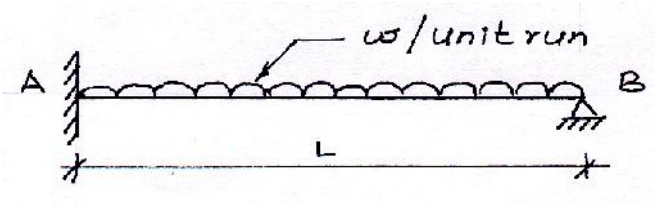
Chapter-2: Slope Deflection Method

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Example: Analyze the propped cantilever shown by using slope defection method. Then draw Bending moment and shear force diagram.



Solution: End A is fixed hence θ_A =0

End B is Hinged hence $\theta_B \neq 0$

Assume both ends are fixed and therefore fixed end moments are

$$F_{AB} = -\frac{wL^2}{12}, \quad F_{BA} = +\frac{wL^2}{12}$$

The Slope deflection equations for final moment at each end are

$$M_{AB} = F_{AB} + \frac{2EI}{L}(2\theta_A + \theta_B)$$

$$= \frac{-wL^2}{12} + \frac{2EI}{L}\theta_B \qquad \rightarrow (1)$$

$$M_{BA} = F_{BA} + \frac{2EI}{L}(2\theta_B + \theta_A)$$

$$= \frac{wL^2}{12} + \frac{4EI}{L}\theta_B \qquad \rightarrow (2)$$

In the above equations there is only one unknown θ_B .

To solve we have boundary condition at B;

Since B is simply supported, the BM at B is zero

ie. M_{BA}=0.

$$\therefore \text{From equation (2) } M_{BA} = \frac{wL^2}{12} + \frac{4EI}{L}\theta_B = 0$$

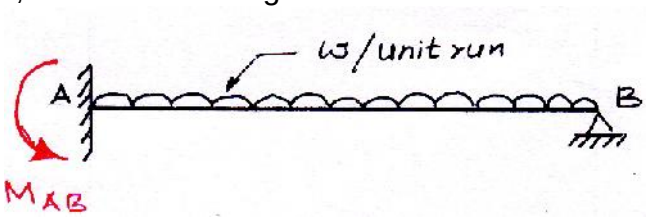
$$\therefore EI\theta_B = -\frac{wL^3}{48}$$ - ve sign indicates the rotation is anticlockwise

Substituting the value of $EI\theta_B$ in equation (1) and (2) we have end moments

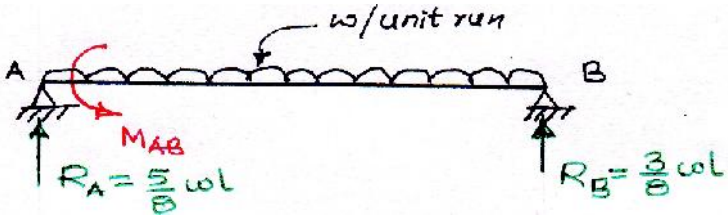
$$M_{AB} = -\frac{wL^2}{12} + \frac{2}{L}\left(-\frac{wL^3}{48}\right) = -\frac{wL^2}{8}$$ -ve sign indicates moment is anticlockwise

$$M_{BA} = +\frac{wL^2}{12} + \frac{4}{L}\left(-\frac{wL^3}{48}\right) = 0$$

M_{BA} has to be zero, because it is hinged.



Now consider the free body diagram of the beam and find reactions using equations of equilibrium.



$$\sum M_B = 0$$

$$R_A \times L = M_{AB} + wL \times \frac{L}{2}$$

$$= \frac{wL^2}{8} + wL \times \frac{L}{2} = +\frac{5}{8}wL$$

$$\therefore \quad R_A = \frac{5}{8}wL$$

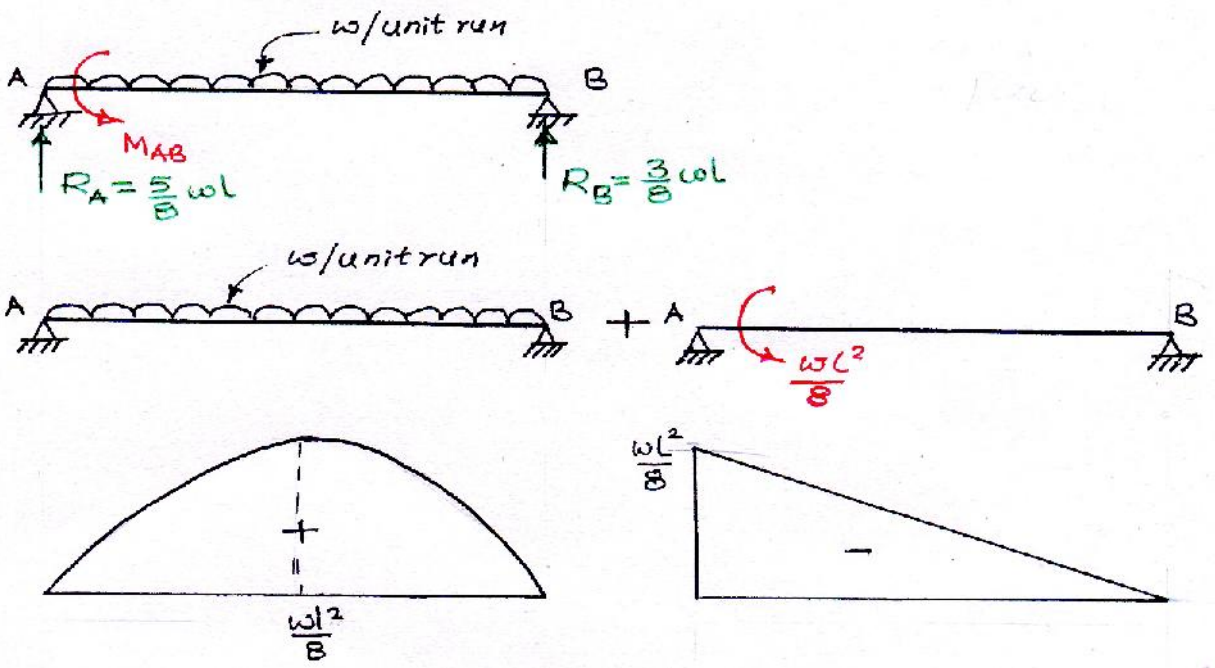
$$\sum V = 0$$

$$R_A + R_B = wL$$

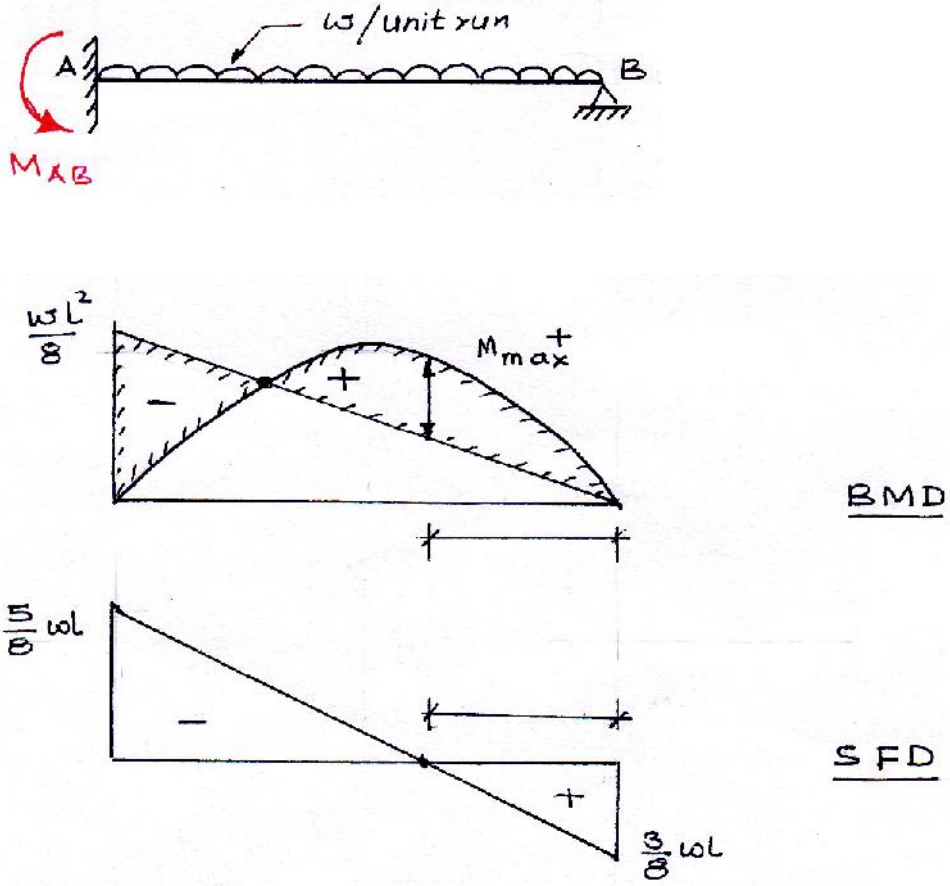
$$R_B = wL - R_A = wL - \frac{5}{8}wL$$

$$= \frac{3}{8}wL$$

Problem can be treated as



The bending moment diagram for the given problem is as below



The max BM occurs where SF=0. Consider SF equation at a distance of x from right support

$$S_X = -\frac{3}{8}wL + wX = 0$$

$$\therefore X = \frac{3}{8}L$$

Hence the max BM occurs at $\frac{3}{8}L$ from support B

$$\therefore M_{max} = M_X = \frac{3}{8}wL \times \frac{3}{8}L - \frac{w}{2}\left(\frac{3}{8}L\right)^2$$

$$= \frac{9}{128}wL^2$$

And point of contra flexure occurs where BM=0, Consider BM equation at a distance of x from right support.

$$M_X = \frac{3}{8}wLX - w\frac{X^2}{2} = 0$$

$$\therefore X = \frac{3}{4}L$$

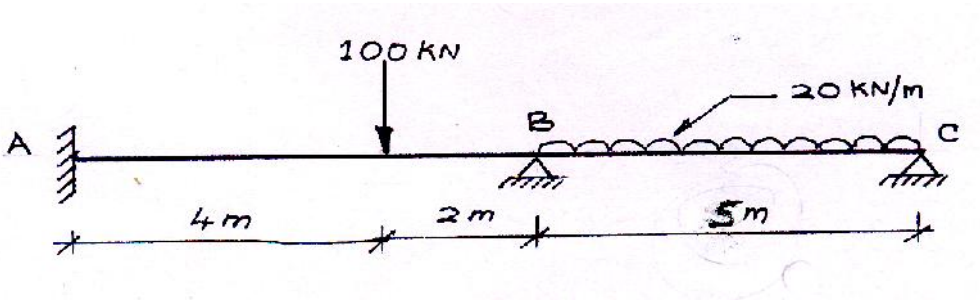
For shear force diagram, consider SF equation from B

$$S_X = +\frac{3}{8}wL - wX$$

$$S_X = 0 = S_B = +\frac{3}{8}wL$$

$$S_X = L = S_A = +\frac{5}{8}wL$$

Example: Analyze two span continuous beam ABC by slope deflection method. Then draw Bending moment & Shear force diagram. Take EI constant



Solution: Fixed end moments are:

$$F_{AB} = \frac{-Wab^2}{L^2} = -\frac{100\times 4\times 2^2}{6^2} = -44.44\text{KNM}$$

$$F_{BA} = +\frac{Wa^2b}{L^2} = +\frac{100\times 4^2\times 2}{6^2} = +88.89\text{KNM}$$

$$F_{BC} = \frac{-wL^2}{12} = -\frac{20\times 5^2}{12} = -41.67\text{KNM}$$

$$F_{CB} = \frac{+wL^2}{12} = \frac{20\times 5^2}{12} = 41.67\text{KNM}$$

Since A is fixed $\theta_A = 0\,,\theta_B \neq 0, \theta_C \neq 0,$

Slope deflection equations are:

$$M_{AB} = F_{AB} + \frac{2EI}{L}(2\theta_A + \theta_B)$$

$$= -44.44 + \frac{2EI}{6}\theta_B$$

$$= -44.44 + \frac{1}{3}EI\,\theta_B \qquad -----\rightarrow (1)$$

$$M_{BA} = F_{BA} + \frac{2EI}{L}(2\theta_B + \theta_A)$$

$$= +88.89 + \frac{2EI\times 2\theta_B}{6}$$

$$= 88.89 + \frac{2}{3}EI\theta_B \qquad -----\rightarrow (2)$$

$$M_{BC} = F_{BC} + \frac{2EI}{L}(2\theta_B + \theta_C)$$

$$= -41.67 + \frac{2EI}{5}(2\theta_B + \theta_C)$$

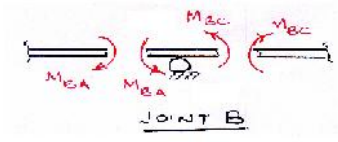
$$= -41.67 + \frac{4}{5}EI\theta_B + \frac{2}{5}EI\theta_C \qquad ----\rightarrow (3)$$

$$M_{CB} = F_{CB} + \frac{2EI}{L}(2\theta_C + \theta_B)$$

$$= +41.67 + \frac{2EI}{5}(2\theta_C + \theta_B)$$

$$= 41.67 + \frac{4EI}{5}\theta_C + \frac{2}{5}EI\theta_B \qquad ----\rightarrow (4)$$

In all the above four equations there are only two unknown θ_B and θ_C. And accordingly the boundary conditions are



i $-M_{BA}-M_{BC}=0$

$M_{BA}+M_{BC}=0$

ii $M_{CB}=0$ since C is end simply support.

Now $$M_{BA} + M_{BC} = 88.89 + \frac{2}{3}EI\theta_B - 41.67 + \frac{4}{5}EI\theta_B + \frac{2}{5}EI\theta_C$$

$$= 47.22 + \frac{22}{15}EI\theta_B + \frac{2}{5}EI\theta_C = 0 \qquad ----->(5)$$

$$M_{CB} = 41.67 + \frac{2}{5}EI\theta_B + \frac{4}{5}EI\theta_C = 0 \qquad ----->(6)$$

Solving simultaneous equations 5 & 6 we get

$EI\theta_B$ = – 20.83 Rotation anticlockwise.

$EI\theta_C$ = – 41.67 Rotation anticlockwise.

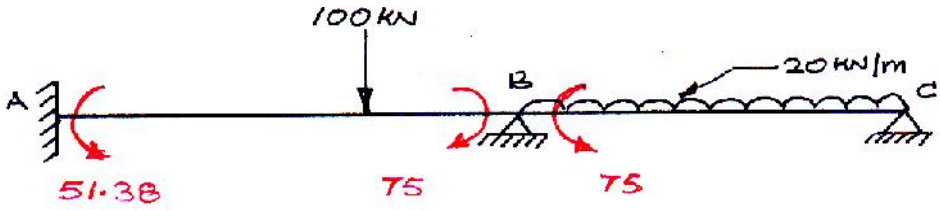
Substituting in the slope definition equations

$$M_{AB} = -44.44 + \frac{1}{3}(-20.83) = -51.38 \text{ KNM}$$

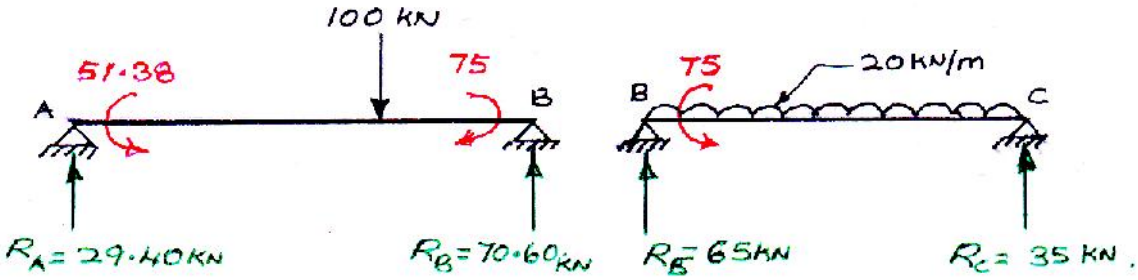
$$M_{BA} = +88.89 + \frac{2}{3}(-20.83) = +75.00 \text{ KNM}$$

$$M_{BC} = -41.67 + \frac{4}{5}(-20.83) + \frac{2}{5}(-41.67) = -75.00 \text{ KNM}$$

$$M_{CB} = +41.67 + \frac{2}{5}(-20.83) + \frac{4}{5}(-41.67) = 0$$



Reactions: Consider the free body diagram of the beam.



Find reactions using equations of equilibrium.

