10^3)\ \text{k/in}^2]} + \frac{1\ \text{k}(0)(96\ \text{in.})}{80\ \text{in}^2[29(10^3)\ \text{k/in}^2]}$$

$$= 0.001616\ \text{in.}\cdot\text{k}$$

**Shear.** Applying Eq. 9–25 with $K = 1.2$ for rectangular cross sections, and using the shear functions shown in Fig. 9–25*b* and 9–25*c*, we have

$$U_s = \int_0^L K\left(\frac{vV}{GA}\right) dx$$

$$= \int_0^{10} \frac{1.2(1)(40 - 4x_1)\, dx_1}{GA} + \int_0^8 \frac{1.2(-1.25)(-25)\, dx_2}{GA}$$

$$= \frac{540\ \text{k}^2 \cdot \text{ft}(12\ \text{in./ft})}{[12(10^3)\ \text{k/in}^2](80\ \text{in}^2)} = 0.00675\ \text{in.} \cdot \text{k}$$

Applying the equation of virtual work, we have

$$1\ \text{k} \cdot \Delta_{C_h} = 1.357\ \text{in.} \cdot \text{k} + 0.001616\ \text{in.} \cdot \text{k} + 0.00675\ \text{in.} \cdot \text{k}$$

$$\Delta_{C_h} = 1.37\ \text{in.} \qquad \textit{Ans.}$$

Including the effects of shear and axial load contributed only a 0.6% increase in the answer to that determined only from bending.

## 9.9 Castigliano's Theorem for Beams and Frames

The internal bending strain energy for a beam or frame is given by Eq. 9–11 ($U_i = \int M^2\,dx/2EI$). Substituting this equation into Eq. 9–20 ($\Delta_i = \partial U_i/\partial P_i$) and omitting the subscript $i$, we have

$$\Delta = \frac{\partial}{\partial P}\int_0^L \frac{M^2\,dx}{2EI}$$

Rather than squaring the expression for internal moment $M$, integrating, and then taking the partial derivative, it is generally easier to differentiate prior to integration. Provided $E$ and $I$ are constant, we have

$$\Delta = \int_0^L M\left(\frac{\partial M}{\partial P}\right)\frac{dx}{EI} \qquad (9\text{–}28)$$

where

$\Delta$ = external displacement of the point caused by the real loads acting on the beam or frame.

$P$ = external force applied to the beam or frame in the direction of $\Delta$.

$M$ = internal moment in the beam or frame, expressed as a function of $x$ and caused by both the force $P$ and the real loads on the beam.

$E$ = modulus of elasticity of beam material.

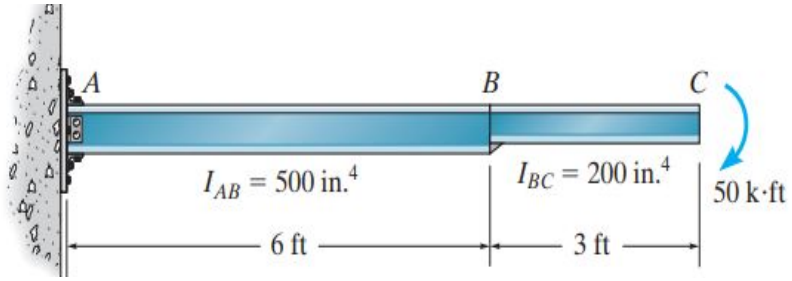
$I$ = moment of inertia of cross-sectional area computed about the neutral axis.

If the slope $\theta$ at a point is to be determined, we must find the partial derivative of the internal moment $M$ with respect to an *external couple moment* $M'$ acting at the point, i.e.,

$$\theta = \int_0^L M\left(\frac{\partial M}{\partial M'}\right)\frac{dx}{EI} \qquad (9\text{–}29)$$

The above equations are similar to those used for the method of virtual work, Eqs. 9–22 and 9–23, except $\partial M/\partial P$ and $\partial M/\partial M'$ replace $m$ and $m_\theta$, respectively. As in the case for trusses, slightly more calculation is generally required to determine the partial derivatives and apply Castigliano's theorem rather than use the method of virtual work. Also, recall that this theorem applies only to material having a linear elastic response. If a more complete accountability of strain energy in the structure is desired, the strain energy due to shear, axial force, and torsion must be included. The derivations for shear and torsion follow the same development as Eqs. 9–25 and 9–26. The strain energies and their derivatives are, respectively,

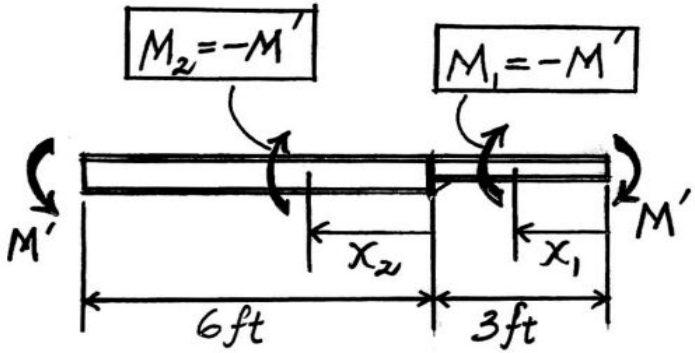
***9–32.** Solve Prob. 9–31 using Castigliano's theorem.