Span AB: $\Sigma M_A = 0$ $\quad R_B \times 6 = 100 \times 4 + 75 - 51.38$

$\therefore R_B = 70.60$ KN

$\Sigma V = 0$ $\quad R_A + R_B = 100$KN

$\therefore R_A = 100 - 70.60 = 29.40$ KN

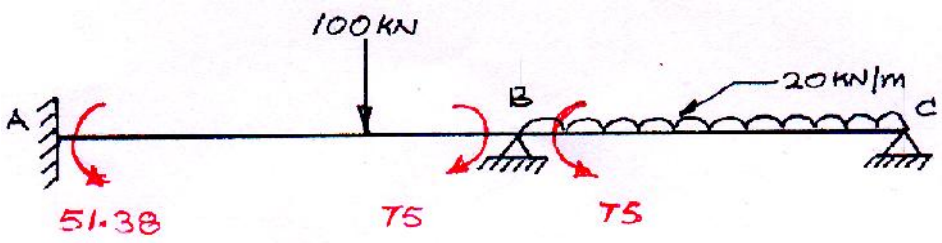
Span BC: $\Sigma M_C = 0$ $\quad R_B \times 5 = 20 \times 5 \times \frac{5}{2} + 75$

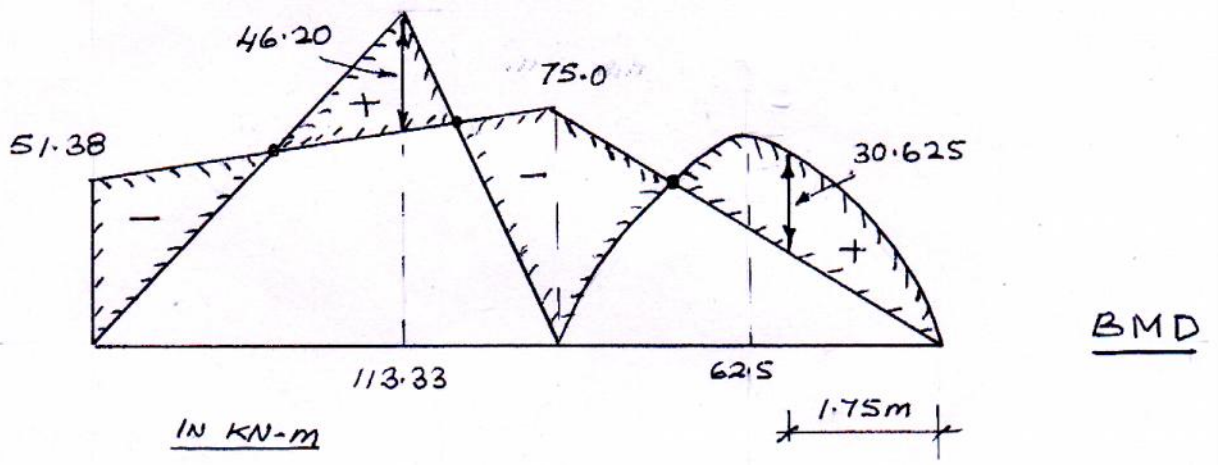
$\therefore R_B = 65$ KN

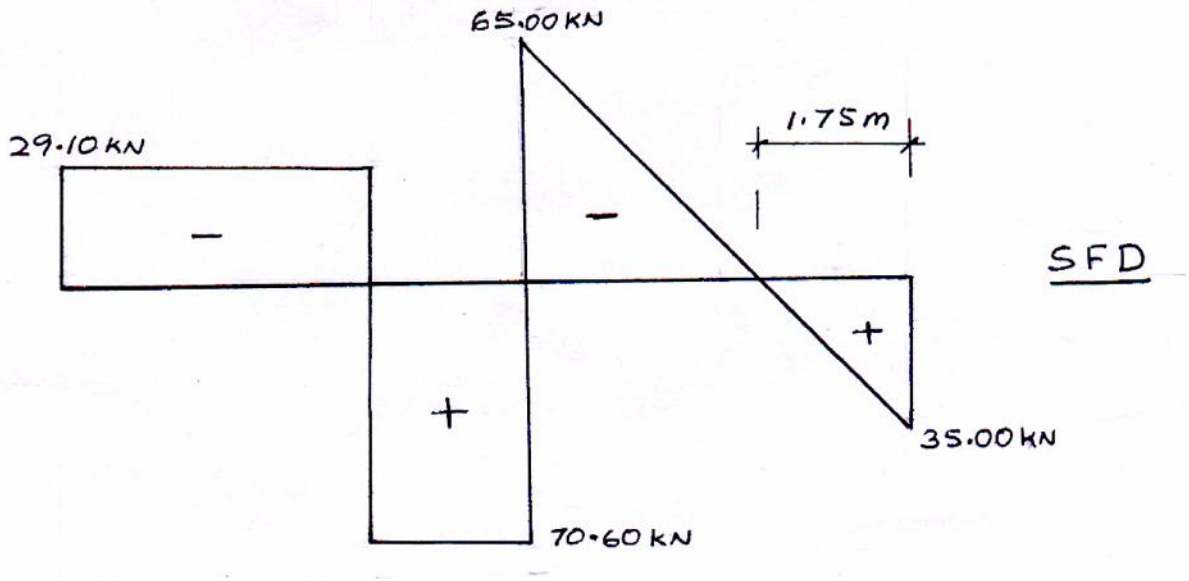
$\Sigma V = 0$ $\quad R_B + R_C = 20 \times 5 = 100$KN

$R_C = 100 - 65 = 35$ KN

Using these data BM and SF diagram can be drawn.







Max BM:

Span AB: Max BM in span AB occurs under point load and can be found geometrically

$$M_{max}=113.33-51.38 - \frac{(75-51.38)}{6}\times 4 = 46.20 \text{ KNM}$$

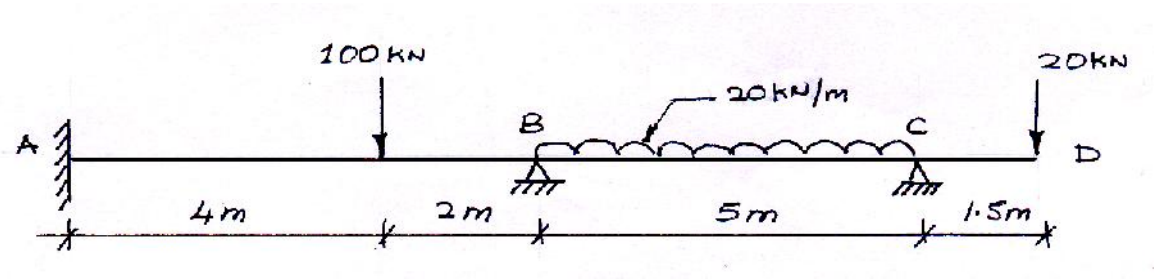
Span BC: Max BM in span BC occurs where shear force is zero or changes its sign. Hence consider SF equation w.r.t C

$$S_x = 35-20x = 0 \qquad \therefore x = \frac{35}{20} = 1.75\text{m}$$

Max BM occurs at 1.75m from C

$$\therefore M_{max} = 35 \times 1.75 - 20 \times \frac{1.75^2}{2} = 30.625 \text{ KNM}$$

Example: Analyze continuous beam ABCD by slope deflection method and then draw bending moment diagram. Take EI constant.



Solution:

$$\theta_A = 0, \quad \theta_B = 0, \quad \theta_C = 0$$

FEMS

$$F_{AB} = -\frac{Wab^2}{L^2} = -\frac{100 \times 4 \times 2^2}{6^2} = -44.44 \text{ KNM}$$

$$F_{BA} = +\frac{Wa^2b}{L^2} = +\frac{100 \times 4^2 \times 2}{6^2} = +88.88 \text{ KNM}$$

$$F_{BC} = -\frac{wL^2}{12} = -\frac{20 \times 5^2}{12} = -41.67 \text{ KNM}$$

$$F_{CB} = +\frac{wL^2}{12} = +\frac{20 \times 5^2}{12} = +41.67 \text{ KNM}$$

$$F_{CD} = -20 \times 1.5 = -30 \text{ KNM}$$

Slope deflection equations:

$$M_{AB} = F_{AB} + \frac{2EI}{L}(2\theta_A + \theta_B) = -44.44 + \frac{1}{3}EI\theta_B \quad \text{--------->} (1)$$

$$M_{BA} = F_{BA} + \frac{2EI}{L}(2\theta_B + \theta_A) = +88.89 + \frac{2}{3}EI\theta_B \quad \text{--------->} (2)$$

$$M_{BC} = F_{BC} + \frac{2EI}{L}(2\theta_B + \theta_C) = -41.67 + \frac{4}{5}EI\theta_B + \frac{2}{5}EI\theta_C \quad \text{--------->} (3)$$

$$M_{CB} = F_{CB} + \frac{2EI}{L}(2\theta_C + \theta_B) = +41.67 + \frac{4}{5}EI\theta_C + \frac{2}{5}EI\theta_B \quad \text{--------->} (4)$$

$$M_{CD} = -30 \text{ KNM}$$

In the above equations we have two unknown rotations θ_B and θ_C, accordingly the boundary conditions are:

$$M_{BA}+M_{BC}=0$$
$$M_{CB}+M_{CD}=0$$

Now, $$M_{BA}+M_{BC}=88.89+\frac{2}{3}EI\theta_B-41.67+\frac{4}{5}EI\theta_B+\frac{2}{5}EI\theta_C$$
$$=47.22+\frac{22}{15}EI\theta_B+\frac{2}{5}EI\theta_C=0 \qquad \text{--------} > (5)$$

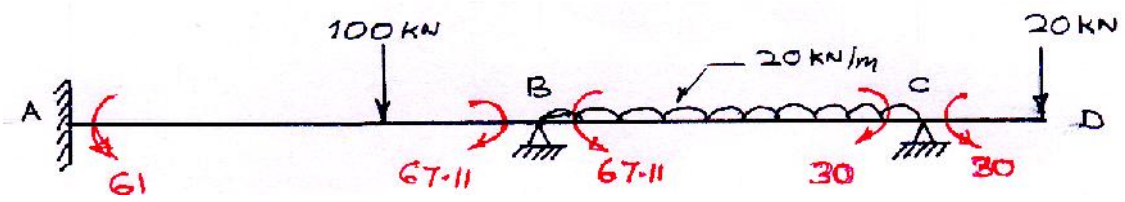
And, $$M_{CB}+M_{CD}=+41.67+\frac{4}{5}EI\theta_C+\frac{2}{5}EI\theta_B-30$$
$$=11.67+\frac{2}{5}EI\theta_B+\frac{4}{5}EI\theta_C \qquad \text{-------} > (6)$$

Solving (5) and (6) we get

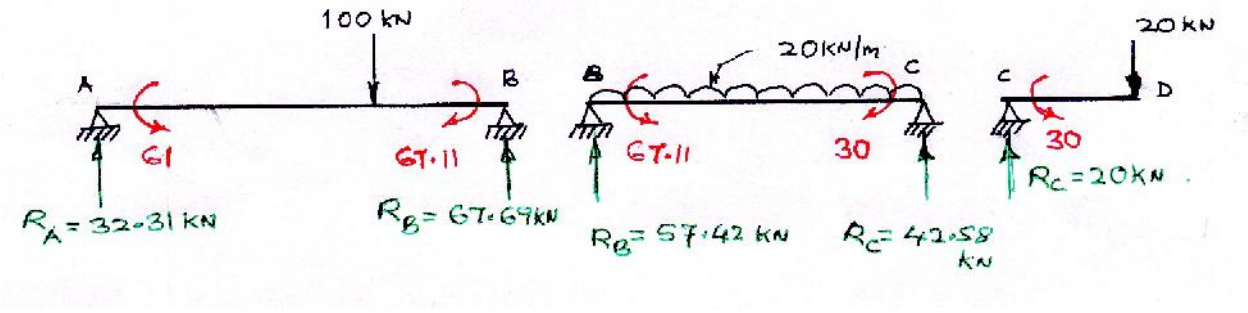
$EI\theta_B=-32.67$ Rotation @ B anticlockwise
$EI\theta_C=+1.75$ Rotation @ B clockwise

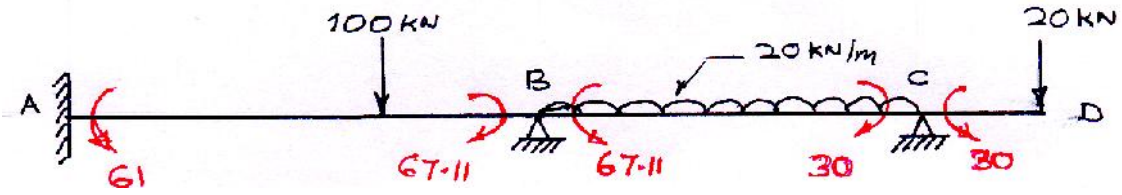
Substituting value of $EI\theta_B$ and $EI\theta_C$ in slope deflection equations we have

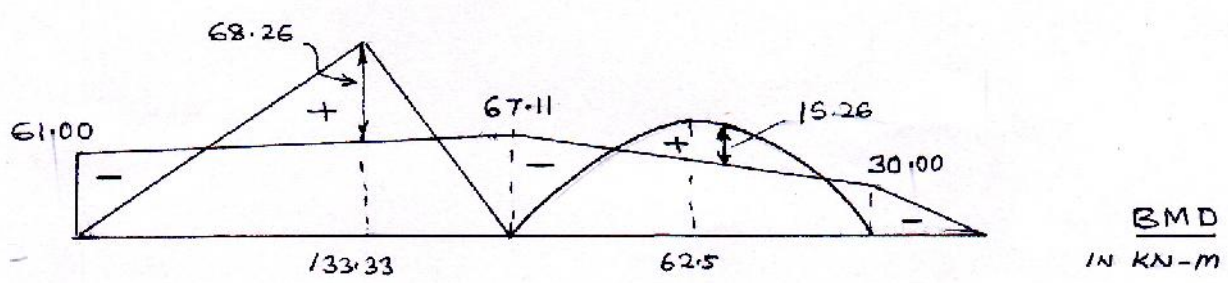
$$M_{AB}=-44.44+\frac{1}{2}(-32.67)=-61.00 \text{ KNM}$$
$$M_{BA}=+88.89+\frac{2}{3}(-32.67)=+67.11 \text{ KNM}$$
$$M_{BC}=-41.67+\frac{4}{5}(-32.67)=+\frac{2}{5}(1.75)=-67.11 \text{ KNM}$$
$$M_{CB}=+41.67+\frac{4}{5}(1.75)+\frac{2}{5}(-32.67)=+30.00 \text{ KNM}$$
$M_{CD}=-30$ KNM

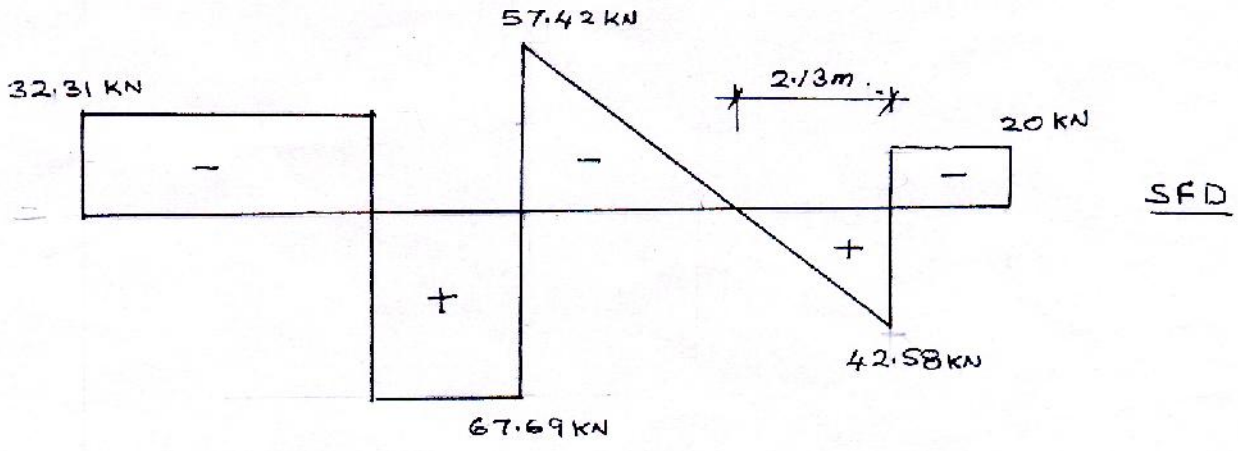


Reactions: Consider free body diagram of beam AB, BC and CD as shown









Span AB

$$R_B \times 6 = 100 \times 4 + 67.11 - 61$$
$$R_B = 67.69 \quad KN$$
$$R_A = 100 - R_B = 32.31 \quad KN$$

Span BC

$$R_C \times 5 = 20 \times \frac{5}{2} \times 5 + 30 - 67.11$$
$$R_C = 42.58 \quad KN$$
$$R_B = 20 \times 5 - R_B = 57.42 \quad KN$$

Maximum Bending Moments:

Span AB: Occurs under point load

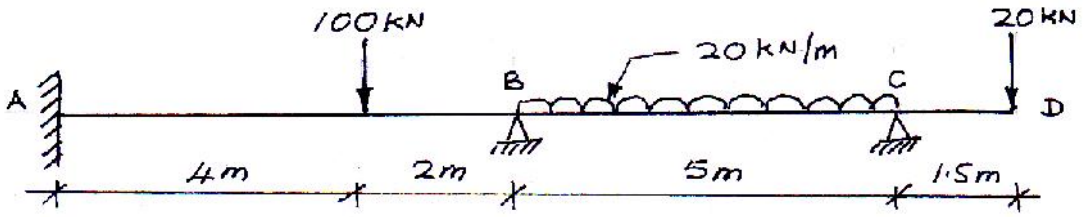
$$Max = 133.33 - 61 - \left(\frac{67.11 - 61}{6} \times 4 = 68.26 \quad KNM \right)$$

Span BC: where SF=0, consider SF equation with C as reference

$$S_X = 42.58 - 20x = 0$$
$$x = \frac{42.58}{20} = 2.13 \quad m$$
$$\therefore \; M_{max} = 42.58 \times 2.13 - 20 \times \frac{2.13^2}{2} - 30 = 15.26 \quad KNM$$

Example: Analyse the continuous beam ABCD shown in figure by slope deflection method. The support B sinks by 15mm.

Take $E = 200\times10^5$ KN/m^2 and $I = 120\times10^{-6}$m^4



Solution:

In this problem θ_A=0, $\theta_B\neq0$, $\theta_C\neq0$, δ=15mm

FEMs:

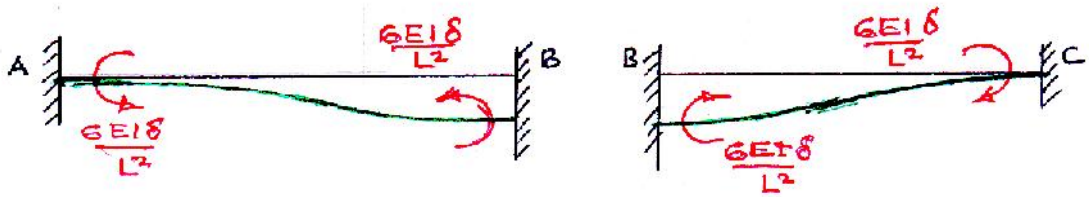
$$F_{AB} = -\frac{Wab^2}{L^2} = -44.44 \text{ KNM}$$

$$F_{BA} = +\frac{Wa^2b}{L^2} = +88.89 \text{ KNM}$$

$$F_{BC} = -\frac{wL^2}{8} = -41.67 \text{ KNM}$$

$$F_{CB} = +\frac{wL^2}{8} = +41.67 \text{ KNM}$$

FEM due to yield of support B



For span AB:

$$m_{ab} = m_{ba} = -\frac{6EI}{L^2}\delta$$

$$= -\frac{6\times200}{6^2}\times10^5\times120\times10^{-6}\times\frac{15}{1000} = -6 \text{ KNM}$$

For span BC:

$$m_{bc} = m_{cb} = +\frac{6EI}{L^2}\delta$$

$$= +\frac{6\times 200}{5^2}\times 10^5 \times 120 \times 10^{-6} \times \frac{15}{1000} = +8.64\text{KNM}$$

Slope deflection equation

$$M_{AB} = F_{AB} + \frac{2EI}{L}(2\theta_A + \theta_B - \frac{3\delta}{L})$$

$$= F_{AB} + \frac{EI}{L}(2\theta_A + \theta_B) - \frac{6EI\delta}{L^2}$$

$$= -44.44 + \frac{1}{3}EI\theta_B - 6$$

$$= -50.44 + \frac{1}{3}EI\theta_B \quad \text{--------->} (1)$$

$$M_{BA} = F_{BA} + \frac{2EI}{L}(2\theta_B + \theta_A) - \frac{6EI\delta}{L^2}$$

$$= +88.89 + \frac{2}{3}EI\theta_B - 6$$

$$= +82.89 + \frac{2}{3}EI\theta_B \quad \text{--------->} (2)$$

$$M_{BC} = F_{BC} + \frac{2EI}{L}(2\theta_B + \theta_C) + \frac{6EI\delta}{L^2}$$

$$= -41.67 + \frac{2}{5}EI(2\theta_B + \theta_C) + 8.64$$

$$= -33.03 + \frac{4}{5}EI\theta_B + \frac{2}{5}EI\theta_C \quad \text{--------->} (3)$$

$$M_{CB} = F_{CB} + \frac{2EI}{L}(2\theta_C + \theta_B) + \frac{6EI\delta}{L^2}$$

$$= +41.67 + \frac{2}{5}EI(2\theta_C + \theta_B) + 8.64$$

$$= +50.31 + \frac{4}{5}EI\theta_C + \frac{2}{5}EI\theta_B \quad \text{--------->} (4)$$

$$M_{CD} = -30 \quad \text{KNM} \quad \text{--------->} (5)$$

There are only two unknown rotations θ_B and θ_C. Accordingly the boundary conditions are

$$M_{BA}+M_{BC}=0$$
$$M_{CB}+M_{CD}=0$$

Now, $M_{BA}+M_{BC}=49.86+\frac{22}{15}EI\theta_B+\frac{2}{5}EI\theta_C=0$

$$M_{CB}+M_{CD}=20.31+\frac{2}{5}EI\theta_B+\frac{4}{5}EI\theta_C=0$$

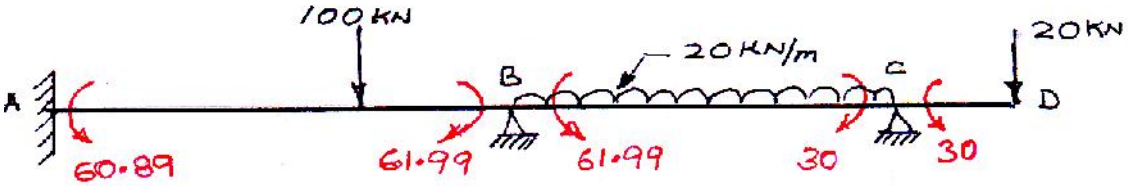
Solving these equations we get

$EI\theta_B=-31.35$ Anticlockwise

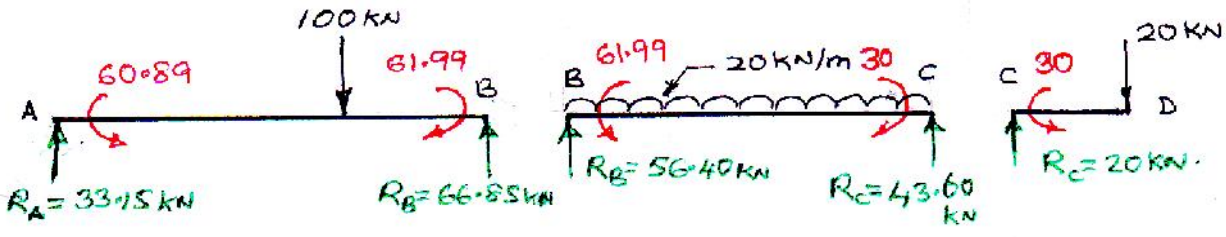
$EI\theta_C=-9.71$ Anticlockwise

Substituting these values in slope deflections we get the final moments:

$$M_{AB}=-50.44+\frac{1}{3}(-31.35)=-60.89 \text{ KNM}$$
$$M_{BA}=+82.89+\frac{2}{3}(-31.35)=+61.99 \text{ KNM}$$
$$M_{BC}=-33.03+\frac{4}{5}(-31.35)+\frac{2}{5}(-9.71)=-61.99 \text{ KNM}$$
$$M_{CB}=+50.31+\frac{4}{5}(-9.71)+\frac{2}{5}(-31.35)=+30.00 \text{ KNM}$$
$$M_{CD}=-30 \text{ KNM}$$



Consider the free body diagram of continuous beam for finding reactions



Reactions:

Span AB:

$R_B \times 6 = 100 \times 4 + 61.99 - 60.89$

$R_B = 66.85$

$R_A = 100 - R_B$

$=33.15$ KN

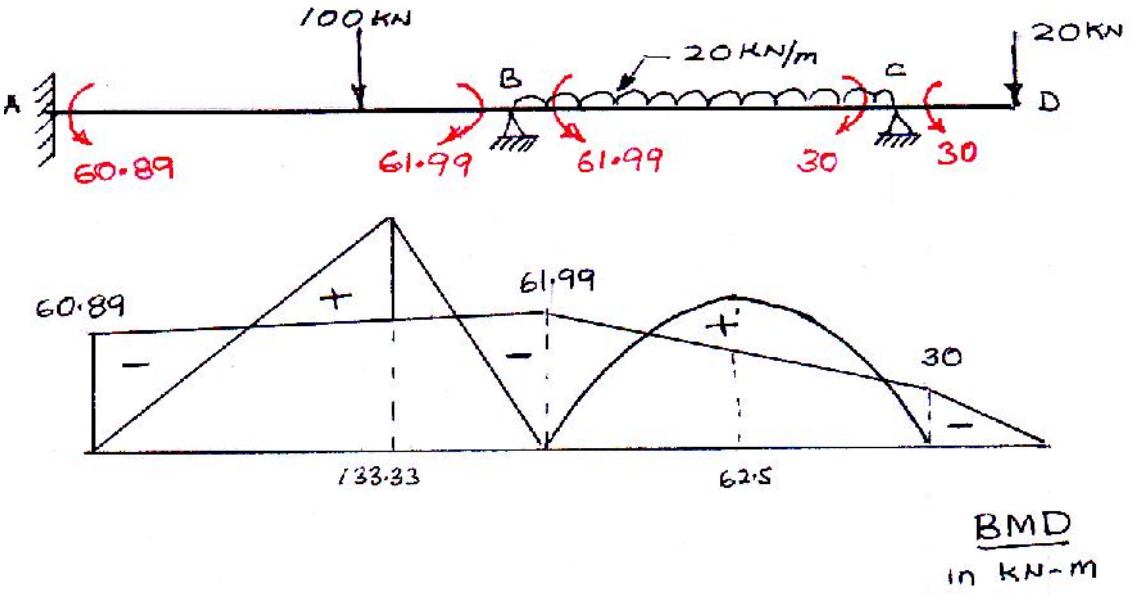
Span BC:

$R_B \times 5 = 20 \times 5 \times \frac{5}{2} + 61.99 - 30$

$R_B = 56.40$ KN

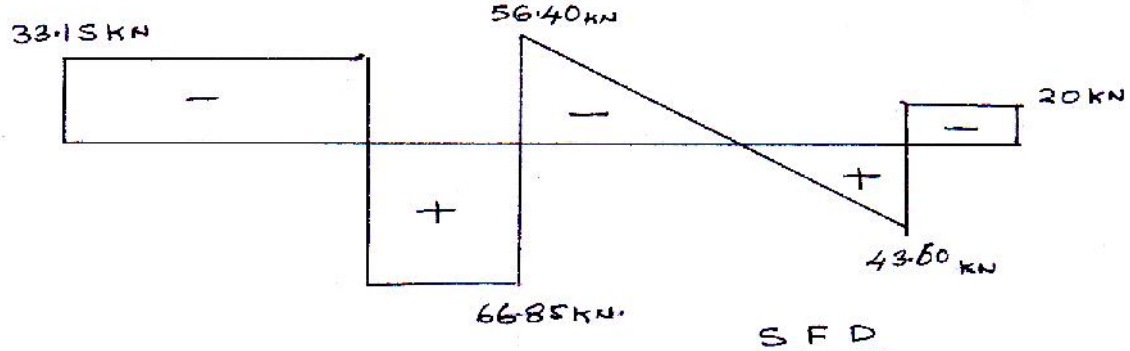
$R_C = 20 \times 5 - R_B$

$=43.60$ KN



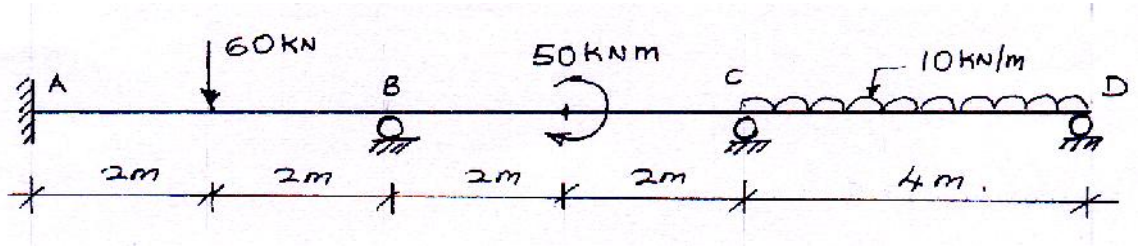
BMD

in KN-m



SFD

Example: Three span continuous beam ABCD is fixed at A and continuous over B, C and D. The beam subjected to loads as shown. Analyse the beam by slope deflection method and draw bending moment and shear force diagram.



Solution:

Since end A is fixed $\theta_A = 0, \theta_B \neq 0, \theta_c \neq 0, \theta_D = 0$

FEMs:

$$F_{AB} = -\frac{Wl}{8} = -\frac{60 \times 4}{8} = -30 \text{ KNM}$$

$$F_{BA} = +\frac{Wl}{8} = +\frac{60 \times 4}{8} = +30 \text{ KNM}$$

$$F_{BC} = +\frac{M}{4} = +12.5 \text{ KNM}$$

$$F_{CB} = +\frac{M}{4} = +12.5 \text{ KNM}$$

$$F_{CD} = -\frac{wl^2}{12} = -\frac{10 \times 4^2}{12} = -13.33 \text{ KNM}$$

$$F_{DC} = +\frac{wl^2}{12} = +\frac{10 \times 4^2}{12} = +13.33 \text{ KNM}$$

Slope deflection equations:

$$M_{AB} = F_{AB} + \frac{2EI}{L}(2\theta_A + \theta_B)$$

$$= -30 + \frac{2EI}{4}(0 + \theta_B)$$

$$= -30 + 0.5EI\theta_B \quad \text{--------} > (1)$$

$$M_{BA} = F_{BA} + \frac{2EI}{L}(2\theta_B + \theta_A)$$

$$= 30 + \frac{2EI}{4}(2\theta_B + 0)$$

$$= +30 + EI\theta_B \quad \text{--------} > (2)$$

$$M_{BC} = F_{BC} + \frac{2EI}{L}(2\theta_B + \theta_C)$$

$$= 12.5 + \frac{2EI}{4}(2\theta_B + \theta_C)$$

$$= 12.5 + EI\theta_B + 0.5EI\theta_C \quad \text{--------->} (3)$$

$$M_{CB} = F_{CB} + \frac{2EI}{L}(2\theta_C + \theta_B)$$

$$= 12.5 + \frac{2EI}{4}(2\theta_C + \theta_B)$$

$$= 12.5 + EI\theta_C + 0.5\,EI\theta_B \quad \text{--------->} (4)$$

$$M_{CD} = F_{CD} + \frac{2EI}{L}(2\theta_C + \theta_D)$$

$$= -13.33 + \frac{2EI}{4}(2\theta_C + \theta_D)$$

$$= -13.33 + EI\,\theta_C + 0.5EI\,\theta_D \quad \text{---------->} (5)$$

$$M_{DC} = F_{DC} + \frac{2EI}{L}(2\theta_D + \theta_C)$$

$$= 13.33 + \frac{2EI}{4}(2\theta_D + \theta_C)$$

$$= 13.33 + 0.5EI\,\theta_C + EI\,\theta_D \quad \text{---------->} (6)$$

In the above Equations there are three unknowns, $EI\,\theta_B$, $EI\theta_C$ & $EI\theta_D$, accordingly the boundary conditions are:

i $M_{BA} + M_{BC} = 0$

ii $M_{CB} + M_{CD} = 0$

iii $M_{DC} = 0$ (∵ hinged)

Now

$$M_{BA} + M_{BC} = 0$$

$$30 + EI\theta_B + 12.5 + EI\theta_B + 0.5EI\theta_C = 0$$

$$2EI\theta_B + 0.5EI\theta_C + 42.5 = 0 \quad \text{------>} (7)$$

$$M_{CB} + M_{BC} = 0$$

$$+12.5 + EI\theta_C + 0.5EI\theta_B - 13.33 + EI\theta_C + 0.5EI\theta_D = 0$$

$$0.5EI\theta_B + 2EI\theta_C + 0.5EI\theta_D - 0.83 = 0 \quad \text{------>} (8)$$

$$M_{DC} = 0$$

$$13.33 + 0.5EI\theta_C + EI\theta_D = 0 \quad \text{------>} (9)$$

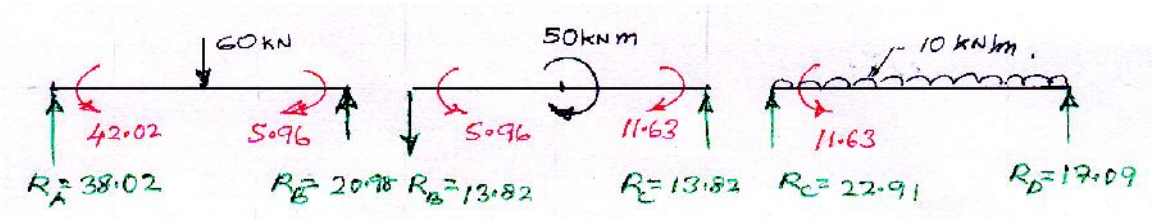
By solving (7), (8) & (9), we get

$$EI\theta_B = -24.04$$
$$EI\theta_C = +11.15$$
$$EI\theta_D = -18.90$$

By substituting the values of θ_B, θ_c and θ_D in respective equations we get

$$M_{AB} = -30 + 0.5\left(-24.04\right) = -42.02 \text{ KNM}$$
$$M_{BA} = +30 + \left(-24.04\right) = +5.96 \text{ KNM}$$
$$M_{BC} = +12.5 + \left(-24.04\right) + 0.5\left(+11.15\right) = -5.96 \text{ KNM}$$
$$M_{CB} = +12.5 + 11.15 + 0.5\left(-24.04\right) = +11.63 \text{ KNM}$$
$$M_{CD} = -13.33 + 11.15 + 0.5\left(-18.90\right) = -11.63 \text{ KNM}$$
$$M_{DC} = +13.33 + 0.5\left(11.15\right) + \left(-18.90\right) = 0 \text{ KNM}$$

Reactions: Consider the free body diagram of beam.