For the slope, the moment functions are shown in Fig. *a*. Here, $\frac{\partial M_1}{\partial M'} = \frac{\partial M_2}{\partial M'} = -1$.

Also, set $M' = 50\ \text{k} \cdot \text{ft}$, then $M_1 = M_2 = -50\ \text{k} \cdot \text{ft}$.

Thus,




$$\theta_C = \int_0^L M\left(\frac{\partial M}{\partial M'}\right)\frac{dx}{EI} = \int_0^{3\,\text{ft}} \frac{-50(-1)dx}{EI_{BC}} + \int_0^{6\,\text{ft}} \frac{-50(-1)dx}{EI_{AB}}$$

$$= \frac{150\ \text{k} \cdot \text{ft}^2}{EI_{BC}} + \frac{300\ \text{k} \cdot \text{ft}^2}{EI_{AB}}$$

$$= \frac{150(12^2)\ \text{k} \cdot \text{in}^2}{[29(10^3\ \text{k/in}^2)](200\ \text{in}^4)} + \frac{300(12^2)\ \text{k} \cdot \text{in}^2}{[29(10^3)\ \text{k/in}^2](500\ \text{in}^4)}$$

$$= 0.00670 \quad \text{Ans.}$$

For the displacement, the moment functions are shown in Fig. *b*. Here, $\frac{\partial M_1}{\partial P} = -x_1$

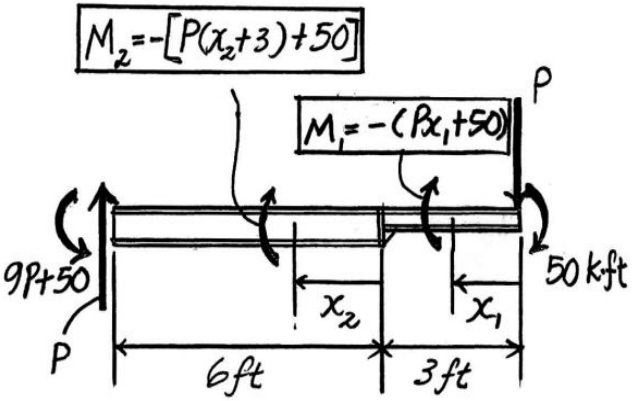
and $\frac{\partial M_2}{\partial P} = -(x_2 + 3)$. Also, set $P = 0$, then $M_1 = M_2 = -50\ \text{k} \cdot \text{ft}$. Thus,

$$\Delta_C = \int_0^L M\left(\frac{\partial M}{\partial P}\right)\frac{dx}{EI} = \int_0^{3\,\text{ft}} \frac{(-50)(-x)dx}{EI_{BC}} + \int_0^{6\,\text{ft}} \frac{(-50)[-(x_2 + 3)]dx}{EI_{AB}}$$

$$= \frac{225\ \text{k} \cdot \text{ft}^3}{EI_{BC}} + \frac{1800\ \text{k} \cdot \text{ft}^3}{EI_{AB}}$$

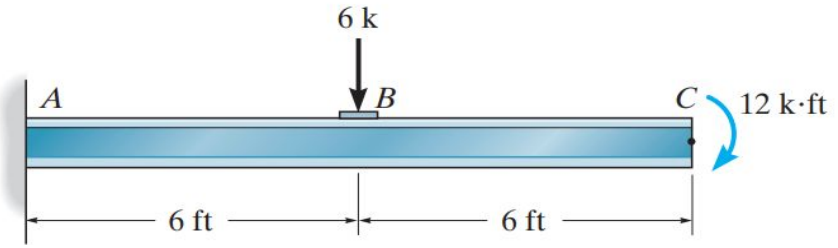
$$= \frac{225(12^3)\ \text{k} \cdot \text{in}^3}{[29(10^3)\ \text{k/in}^2](200\ \text{in}^4)} + \frac{1800(12^3)\ \text{k} \cdot \text{in}^3}{[29(10^3)\ \text{k/in}^2](500\ \text{in}^4)}$$