Beam AB:

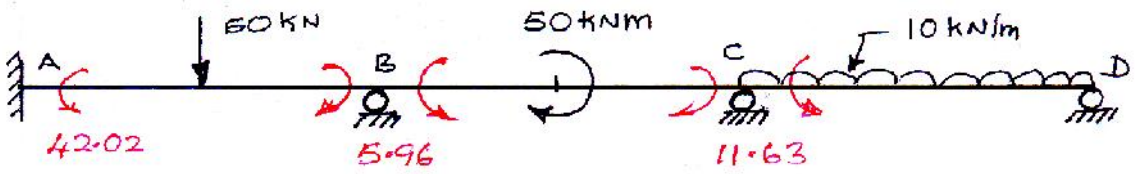
$$R_B = \frac{60 \times 2 + 5.96 - 42.02}{4} = 20.985 \text{ KN}$$
$$\therefore R_A = 60 - R_B = 30.015 \text{ KN}$$

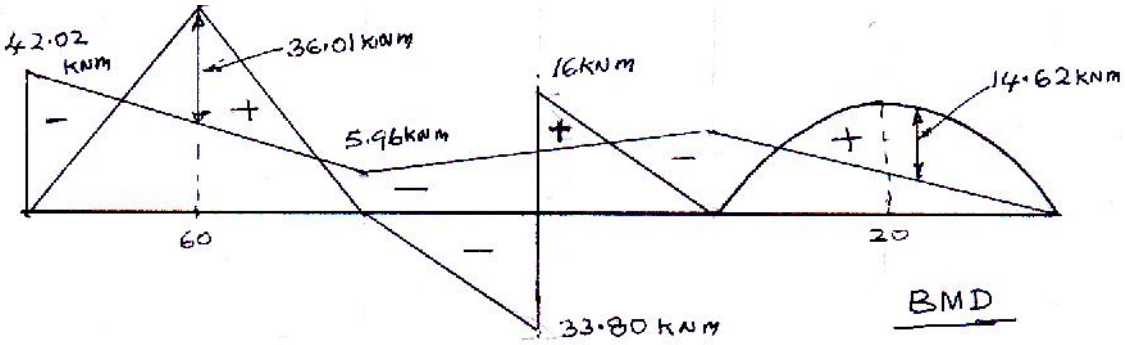
Beam BC:

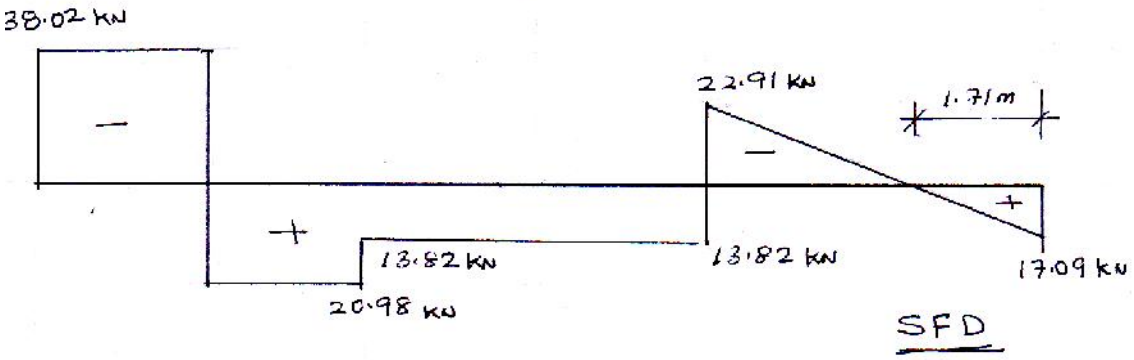
$$R_C = \frac{11.63 + 50 - 5.96}{4} = 13.92 \text{ KN}$$
$$\therefore R_B = -R_C = -13.92 \text{ KN} \quad \therefore R_B \text{ is downward}$$

Beam CD:

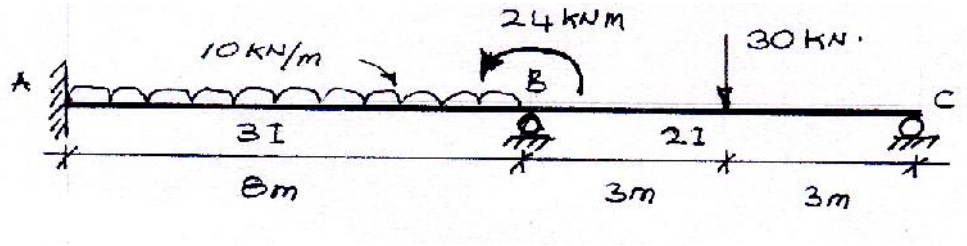
$$R_D = \frac{10 \times 4 \times 2 - 11.63}{4} = 17.09 \text{ KN}$$
$$\therefore R_C = 10 \times 4 - R_D = 22.91 \text{KN}$$


60 KN
50 KNm
10 KN/m
A
B
C
D
42.02
5.96
11.63


42.02 KNm
36.01 KNm
16KNm
14.62KNm
5.96KNm
60
20
33.80 KNm
BMD


38.02 KN
22.91 KN
1.71m
13.82 KN
13.82 KN
20.98 KN
17.09 KN
SFD

Example: Analyse the continuous beam shown using slope deflection method. Then draw bending moment and shear force diagram.



Solution: In this problem $\theta_A = 0$, $\because$ end A is fixed

FEMs:

$$F_{AB} = -\frac{wl^2}{12} = -\frac{10 \times 8^2}{12} = \text{ - } 53.33 \text{ KNM}$$

$$F_{BA} = +\frac{wl^2}{12} = +53.33 \text{ KNM}$$

$$F_{BC} = -\frac{Wl}{8} = -\frac{30 \times 6}{8} = \text{ - } 22.50 \text{ KNM}$$

$$F_{CD} = +\frac{WL}{8} = +\ 22.50 \text{ KNM}$$

Slope deflection equations:

$$M_{AB} = F_{AB} + \frac{2EI}{L}(2\theta_A + \theta_B)$$

$$= \text{ - } 53.33 + \frac{2E \times 3I}{8}(0 + \theta_B)$$

$$= \text{ - } 53.33 + \frac{3}{4}\ EI\theta_B \qquad \text{--------} > (1)$$

$$M_{BA} = F_{BA} + \frac{2EI}{L}(2\theta_B + \theta_A)$$

$$= +53.33 + \frac{2E \times 3I}{8}(2\theta_B + 0)$$

$$= 53.33 + \frac{3}{2}EI\theta_B \qquad \text{--------} > (2)$$

$$M_{BC} = F_{BC} + \frac{2EI}{L}(2\theta_B + \theta_C)$$

$$= -22.5 + \frac{2E2I}{6}(2\theta_B + \theta_C)$$

$$= -22.5 + \frac{4}{3}EI\theta_B + \frac{2}{3}EI\theta_C \qquad \text{--------} > (3)$$

$$M_{CB} = F_{CB} + \frac{2EI}{L}(2\theta_C + \theta_B)$$

$$= +22.5 + \frac{2E2I}{6}(2\theta_C + \theta_B)$$

$$= +22.5 + \frac{4}{3}EI\theta_C + \frac{2}{3}\ EI\theta_B \qquad \text{--------} > (4)$$

In the above equation there are two unknown θ_B and θ_C, accordingly the boundary conditions are:

i $-M_{BA} - M_{BC} - 24 = 0$

ii $M_{CB} = 0$

$$\text{Now, } M_{BA} + M_{BC} - 24 = 53.33 + \frac{3}{2}EI\theta_B - 22.5 + \frac{4}{3}EI\theta_B + \frac{2}{3}EI\theta_C + 24$$

$$= +54.83 + \frac{17}{6}EI\theta_B + \frac{2}{3}EI\theta_C = 0 \qquad ------ > (5)$$

and $M_{CB} = 22.5 + \frac{4}{3}EI\theta_C + \frac{2}{3}EI\theta_B = 0$

$\therefore$

$$\frac{2}{3}EI\theta_C = -11.25 - \frac{1}{3}EI\theta_B \qquad \text{----------} > (6)$$

Substituting in eqn. (5)

$$54.83 + \frac{17}{6}EI\theta_B - 11.25 - \frac{1}{3}EI\theta_B = 0$$

$$+44.58 + \frac{15}{6}EI\theta_B = 0$$

$$\therefore EI\theta_B = -\frac{44.58 \times 6}{15} = -17.432$$ rotation anticlockwise

$\therefore$ from equation (6)

$$EI\theta_C = \frac{3}{2}\left[-11.25 - \frac{1}{3}(-17.432)\right]$$

$$= -8.159 \quad \text{rotation anticlockwise}$$

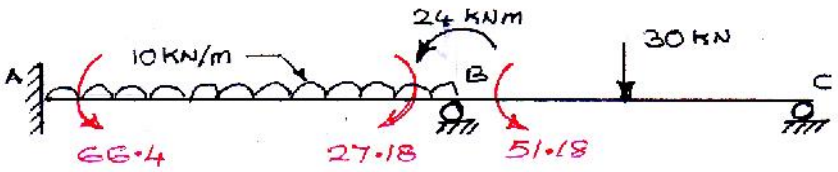
Substituting $EI\theta_B = -17.432$ and $EI\theta_C = -8.159$ in the slope deflection equation we get Final Moments:

$$M_{AB} = -53.33 + \frac{3}{4}(-17.432) = -66.40 \quad \text{KNM}$$

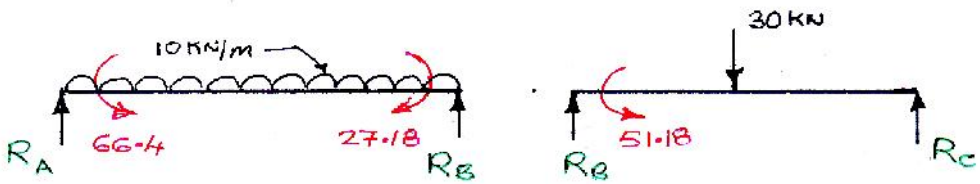
$$M_{BA} = +53.33 + \frac{3}{2}(-17.432) = +27.18 \quad \text{KNM}$$

$$M_{BC} = -22.5 + \frac{4}{3}(-17.432) + \frac{2}{3}(-8.159) = -51.18 \quad \text{KNM}$$

$$M_{CB} = +22.5 + \frac{4}{3}(-8.159) + \frac{2}{3}(-17.432) = 0.00$$



Reactions: Consider free body diagram of beams as shown



Span AB:

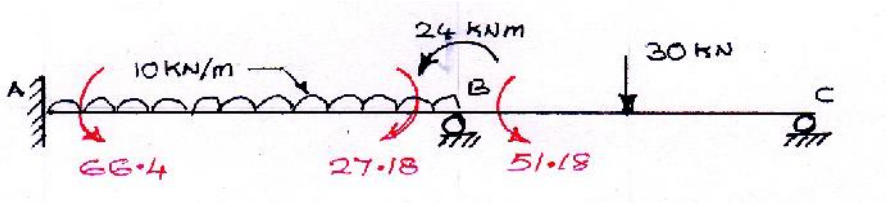
$$R_B = \frac{27.18 - 66.40 + 10 \times 8 \times 4}{8} = 35.13 \text{ KN}$$

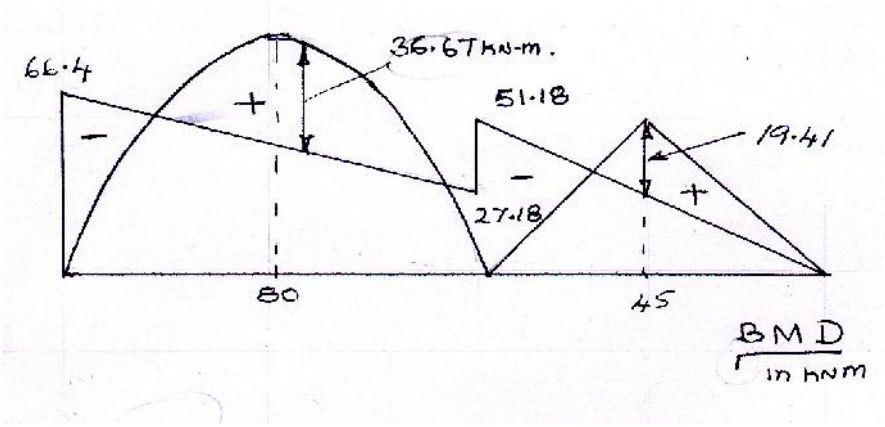
$$\therefore R_A = 10 \times 8 - R_B = 44.87 \text{ KN}$$

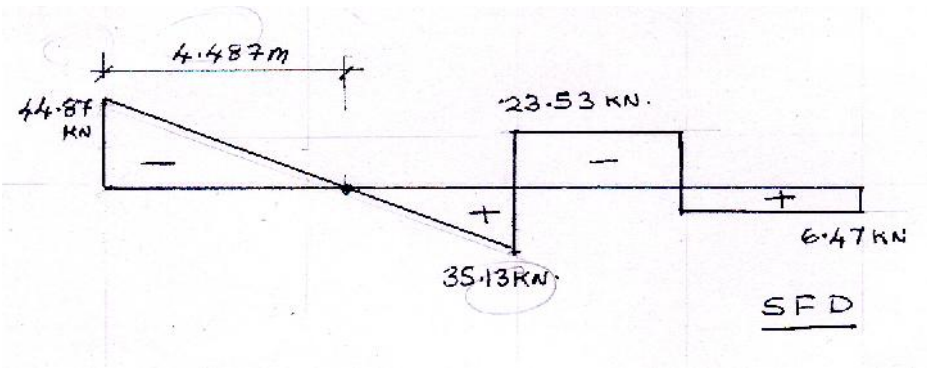
Span BC:

$$R_B = \frac{51.18 + 30 \times 3}{6} = 23.53 \text{ KN}$$

$$R_C = 30 - R_B = 6.47 \quad \text{KN}$$







Max BM

Span AB: Max BM occurs where SF=0, consider SF equation with A as origin

$$S_x = 44.87 - 10x = 0$$

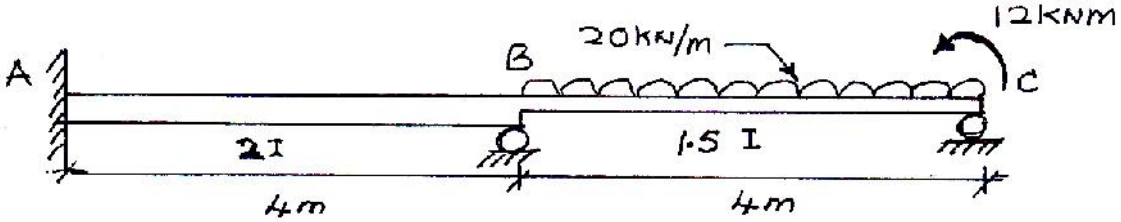
$$x = 4.487\ \text{m}$$

$$\therefore M_{max} = 44.87 \times 4.487 - 10 \times \frac{4.487^2}{2} - 64 = 36.67\ \text{KNM}$$

Span BC: Max BM occurs under point load

$$\text{BC}\ M_{max} = 45 - \frac{51.18}{2} = 19.41\text{KN M}$$

Example: Analyse the beam shown in figure. End support C is subjected to an anticlockwise moment of 12 KNM.



Solution: In this problem $\theta_A = 0, \therefore$ end is fixed

FEMs:

$$F_{BC} = -\frac{wl^2}{12} = -\frac{20 \times 4^2}{12} = -26.67 \quad KNM$$

$$F_{CB} = +\frac{wl^2}{12} = +\ 26.67 \quad KNM$$

Slope deflection equations:

$$M_{AB} = F_{AB} + \frac{2EI}{L}(2\theta_A + \theta_B)$$

$$= 0 + \frac{2E2I}{4}(0 + \theta_B)$$

$$= EI\theta_B \qquad \text{---------} > (1)$$

$$M_{BA} = F_{BA} + \frac{2EI}{L}(2\theta_B + \theta_A)$$

$$= 0 + \frac{2E2I}{4}(2\theta_B + 0)$$

$$= 2EI\theta_B \qquad \text{---------} > (2)$$

$$M_{BC} = F_{BC} + \frac{2EI}{L}(2\theta_B + \theta_C)$$

$$= -26.67 + \frac{2E \times 1.5I}{4}(2\theta_B + \theta_C)$$

$$= -26.67 + \frac{3}{2}EI\theta_B + \frac{3}{4}EI\theta_C \qquad \text{---------} > (3)$$

$$M_{CB} = F_{CB} + \frac{2EI}{L}(2\theta_C + \theta_B)$$

$$= +26.67 + \frac{2E \times 1.5I}{4}(2\theta_C + \theta_B)$$

$$= +26.67 + \frac{3}{2}EI\theta_C + \frac{3}{4}\ EI\theta_B \qquad \text{---------} > (4)$$

In the above equation there are two unknowns θ_B and θ_C, accordingly the boundary conditions are

$$M_{BA}+M_{BC}=0$$

$$M_{CB}+12=0$$

$$\text{Now, } M_{BA}+M_{BC}=2EI\theta_B-26.67+\frac{3}{2}EI\theta_B+\frac{3}{4}EI\theta_C$$

$$=\frac{7}{2}EI\theta_B+\frac{3}{4}EI\theta_C-26.67=0 \qquad \text{--------->(5)}$$

$$\text{and, } M_{CB}+12=26.67+\frac{3}{2}EI\theta_C+\frac{3}{4}EI\theta_B+12$$

$$=38.67+\frac{3}{4}EI\theta_B+\frac{3}{2}EI\theta_C=0 \qquad \text{--------->(6)}$$

From (5) and (6)

$$\frac{7}{2}EI\theta_B+\frac{3}{4}EI\theta_C-26.67=0$$

$$\frac{3}{8}EI\theta_B+\frac{3}{4}EI\theta_C+19.33=0$$

$$\frac{25}{8}EI\theta_B-46=0$$

$$EI\theta_B=+46\times\frac{8}{25}=+14.72$$

From (6)

$$EI\theta_C=-\frac{2}{3}\left(38.67+\frac{3}{4}(14.72)\right)$$

$$=-33.14$$ - ve sign indicates rotation anticlockwise

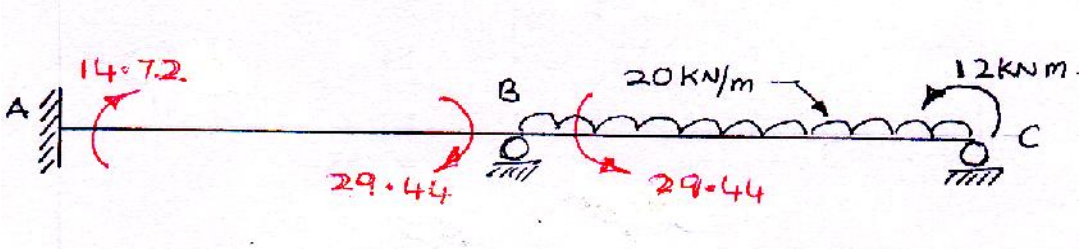
Substituting $EI\theta_B$ and $EI\theta_C$ is slope deflection equations

$$M_{AB}=EI\theta_B=+14.72 \text{ KNM}$$

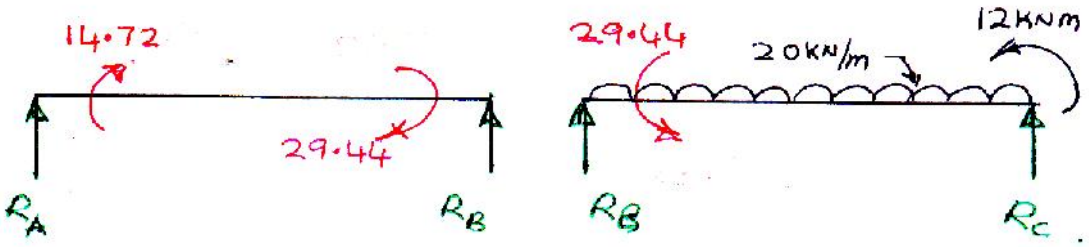
$$M_{BA}=2EI\theta_B=2(14.72)=29.42 \text{ KNM}$$

$$M_{BC}=-26.67+\frac{3}{2}(14.72)+\frac{3}{4}(-33.14)=29.44 \text{ KNM}$$

$$M_{CB}=+26.67+\frac{3}{2}(-33.14)+\frac{3}{4}(14.72)=-12 \text{ KNM}$$



Reaction: Consider free body diagrams of beam



Span AB:

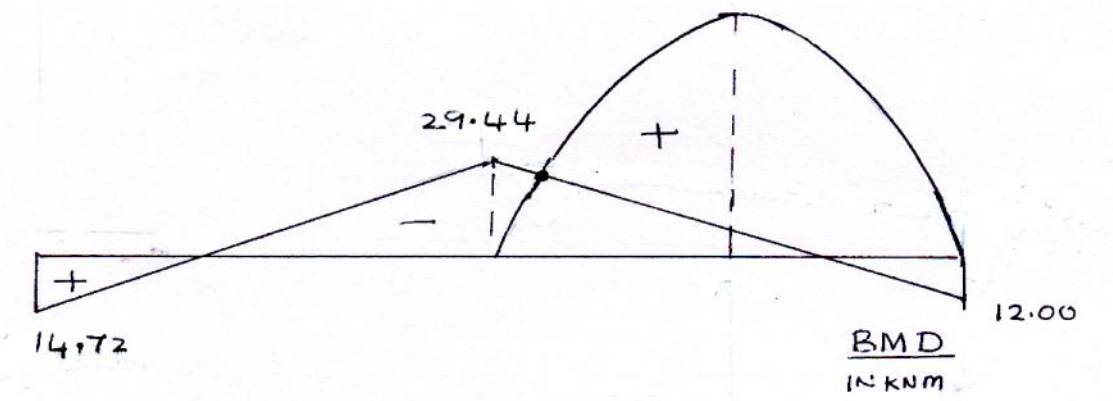
$$R_B = \frac{14.72 + 29.44}{4} = 11.04 \quad KN$$

$$R_A = -R_B = -11.04\ KN$$

Span BC:

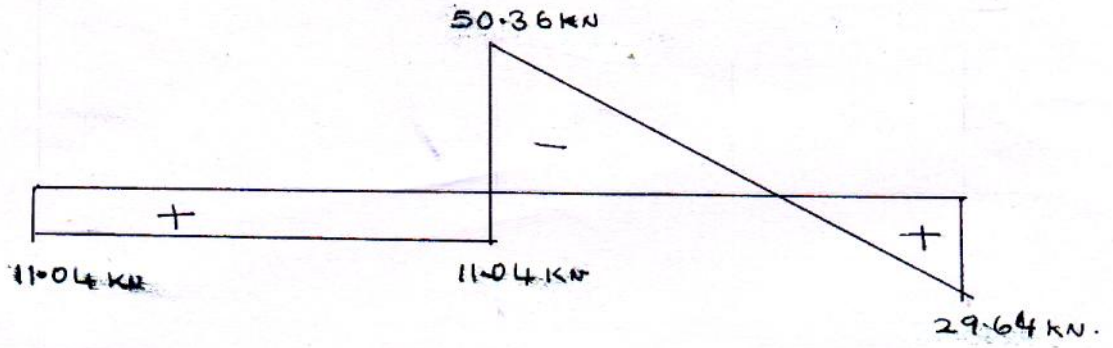
$$R_B = \frac{29.44 + 12 + 20 \times 4 \times 2}{4} = 50.36\ KN$$

$$R_C = 20 \times 4 - R_B = 29.64\ KN$$



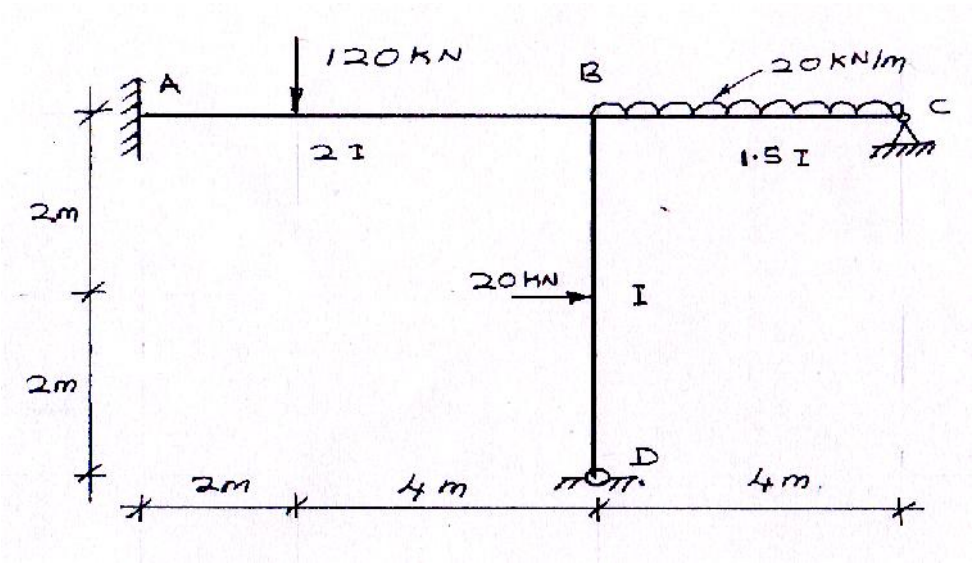
BMD

IN KNM



SFD

Example: Analyse the simple frame shown in figure. End A is fixed and ends B & C are hinged. Draw the bending moment diagram.



Solution:

In this problem $\theta_A = 0, \theta_B \neq 0, \theta_C \neq 0, \theta_D \neq 0,$

FEMS:-

$$F_{AB} = -\frac{Wab^2}{L^2} = -\frac{120 \times 2 \times 4^2}{6^2} = -106.67 \text{ KNM}$$

$$F_{BA} = +\frac{Wa^2b}{L^2} = +\frac{120 \times 2^2 \times 4}{6^2} = +53.33 \text{ KNM}$$

$$F_{BC} = -\frac{wl^2}{12} = -\frac{20 \times 4^2}{12} = -26.67 \text{ KNM}$$

$$F_{CB} = +\frac{wl^2}{12} = +\frac{20 \times 4^2}{12} = +26.67 \text{ KNM}$$

$$F_{CD} = +\frac{WL}{8} = +\frac{20 \times 4}{8} = +10 \text{ KNM}$$

$$F_{DB} = -\frac{WL}{8} = -10 \text{ KNM}$$

Slope deflections are

$$M_{AB} = F_{AB} + \frac{2EI}{L}(2\theta_A + \theta_B)$$

$$= -106.67 + \frac{2E2I}{6}(\theta_B) = -106.67 + \frac{2}{3}EI\theta_B \qquad ----> (1)$$

$$M_{BA} = F_{BA} + \frac{2EI}{L}\left(2\theta_B + \theta_B\right)$$

$$= +53.33 + \frac{2E2I}{6}\left(2\theta_B\right) = +53.33 + \frac{4}{3}EI\theta_B \qquad ----> (2)$$

$$M_{BC} = F_{CB} + \frac{2EI}{L}\left(2\theta_B + \theta_C\right)$$

$$= -26.67 + \frac{2E}{4} \times \frac{3I}{2}\left(2\theta_B + \theta_C\right) = -26.67 + \frac{3}{2}EI\theta_B + \frac{3}{4}EI\theta_C \qquad ----> (3)$$

$$M_{CB} = F_{CB} + \frac{2EI}{L}\left(2\theta_C + \theta_B\right)$$

$$= +26.67 + \frac{2E}{4} \times \frac{3I}{2}\left(2\theta_C + \theta_B\right) = +26.67 + \frac{3}{2}EI\theta_C + \frac{3}{4}EI\theta_B \qquad ----> (4)$$

$$M_{BD} = F_{BD} + \frac{2EI}{L}\left(2\theta_B + \theta_D\right)$$

$$= +10 + \frac{2EI}{4}\left(2\theta_B + \theta_D\right) = +10 + EI\theta_B + \frac{1}{2}EI\theta_D \qquad ----> (5)$$

$$M_{DB} = F_{DB} + \frac{2EI}{L}\left(2\theta_D + \theta_B\right)$$

$$= -10 + \frac{2EI}{4}\left(2\theta_D + \theta_B\right) = -10 + EI\theta_D + \frac{1}{2}EI\theta_B \qquad ----> (6)$$

In the above equations we have three unknown rotations $\theta_B, \theta_C, \theta_D$ accordingly we have three boundary conditions.

$$M_{BA} + M_{BC} + M_{BD} = 0$$

$$M_{CB} = 0 \qquad \text{Since C and D are hinged}$$

$$M_{DB} = 0$$

Now

$$M_{BA} + M_{BC} + M_{BD} = 53.33 + \frac{4}{3}EI\theta_B - 26.67 + \frac{3}{2}EI\theta_B + \frac{3}{4}EI\theta_C + 10 + EI\theta_B + \frac{1}{2}EI\theta_D$$

$$= 36.66 + \frac{23}{6}EI\theta_B + \frac{3}{4}EI\theta_C + \frac{1}{2}EI\theta_D = 0 \qquad ----> (7)$$

$$M_{CB} = 26.67 + \frac{3}{4}EI\theta_B + \frac{3}{2}EI\theta_C = 0 \qquad ----> (8)$$

$$M_{DB} = -10 + \frac{1}{2}EI\theta_B + EI\theta_D = 0 \qquad ----> (9)$$

Solving equations 7, 8, & 9 we get

$$EI\theta_B = -8.83$$
$$EI\theta_C = -13.36$$
$$EI\theta_D = +14.414$$

Substituting these values in slope equations

$$M_{AB} = -106.67 + \frac{2}{3}(-8.83) = -112.56 \text{ KNM}$$

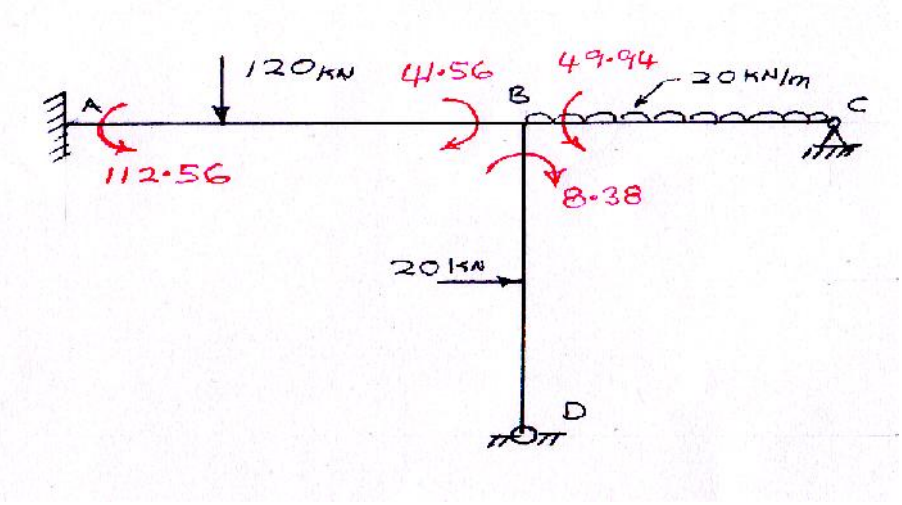
$$M_{BA} = 53.33 + \frac{4}{3}(-8.83) = 41.56 \text{ KNM}$$

$$M_{BC} = -26.67 + \frac{3}{2}(-8.3) + \frac{3}{4}(-13.36) = -49.94 \text{ KNM}$$

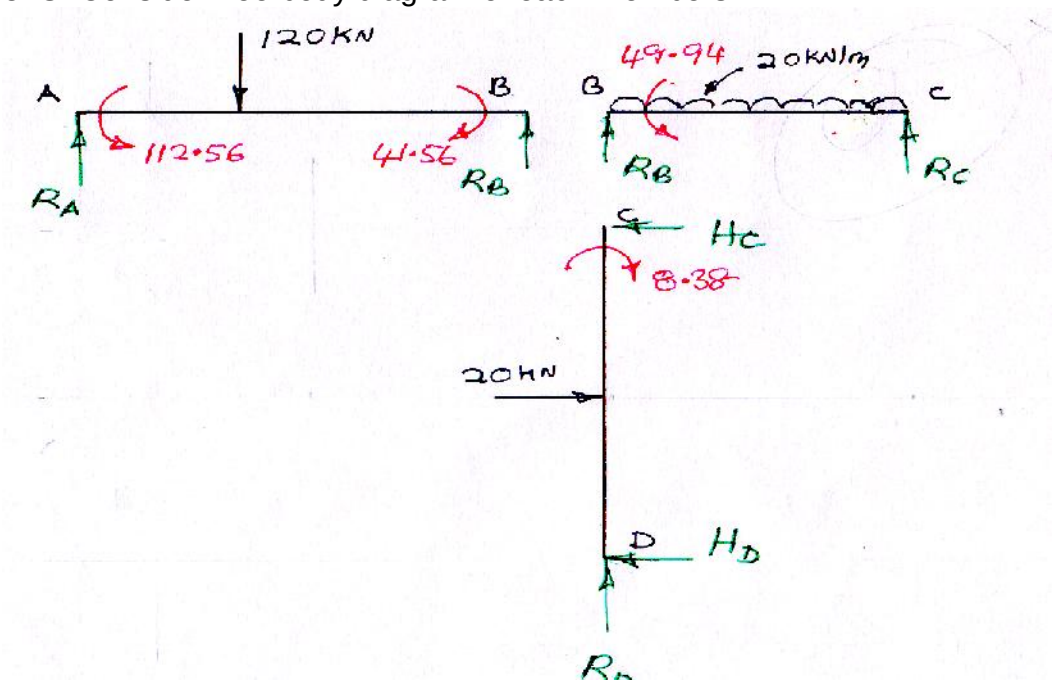
$$M_{CB} = +26.67 + \frac{3}{2}(-13.36) + \frac{3}{4}(-8.83) = 0$$

$$M_{BD} = 10 + (-8.83) + \frac{1}{2}(+14.414) = 8.38 \text{ KNM}$$

$$M_{DB} = -10 + (14.414) + \frac{1}{2}(-8.83) = 0$$



Reactions: Consider free body diagram of each members



Span AB:

$$R_B = \frac{41.56 - 112.56 + 120 \times 2}{6} = 28.17 \text{ KN}$$

$$\therefore R_A = 120 - R_B = 91.83 \text{ KN}$$

Span BC:

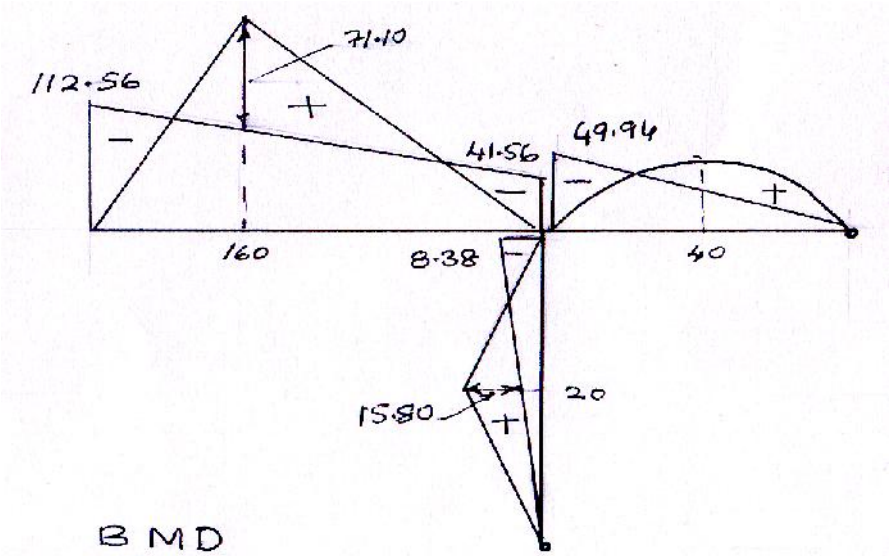
$$R_B = \frac{49.94 + 20 \times 4 \times 2}{4} = 52.485 \text{ KN}$$

$$\therefore R_C = 20 \times 4 - R_B = 27.515 \text{ KN}$$

Column BD:

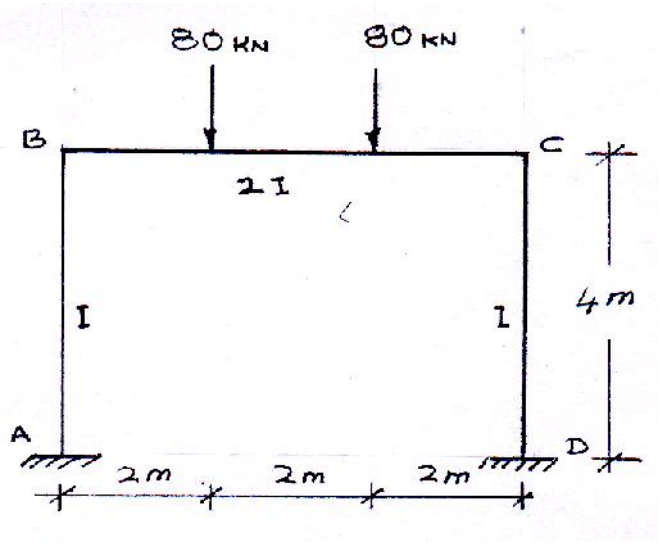
$$H_D = \frac{20 \times 2 - 8.33}{4} = 7.92 \text{ KN}$$

$$\therefore H_B = 12.78 \text{ KN} \qquad \left[\because H_A + H_D = 20\right]$$



BMD
in kN-m.