$$= 0.282\ \text{in} \downarrow$$ **Ans.**



# Structural Analysis & Design I

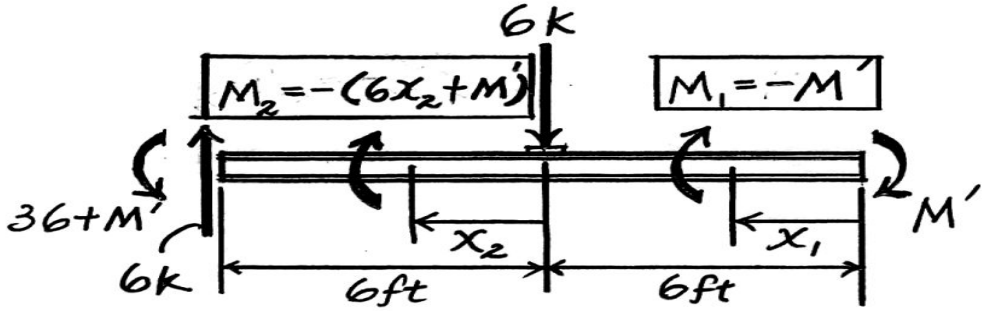
**9–30.** Solve Prob. 9–29 using Castigliano's theorem.




For the slope, the moment functions are shown in Fig. *a*. Here, $\frac{\partial M_1}{\partial M'} = -1$ and $\frac{\partial M_2}{\partial M'} = -1$. Also, set $M' = 12$ kft, then $M_1 = -12\ \text{k}\cdot\text{ft}$ and $M_2 = -(6x_2 + 12)\ \text{k}\cdot\text{ft}$. Thus,

$$\theta_c = \int_0^L M\left(\frac{\partial M}{\partial M'}\right)\frac{dx}{EI} = \int_0^{6\ \text{ft}} \frac{(-12)(-1)}{EI}dx_2 + \int_0^{6\ \text{ft}} \frac{-(6x_2 + 12(-1)}{EI}dx_2$$

$$\theta_c = \frac{252\ \text{k}\cdot\text{ft}^2}{EI} = \frac{252(12^2)\ \text{k}\cdot\text{in}^2}{[29(10^3\ \text{k/in}^2](800\ \text{in}^4)} = 0.00156\ \text{rad}$$

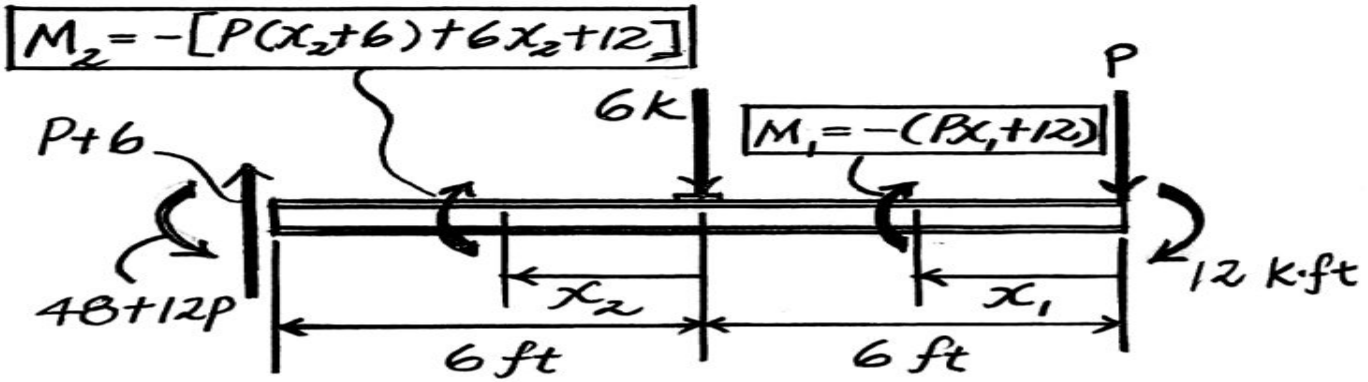


For the displacement, the moment functions are shown in Fig. *b*. Here, $\frac{\partial M_1}{\partial P} = -x_1$ and $\frac{\partial M_2}{\partial P} = -(x_2 + 6)$. Also set, $P = 0$, then $M_1 = -12\ \text{k} \cdot \text{ft}$ and $M_2 = -(6x_2 + 12)\ \text{k} \cdot \text{ft}$. Thus,

$$\Delta_C = \int_0^L \left(\frac{\partial M}{\partial P}\right)\frac{dx}{EI} = \int_0^{6\ \text{ft}} \frac{(-12)(-x_1)}{EI} dx_1 + \int_0^{6\ \text{ft}} \frac{-(6x_2 + 12)[-(x_2 + 6)]}{EI} dx_2$$

$$= \frac{1944\ \text{k} \cdot \text{ft}^3}{EI}$$

$$= \frac{1944(12^3)\ \text{k} \cdot \text{in}^3}{[29(10^3)\ \text{k/in}^2](800\ \text{in}^4)} = 0.145\ \text{in} \downarrow$$



# Structural Analysis & Design I

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