Example: Analyse the portal frame shown in figure and also drawn bending moment and shear force diagram



Solution:

Symmetrical problem
- Sym frame + Sym loading

$\theta_A = 0,\ \theta_B \neq 0,\ \theta_C \neq 0,\ \theta_D = 0$

FEMS

$$F_{BC} = -\frac{W_1 ab^2}{L^2} - \frac{W_2 cd^2}{L^2}$$

$$= \frac{80 \times 2 \times 4^2}{6^2} - \frac{80 \times 4 \times 2^2}{6^2} = -106.67 \quad \text{KNM}$$

$$F_{CB} = +\frac{Wa^2 b}{L^2} + \frac{W_2 c^2 d}{L^2} = +106.67 \quad \text{KNM}$$

Slope deflection equations:

$$M_{AB} = F_{AB} + \frac{2EI}{L}(2\theta_A + \theta_B) = 0 + \frac{2EI}{4}(0 + \theta_B) = \frac{1}{2}EI\theta_B \qquad \text{--------} > (1)$$

$$M_{BA} = F_{BA} + \frac{2EI}{L}(2\theta_B + \theta_A) = 0 + \frac{2EI}{4}(2\theta_B + 0) = EI\theta_B \qquad \text{-------} > (2)$$

$$M_{BC}=F_{BC}+\frac{2EI}{L}(2\theta_B+\theta_C)$$

$$=-106.67+\frac{2E2I}{6}(2\theta_B+\theta_C)=-106.67+\frac{4}{3}EI\theta_B+\frac{2}{3}EI\theta_C \quad ------>(3)$$

$$M_{CB}=F_{CB}+\frac{2EI}{L}(2\theta_C+\theta_B)$$

$$=+106.67+\frac{2E2I}{6}(2\theta_C+\theta_B)=+106.67+\frac{4}{3}EI\theta_C+\frac{2}{3}EI\theta_B \quad ------>(4)$$

$$M_{CD}=F_{CD}+\frac{2EI}{L}(2\theta_C+\theta_D)$$

$$=0+\frac{2EI}{4}(2\theta_C+0)=EI\theta_C \quad -------->(5)$$

$$M_{DC}=F_{DC}+\frac{2EI}{L}(2\theta_D+\theta_C)$$

$$=0+\frac{2EI}{4}(0+\theta_C)=\frac{1}{2}EI\theta_C \quad -------->(6)$$

In the above equation there are two unknown rotations. Accordingly the boundary conditions are

$$M_{BA}+M_{BC}=0$$

$$M_{CB}+M_{CD}=0$$

Now $M_{BA}+M_{BC}=-106.67+\frac{7}{3}EI\theta_B+\frac{2}{3}EI\theta_C=0$ --------> (7)

$$M_{CB}+M_{CD}=+106.67+\frac{2}{3}EI\theta_B+\frac{7}{3}EI\theta_C=0 \quad -------->(8)$$

Multiply by (7) and (8) by 2

$$\left.\begin{aligned}-746.69+\frac{49}{3}EI\theta_B+\frac{14}{3}EI\theta_C=0\\ +213.34+\frac{4}{3}EI\theta_B+\frac{14}{3}EI\theta_C=0\end{aligned}\right\}\text{subtracts}$$

$$-960.03+\frac{45}{3}EI\theta_B=0$$

$$EI\theta_B=+960.03\times\frac{3}{45}=+\ 64 \quad \text{Clockwise}$$

Using equation (7)

$$EI\theta_C = -\frac{3}{2}\left[-106.67 + \frac{7}{3}EI\theta_B\right]$$

$$= -\frac{3}{2}\left[-106.67 + \frac{7}{3}\times 64\right] = -\ 64 \quad \text{Anticlockwise}$$

Here we find $\theta_B = -\ \theta_C$. It is obvious because the problem is symmetrical.

$\therefore$ Final moments are

$$M_{AB} = +\frac{64}{2} = +32 \quad \text{KNM}$$

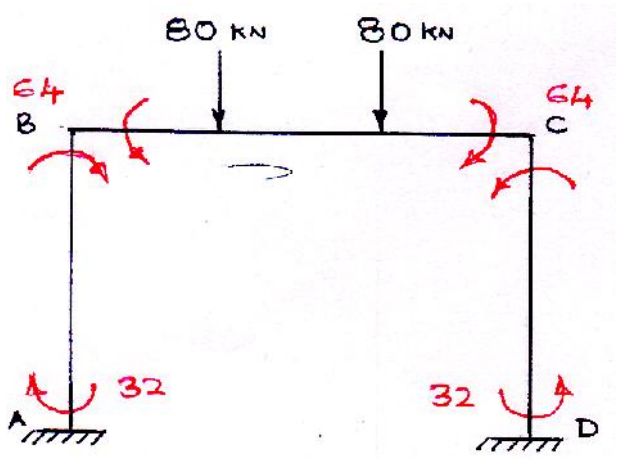
$$M_{BA} = 64 \quad \text{KNM}$$

$$M_{BC} = -106.67 + \frac{4}{3}64 + \frac{2}{3}(-64) = -\ 64 \quad \text{KNM}$$

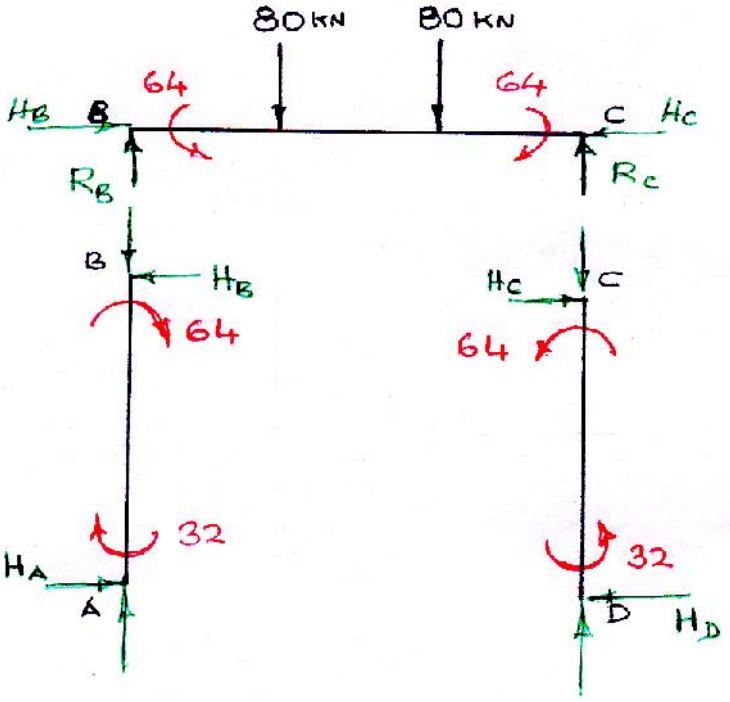
$$M_{CB} = +106.67 + \frac{4}{3}(-64) + \frac{2}{3}(64) = +\ 64 \quad \text{KNM}$$

$$M_{CD} = -\ 64 \quad \text{KNM}$$

$$M_{DC} = -\ \frac{1}{2}\ 64 = -32 \quad \text{KNM}$$



Consider free body diagram's of beam and columns as shown



By symmetrical we can write

$$R_A = R_B = 60 \quad KNM$$

$$R_D = R_C = 80 \quad KNM$$

Now consider free body diagram of column AB

Apply

$$\sum M_B = 0$$

$$H_A \times 4 = 64 + 32$$

$$\therefore H_A = 24 \quad KN$$

Similarly from free body diagram of column CD

Apply

$$\sum M_C = 0$$

$$H_A \times 4 = 64 + 32$$

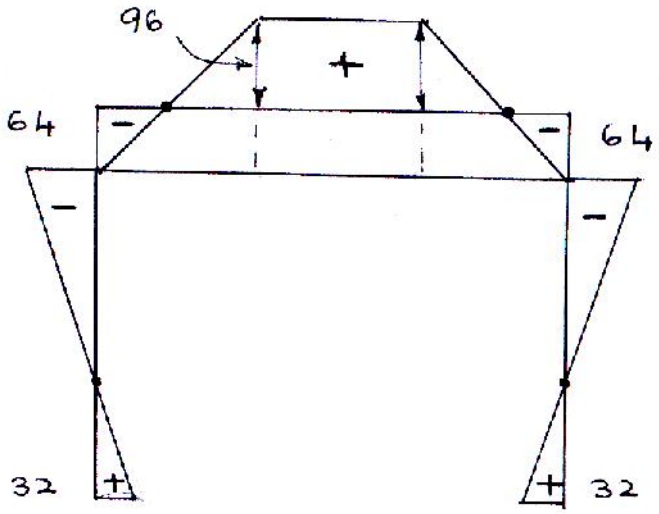
$$\therefore H_D = 24 \quad KN$$

Check:

$\sum H = 0$

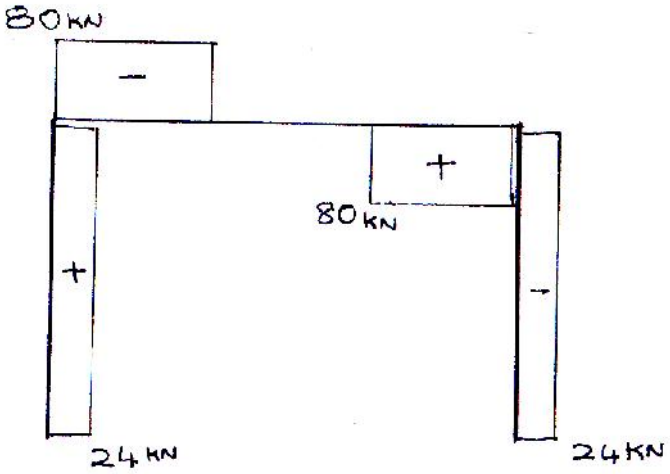
$H_A + H_D = 0$

Hence okay



BMD

In KN-m

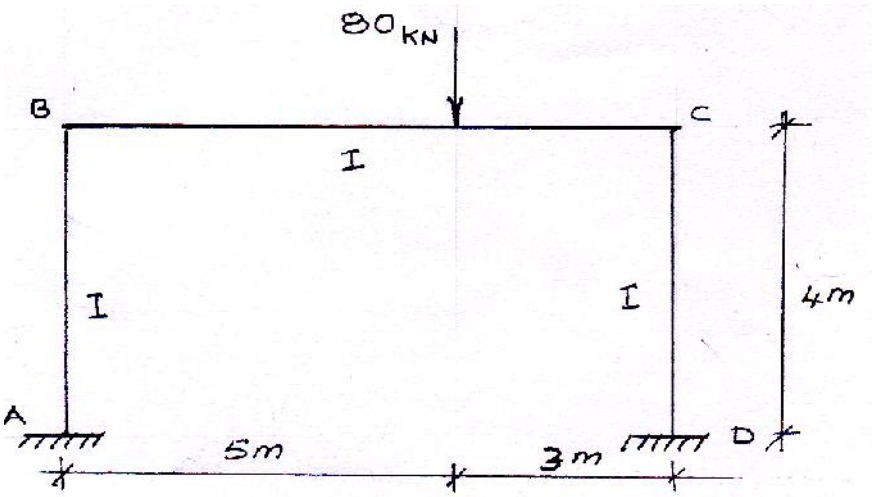


SFD

Note: Since symmetrical, only half frame may be analysed. Using first three equations

and taking $\theta_B = -\theta_C$

Example: Analyse the portal frame and then draw the bending moment diagram



Solution:

This is a symmetrical frame and unsymmetrically loaded, thus it is an unsymmetrical problem and there is a sway

Assume sway to right.

Here $\theta_A = 0, \theta_D = 0, \theta_B \neq 0, \theta_D = 0$

FEMS:

$$F_{BC} = -\frac{Wab^2}{L^2} = -\frac{80 \times 5 \times 3^2}{8^2} = -56.25 \quad \text{KNM}$$

$$F_{CB} = +\frac{Wa^2b}{L^2} = +\frac{80 \times 5^2 \times 3}{8^2} = +93.75 \quad \text{KNM}$$

Slope deflection equations

$$M_{AB} = F_{AB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(0 + \theta_B - \frac{3\delta}{4}\right) = \frac{1}{2}EI\theta_B - \frac{3}{8}EI\delta \qquad \text{--------} > (1)$$

$$M_{BA} = F_{BA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(2\theta_B + 0 - \frac{3\delta}{4}\right) = EI\theta_B - \frac{3}{8}EI\delta \qquad \text{--------} > (2)$$

$$M_{BC}=F_{BC}+\frac{2EI}{L}\left(2\theta_B+\theta_C\right)$$

$$=-56.25+\frac{2EI}{8}\left(2\theta_B+\theta_C\right)=-56.25+\frac{1}{2}EI\theta_B+\frac{1}{4}EI\theta_C \qquad \text{--------->}(3)$$

$$M_{CB}=F_{CB}+\frac{2EI}{L}\left(2\theta_C+\theta_B\right)$$

$$=+93.75+\frac{2EI}{8}\left(2\theta_C+\theta_B\right)=93.75+\frac{1}{2}EI\theta_C+\frac{1}{4}EI\theta_B \qquad \text{--------->}(4)$$

$$M_{CD}=F_{CD}+\frac{2EI}{L}\left(2\theta_C+\theta_D-\frac{3\delta}{L}\right)$$

$$=0+\frac{2EI}{4}\left(2\theta_C+0-\frac{3\delta}{4}\right)=EI\theta_C-\frac{3}{8}EI\delta \qquad \text{--------->}(5)$$

$$M_{DC}=F_{DC}+\frac{2EI}{L}\left(2\theta_D+\theta_C-\frac{3\delta}{L}\right)$$

$$=0+\frac{2EI}{4}\left(0+\theta_C-\frac{3\delta}{4}\right)=\frac{1}{2}EI\theta_C-\frac{3}{8}EI\delta \qquad \text{--------->}(6)$$

In the above equation there are three unknowns θ_B, θ_C and δ, accordingly the boundary conditions are,

$M_{BA}+M_{BC}=0$ ---> Joint conditions

$M_{CB}+M_{CD}=0$

$H_A+H_D+\sum P_H=0$ ---> Shear condition

i.e, $\frac{M_{AB}+M_{BA}}{4}+\frac{M_{CD}+M_{DC}}{4}=0$

$\therefore \quad M_{AB}+M_{BA}+M_{CD}+M_{DC}=0$

Now, $M_{BA}+M_{BC}=EI\theta_B-\frac{3}{8}EI\delta-56.25+\frac{1}{2}EI\theta_B+\frac{1}{4}EI\theta_C=0$

$$=-56.25+\frac{3}{2}EI\theta_B+\frac{1}{4}EI\theta_C-\frac{3}{8}EI\delta=0 \qquad \text{------->}(7)$$

And, $M_{CB}+M_{CD}=93.75+\frac{1}{2}EI\theta_C+\frac{1}{4}EI\theta_B+EI\theta_C-\frac{3}{8}EI\delta=0$

$$=93.75+\frac{1}{4}EI\theta_B+\frac{3}{2}EI\theta_C-\frac{3}{8}EI\delta=0 \qquad \text{------->}(8)$$

And, $M_{AB}+M_{BA}+M_{CD}+M_{DC}=\frac{1}{2}EI\theta_B-\frac{3}{8}EI\delta+EI\theta_B-\frac{3}{8}EI\delta+EI\theta_C-\frac{3}{8}EI\delta$

$$+\frac{1}{2}EI\theta_C-\frac{3}{8'}EI\delta$$

$$=\frac{3}{2}EI\theta_B+\frac{3}{2}EI\theta_C-\frac{3}{2}EI\delta=0 \qquad ------>(9)$$

From (9) $EI\delta=EI\theta_B+EI\theta_C$

Substitute in (7) & (8)

Eqn (7)

$$-56.25+\frac{3}{2}EI\theta_B+\frac{1}{4}EI\theta_C-\frac{3}{8}\left[EI\theta_B+EI\theta_C\right]=0$$

$$-56.25+\frac{9}{8}EI\theta_B-\frac{1}{8}EI\theta_C=0 \qquad ------->(10)$$

Eqn(8)

$$+93.75+\frac{1}{4}EI\theta_B+\frac{3}{2}EI\theta_C-\frac{3}{8}\left[EI\theta_B+EI\theta_C\right]=0$$

$$+93.75-\frac{1}{8}EI\theta_B+\frac{9}{8}EI\theta_C=0 \qquad ---------->(11)$$

Solving equations (10) & (11) we get $EI\theta_B=41.25$

By Equation (10)

$$EI\theta_C=8\left[-56.25+\frac{9}{8}EI\theta_B\right]$$

$$=8\left[-56.25+\frac{9}{8}41.25\right]=-78.75$$

$$\therefore EI\delta=EI\theta_B+EI\theta_C=41.25-78.75=-37.5$$

Hence

$$EI\theta_B=41.25,\ EI\theta_C=-78.75,\ EI\delta=-37.5$$

Substituting these values in slope deflection equations, we have

$$M_{AB} = \frac{1}{2}(41.25) - \frac{3}{8}(-37.5) = +34.69 \text{ KNM}$$

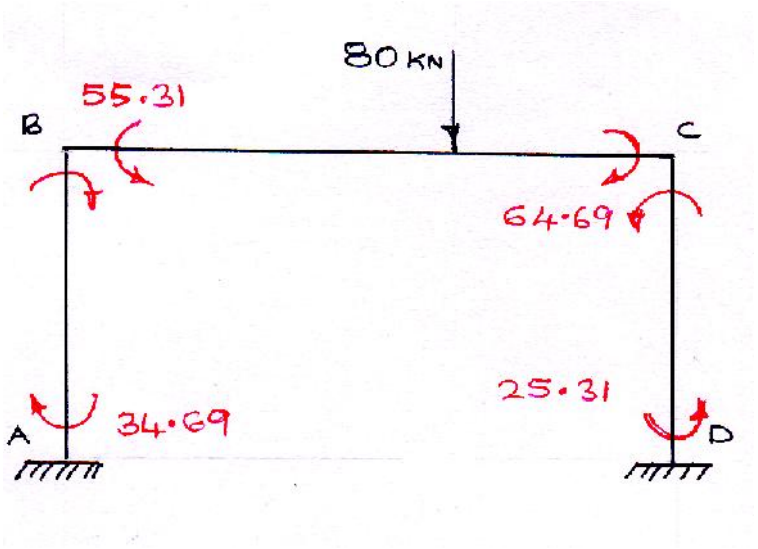
$$M_{BA} = 41.25 - \frac{3}{8}(-37.5) = +55.31 \text{ KNM}$$

$$M_{BC} = -56.25 + \frac{1}{2}(41.25) + \frac{1}{4}(-78.75) = -55.31 \text{ KNM}$$

$$M_{CB} = 93.75 + \frac{1}{2}(-78.75) + \frac{1}{4}(41.75) = +64.69 \text{ KNM}$$

$$M_{CD} = -78.75 - \frac{3}{8}(-37.5) = -64.69 \text{ KNM}$$

$$M_{DC} = \frac{1}{2}(-78.75) - \frac{3}{8}(-37.5) = -25.31 \text{ KNM}$$



Reactions: consider the free body diagram of beam and columns

Column AB:

$$H_A = \frac{34.69 + 55.31}{4} = 22.5 \text{ KN}$$

Span BC:

$$R_B = \frac{55.31 - 64.69 + 80 \times 3}{8} = 28.83 \text{ KN}$$

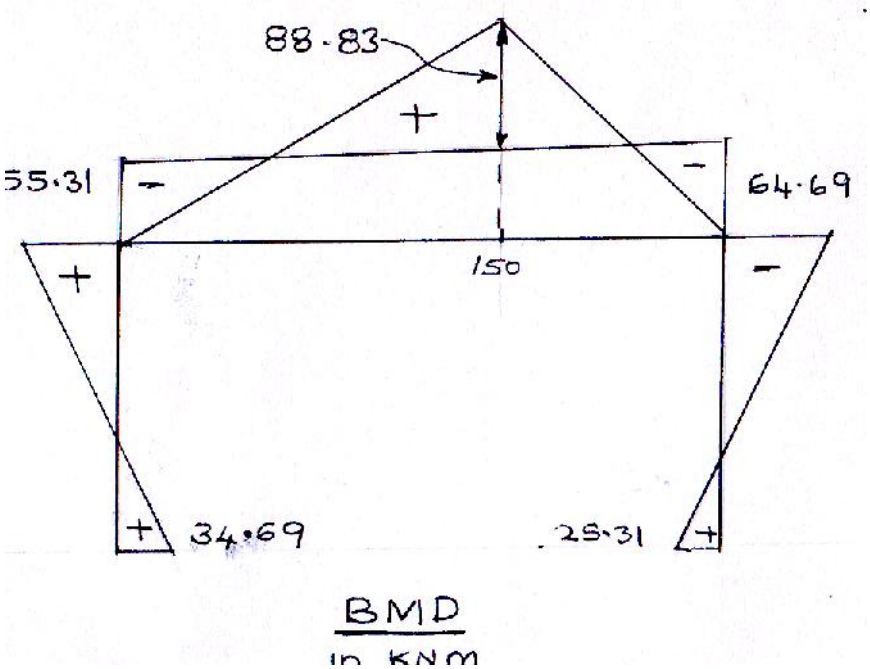
$$\therefore \; R_C = 80 - R_B = 51.17$$

Column CD:

$$H_D = \frac{64.69 + 25.31}{4} = 22.5$$

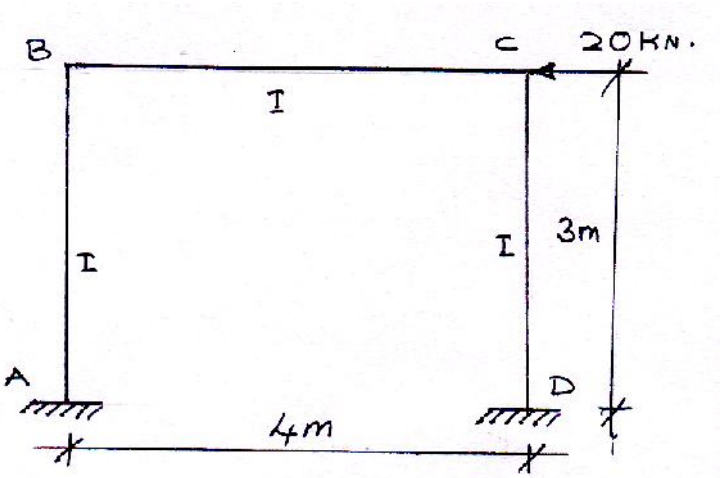
Check:

$\Sigma H = 0$
$H_A + H_D = 0$
$22.5 – 22.5 = 0$
Hence okay



BMD
in KNM

Example: Frame ABCD is subjected to a horizontal force of 20 KN at joint C as shown in figure. Analyse and draw bending moment diagram.



Solution:

Frame is Symmetrical and unsymmetrical loaded hence there is a sway. Assume sway towards right

FEMS

$$F_{AB}=F_{BA}=F_{BC}=F_{CB}=F_{CD}=F_{DC}=0$$

Slope deflection equations are

$$M_{AB}=F_{AB}+\frac{2EI}{L}\left(2\theta_A+\theta_B-\frac{3\delta}{L}\right)$$

$$=\frac{2EI}{3}\left(\theta_B-\frac{3\delta}{3}\right)$$

$$=\frac{2}{3}EI\theta_B-\frac{2}{3}EI\delta \qquad \text{--------->}(1)$$

$$M_{BA}=F_{BA}+\frac{2EI}{L}\left(2\theta_B+\theta_A-\frac{3\delta}{L}\right)$$

$$=\frac{2EI}{3}\left(2\theta_B-\frac{3\delta}{3}\right)$$

$$=\frac{4}{3}EI\theta_B-\frac{2}{3}EI\delta \qquad \text{------->}(2)$$

$$M_{BC} = F_{BC} + \frac{2EI}{L}(2\theta_B + \theta_C)$$

$$= \frac{2EI}{4}(2\theta_B + \theta_C)$$

$$= EI\theta_B + 0.5\, EI\theta_C \qquad \text{--------} > (3)$$

$$M_{CB} = F_{CB} + \frac{2EI}{L}(2\theta_C + \theta_B)$$

$$= \frac{2EI}{4}(2\theta_C + \theta_B)$$

$$= EI\theta_C + 0.5\, EI\theta_B \qquad \text{--------} > (4)$$

$$M_{CD} = F_{CD} + \frac{2EI}{L}\left(2\theta_C + \theta_D - \frac{3\delta}{L}\right)$$

$$= \frac{2EI}{3}\left[2\theta_C - \frac{3\delta}{3}\right]$$

$$= \frac{4}{3}EI\theta_C - \frac{2}{3}EI\delta \qquad ------ > (5)$$

$$M_{DC} = F_{DC} + \frac{2EI}{L}\left[2\theta_D + \theta_C - \frac{3\delta}{L}\right]$$

$$= \frac{2\,EI}{3}\left[\theta_c - \frac{3\delta}{3}\right]$$

$$= \frac{2}{3}EI\theta_C - \frac{2}{3}EI\delta \qquad \text{--------} > (6)$$

The unknown are $\theta_{B,}, \theta_C$ & δ. Accordingly the boundary conditions are

I. $M_{BA} + M_{BC} = 0$

II. $M_{CB} + M_{CD} = 0$

III. $H_A + H_D - 20 = 0$

i.e, $\frac{M_{AB} + M_{BA}}{3} + \frac{M_{CD} + M_{DC}}{3} - 20 = 0$

$$M_{AB} + M_{BA} + M_{CD} + M_{DC} - 60 = 0$$

Now $M_{BA} + M_{BC} = \frac{4}{3}EI\theta_B - \frac{2}{3}EI\delta + EI\theta_B + 0.5EI\theta_C$

$$= \frac{7}{3}EI\theta_B + 0.5EI\theta_C - \frac{2}{3}EI\delta = 0 \quad ------ > (7)$$

$$\text{and } M_{CB}+M_{CD}=EI\theta_C+0.5EI\theta_B+\frac{4}{3}EI\theta_C-\frac{2}{3}EI\delta$$

$$=0.5EI\theta_B+\frac{7}{3}EI\theta_C-\frac{2}{3}EI\delta=0 \quad ------>(8)$$

$$\text{and } M_{AB}+M_{BA}+M_{CD}+M_{DC}-60=\frac{2}{3}EI\theta_B-\frac{2}{3}EI\delta+\frac{4}{3}EI\theta_B-\frac{2}{3}EI\delta+\frac{4}{3}EI\theta_C$$

$$-\frac{2}{3}EI\delta+\frac{2}{3}EI\theta_C-\frac{2}{3}EI\delta-60$$

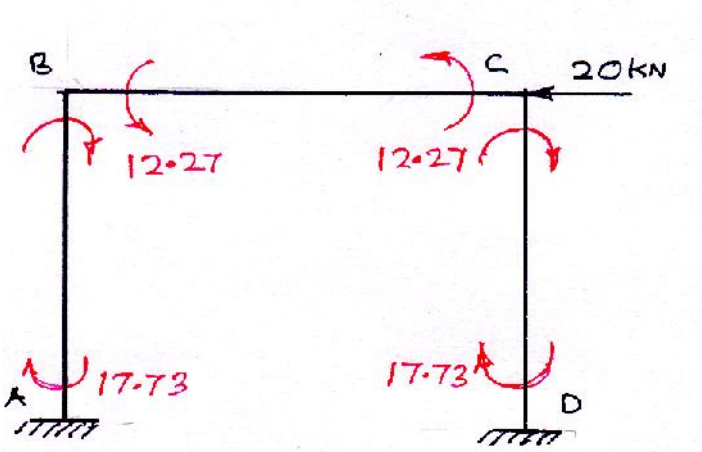
$$=2EI\theta_B+2EI\theta_C-\frac{8}{3}EI\delta-60=0--------(9)$$

Solving (7).(8) & (9) we get

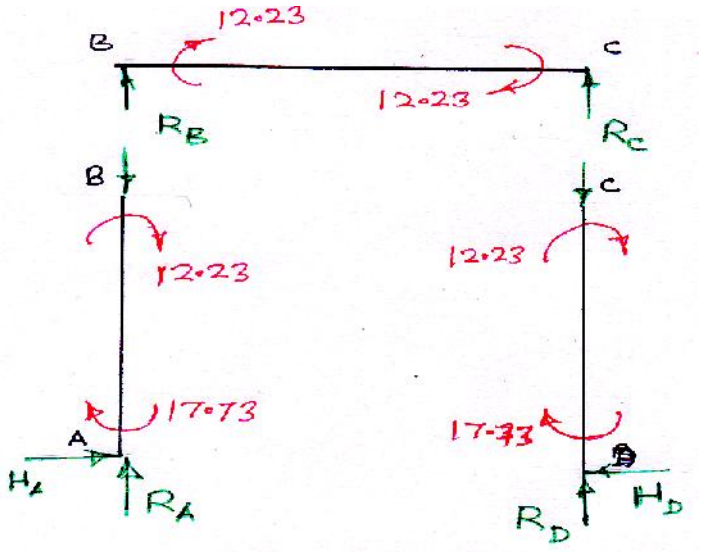
$$EI\theta_B=-8.18,$$
$$EI\theta_C=-8.18,$$
$$EI\delta=-34.77$$

Substituting the value of θ_B, θ_C and δ in slope deflection equations

$$M_{AB}=\frac{2}{3}(-8.18)-\frac{2}{3}(-34.77)=+17.73 \text{ KNM}$$
$$M_{BA}=\frac{4}{3}(-8.18)-\frac{2}{3}(-34.77)=+12.27 \text{ KNM}$$
$$M_{BC}=0-8.18+0.5(-8.18)=-12.27 \text{ KNM}$$
$$M_{CB}=0.5(-8.18)-8.18=-12.27 \text{ KNM}$$
$$M_{CD}=\frac{4}{3}(-8.18)-\frac{2}{3}(-34.77)=+12.27 \text{ KNM}$$
$$M_{DC}=\frac{2}{3}(-8.18)-\frac{2}{3}(-34.77)=+17.73 \text{ KNM}$$



Reactions: Consider the free body diagram of the members



Member AB:

$$H_A = \frac{17.73 + 12.27}{3} = 10 \text{ KN}$$

Member BC:

$$R_C = \frac{12.27 + 12.27}{4} = 6.135 \text{ KN}$$

$\therefore R_B = -R_C = -6.135$ KN - ve sign indicates direction of R_B downwards

Member CD:

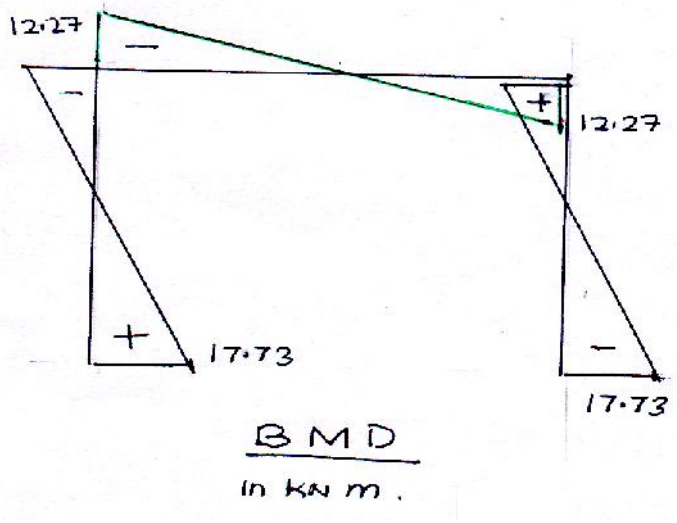
$H_D = \frac{-17.73 - 12.27}{3} = -10\ KN$ - ve sign indicates the direction of H_D is left to right

Check: $\Sigma H = 0$

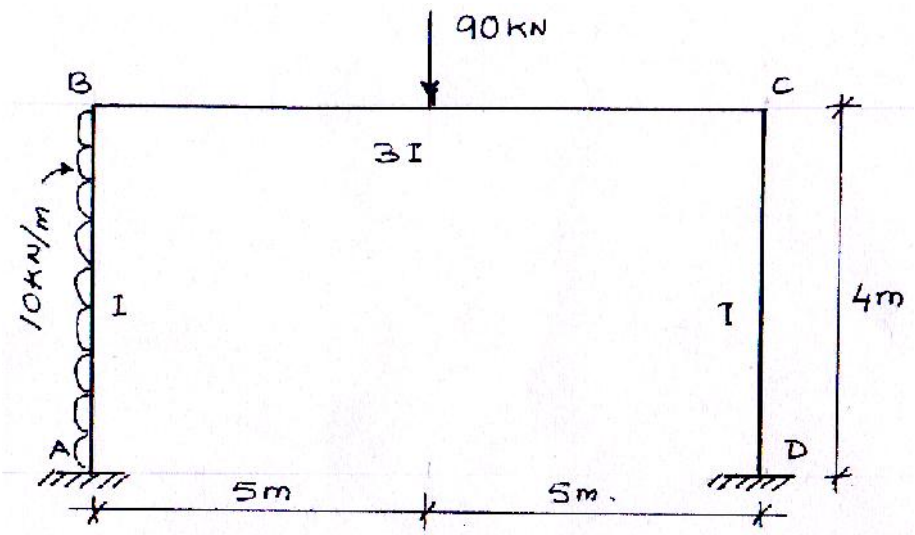
$H_A + H_D + P = 0$

$+10 + 10 - 20 = 0$

Hence okay



Example: Analyse the portal frame subjected to loads as shown. Also draw bending moment diagram.



The frame is symmetrical but loading is unsymmetrical. Hence there is a sway.

Assume sway towards right. In this problem $\theta_A = 0, \theta_B \neq 0, \theta_C \neq 0, \theta_D = 0$

FEMs:

$$F_{AB} = -\frac{wl^2}{12} = -\frac{10 \times 4^2}{12} = \text{ - } 13.33 \quad \text{KNM}$$

$$F_{BA} = +\frac{wl^2}{12} = +\frac{10 \times 4^2}{12} = +13.33 \quad \text{KNM}$$

$$F_{BC} = -\frac{wl}{8} = -\frac{90 \times 10}{8} = \text{ - } 112.5 \quad \text{KNM}$$

$$F_{CB} = \frac{wl}{8} = \frac{90 \times 10}{8} = + \ 112.5 \quad \text{KNM}$$

Slope deflection equations:

$$M_{AB} = F_{AB} + \frac{2EI}{L}\left(2\theta_A + \theta_B + \frac{3\delta}{L}\right)$$

$$= \text{ -}13.33 + \frac{2EI}{4}\left(0 + \theta_B - \frac{3\delta}{4}\right)$$

$$= \text{ -}13.33 + 0.5\,EI\theta_B - 0.375\,EI\delta \qquad \text{--------} > (1)$$

$$M_{BA} = F_{BA} + \frac{2EI}{L}\left(2\theta_B + \theta_A + \frac{3\delta}{L}\right)$$

$$= 13.33 + \frac{2EI}{4}\left(2\theta_B + 0 - \frac{3\delta}{4}\right)$$

$$= 13.33 + EI\theta_B - 0.375\,EI\delta \qquad \text{--------} > (2)$$

$$M_{BC} = F_{BC} + \frac{2EI}{L}(2\theta_B + \theta_C)$$

$$= -112.5 + \frac{2E3I}{10}(2\theta_B + \theta_C)$$

$$= -112.5 + 1.2EI\theta_B + 0.6EI\theta_C \quad \text{--------->} (3)$$

$$M_{CB} = F_{CB} + \frac{2EI}{L}(2\theta_C + \theta_B)$$

$$= +112.5 + \frac{2E3I}{10}(2\theta_C + \theta_B)$$

$$= 112.5 + 1.2EI\theta_C + 0.6\ EI\theta_B \quad \text{--------->} (4)$$

$$M_{CD} = F_{CD} + \frac{2EI}{L}\left(2\theta_C + \theta_D - \frac{3\delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(2\theta_C + 0 - \frac{3\delta}{4}\right)$$

$$= EI\,\theta_C - 0.375\,EI\delta \quad \text{--------->} (5)$$

$$M_{DC} = F_{DC} + \frac{2EI}{L}\left(2\theta_D + \theta_C - \frac{3\delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(0 + 2\theta_C - \frac{3\delta}{4}\right)$$

$$= 0.5EI\,\theta_C - 0.375\,EI\delta \quad \text{--------->} (6)$$

There are 3 unknowns $EI\theta_B$, $EI\theta_C$ and $EI\delta$, accordingly the boundary conditions are

$$M_{BA} + M_{BC} = 0$$

$$M_{CB} + M_{CD} = 0$$

$$H_A + H_D + 40 = 0$$

Here $H_A \times 4 = M_{AB} + M_{BA} - 10 \times 4 \times \frac{4}{2}$

$$H_A = \frac{M_{AB} + M_{BA} - 80}{4}$$

and $H_D \times 4 = M_{CD} + M_{BC}$

$$H_D = \frac{M_{CD} + M_{DC}}{4}$$

$$\therefore \quad \frac{M_{AB} + M_{BA} - 80}{4} + \frac{M_{CD} + M_{DC}}{4} + 40 = 0$$

$$M_{AB} + M_{BA} + M_{CD} + M_{DC} + 80 = 0$$

Now $M_{BA} + M_{BC} = 0$

$$13.33EI\theta_B - 0.375EI\delta - 112.5 + 1.2EI\theta_B + 0.6EI\theta_C = 0$$

$$2.2EI\theta_B + 0.6EI\theta_C - 0.375EI\delta - 99.17 = 0 \quad \text{--------> (7)}$$

and $M_{CB} + M_{DC} = 0 \quad (4) + (5)$

$$112.5 + 1.2EI\theta_C + 0.6EI\theta_B + EI\theta_C - 0.375EI\delta = 0$$

$$112.5 + 2.2EI\theta_C + 0.6EI\theta_B - 0.375EI\delta = 0 \quad \text{--------> (8)}$$

also $M_{AB} + M_{BA} + M_{CB} + M_{DC} + 80 = 0$

$$-13.33 + 0.5EI\theta_B - 0.375EI\delta + 13.33 + EI\theta_B - 0.375EI\theta\delta + EI\theta_C - 0.375EI\delta$$

$$+ 0.5EI\theta_C - 0.375EI\delta + 80 = 0$$

$$1.5EI\theta_B + 1.5EI\theta_C - 1.5EI\delta + 80 = 0 \quad \text{--------> (9)}$$

By solving (7), (8) and (9) we get

$$EI\theta_B = 72.65$$

$$EI\theta_C = -59.64$$

$$EI\delta = +66.34$$

Final moments:

$$M_{AB} = -13.33 + 0.5(72.65) - 0.375(66.34) = \text{-}1.88 \quad \text{KNM}$$

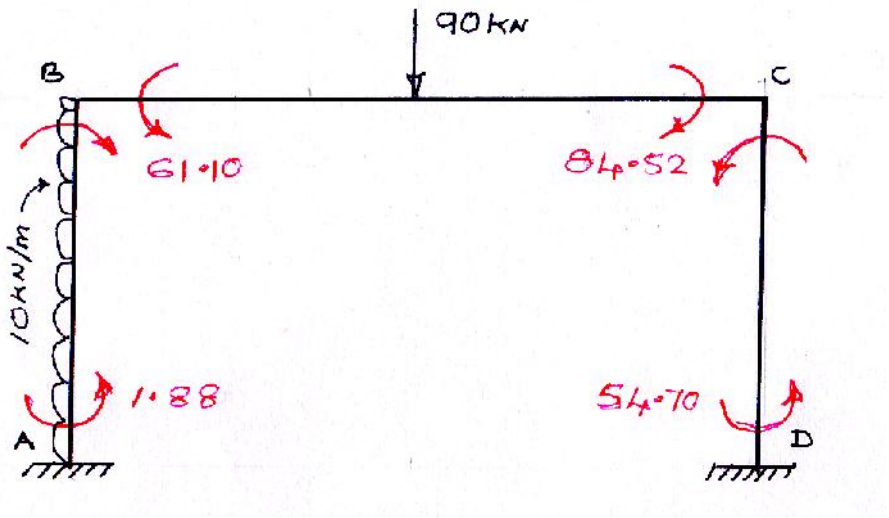
$$M_{BA} = +72.65 - 0.375(66.34) = 61.10 \quad \text{KNM}$$

$$M_{BC} = -112.5 + 1.2(72.65) = +0.6(-59.64) = -61.10 \quad \text{KNM}$$

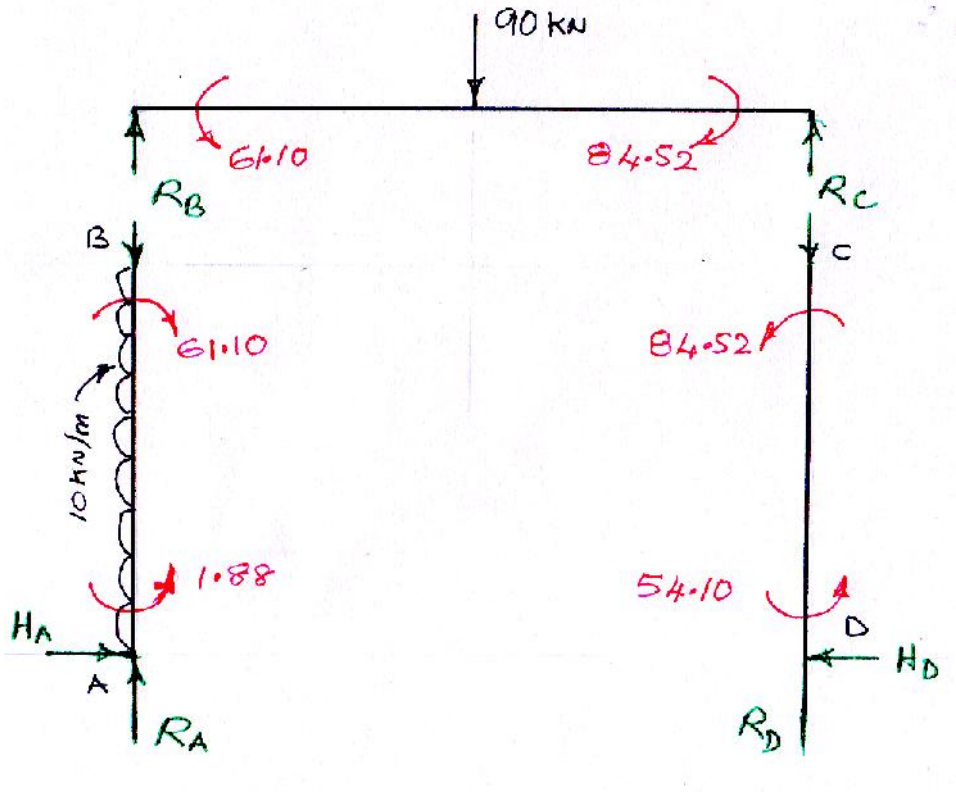
$$M_{CB} = 112.5 + 1.2(-59.64) + 0.6(72.65) = 84.52 \quad \text{KNM}$$

$$M_{CD} = -59.64 - 0.375(66.34) = -84.52 \quad \text{KNM}$$

$$M_{DC} = 0.5(-59.64) - 0.375(66.34) = -54.70 \quad \text{KNM}$$



Reactions: Consider the free body diagrams of various members



Member AB:

$$H_A = \frac{61.10 - 1.88 - 10 \times 4 \times 2}{4}$$

$= -5.195$ KN - ve sign indicates direction of H_A is from right to left

Member BC:

$$R_C = \frac{84.52 - 61.10 + 90 \times 5}{10} = 47.34 \text{ KN}$$

$$\therefore R_B = 90 - R_C = 38.34 \text{ KN}$$

Member CD

$$H_D = \frac{84.54 + 54.7}{4} = 34.81 \text{KN}$$

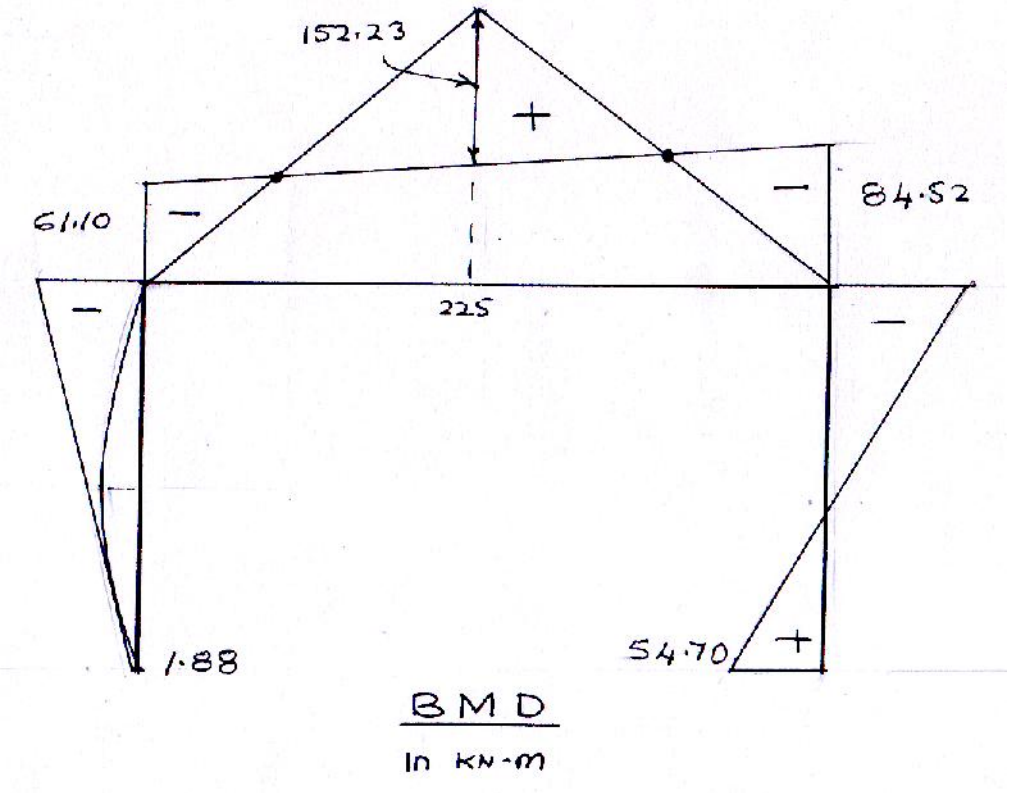
Check

$\Sigma H = 0$

$H_A + H_D + 10 \times 4 = 0$

-5.20 - 34.81+ 40 = 0

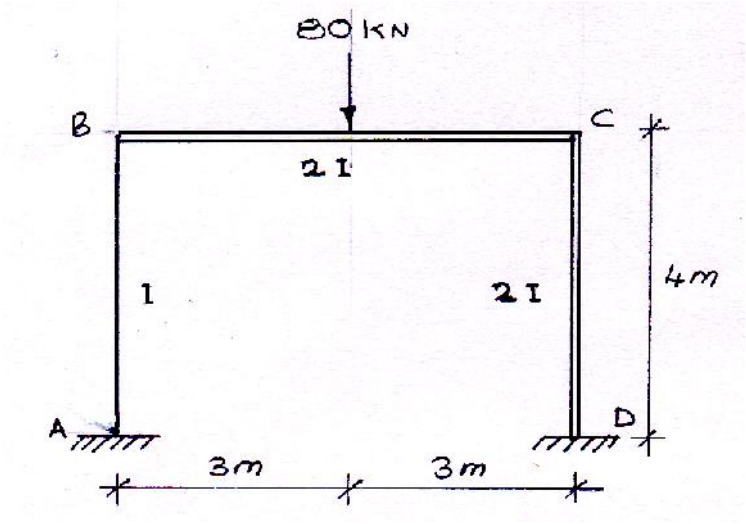
Hence okay



BMD

In KN-m

Example: Analyse the portal frame and then draw the bending moment diagram



Solution:

Since the columns have different moment of inertia, it is an unsymmetrical frame. Assume sway towards right

FEMS:

$$F_{BC} = -\frac{WL}{8} = -\frac{80 \times 6}{8} = -60 \quad KNM$$

$$F_{CB} = +\frac{WL}{8} = +60 \quad KNM$$

Here $\theta_A = 0,\ \theta_D = 0$

Slope deflection equations

$$M_{AB} = F_{AB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(0 + \theta_B - \frac{3\delta}{4}\right) = \frac{1}{2}EI\theta_B - \frac{3}{8}EI\delta \qquad \text{--------} > (1)$$

$$M_{BA} = F_{BA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(2\theta_B + 0 - \frac{3\delta}{4}\right) = EI\theta_B - \frac{3}{8}EI\delta \qquad \text{--------} > (2)$$

$$M_{BC}=F_{BC}+\frac{2EI}{L}(2\theta_B+\theta_C)$$

$$=-60+\frac{2E2I}{6}(2\theta_B+\theta_C)=-60+\frac{4}{3}EI\theta_B+\frac{2}{3}EI\theta_C \qquad ---------->(3)$$

$$M_{CB}=F_{CB}+\frac{2EI}{L}(2\theta_C+\theta_B)$$

$$=60+\frac{2E2I}{6}(2\theta_C+\theta_B)=+60+\frac{2}{3}EI\theta_B+\frac{4}{3}EI\theta_C \qquad ---------->(4)$$

$$M_{CD}=F_{CD}+\frac{2EI}{L}\left(2\theta_C+\theta_D-\frac{3\delta}{L}\right)$$

$$=0+\frac{2E2I}{4}\left(2\theta_C+0-\frac{3\delta}{4}\right)=2EI\theta_C-\frac{3}{4}EI\delta \qquad ---------->(5)$$

$$M_{DC}=F_{DC}+\frac{2EI}{L}\left(2\theta_D+\theta_C-\frac{3\delta}{L}\right)$$

$$=0+\frac{2E2I}{4}\left(0+\theta_C-\frac{3\delta}{4}\right)=EI\theta_C-\frac{3}{4}EI\delta \qquad ---------->(6)$$

In the above equation there are three unknowns θ_B, θ_C and δ, accordingly the boundary conditions are,

$$M_{BA}+M_{BC}=0$$

$$M_{CB}+M_{CD}=0 \qquad -->\text{Jo int conditions}$$

$$H_A+H_D=0 \qquad --->\text{Shear condition}$$

$$\text{i.e, } \frac{M_{AB}+M_{BA}}{4}+\frac{M_{CD}+M_{DC}}{4}=0$$

$$\therefore \quad M_{AB}+M_{BA}+M_{CD}+M_{DC}=0$$

Now, $M_{BA}+M_{BC}=EI\theta_B-\frac{3}{8}EI\delta-60+\frac{4}{3}EI\theta_B+\frac{2}{3}EI\theta_C=0$

$$=-60+\frac{7}{3}EI\theta_B+\frac{2}{3}EI\theta_C-\frac{3}{8}EI\delta=0 \qquad ------>(7)$$

And, $M_{CB}+M_{CD}=+60+\frac{2}{3}EI\theta_B+\frac{4}{3}EI\theta_C+2EI\theta_C-\frac{3}{4}EI\delta=0$

$$=\frac{2}{3}EI\theta_B+\frac{10}{3}EI\theta_C-\frac{3}{4}EI\delta+60=0 \qquad ------>(8)$$

And, $M_{AB}+M_{BC}+M_{CD}+M_{DC}=\frac{1}{2}EI\theta_B-\frac{3}{8}EI\delta+EI\theta_B-\frac{3}{8}EI\delta+2EI\theta_C-\frac{3}{4}EI\delta$

$$+EI\theta_C-\frac{3}{4}EI\delta$$

$$=\frac{3}{2}EI\theta_B+3EI\theta_C-\frac{9}{4}EI\delta=0 \qquad ------>(9)$$

From (9) $\qquad EI\delta=\frac{4}{9}\left(\frac{3}{2}EI\theta_B+3EI\theta_C\right)$

Substituting value of $EI\delta$ in (7)

$$\frac{7}{3}EI\theta_B+\frac{2}{3}EI\theta_C-\frac{3}{8}\left[\frac{4}{9}\left(\frac{3}{2}EI\theta_B+3EI\theta_C\right)\right]-60=0$$

$$\frac{7}{3}EI\theta_B+\frac{2}{3}EI\theta_C-\frac{1}{4}EI\theta_B-\frac{1}{2}EI\theta_C-60=0$$

$$\frac{25}{12}EI\theta_B+\frac{1}{6}EI\theta_C-60=0 \qquad ------>(10)$$

Substituting value of $EI\delta$ in (8)

$$\frac{2}{3}EI\theta_B+\frac{10}{3}EI\theta_C-\frac{3}{8}\left[\frac{4}{9}\left(\frac{3}{2}EI\theta_B+3EI\theta_C\right)\right]+60=0$$

$$\frac{2}{3}EI\theta_B+\frac{10}{3}EI\theta_C-\frac{1}{2}EI\theta_B-EI\theta_C+60=0$$

$$\frac{1}{6}EI\theta_B+\frac{7}{3}EI\theta_C+60=0 \qquad ------>(11)$$

Solving (10) & (11) we get $EI\theta_B$ =31.03

By Equation (11)

$$EI\theta_C=\frac{3}{7}\left[\frac{1}{6}EI\theta_B+60\right]$$

$$=-27.3$$

Now

$$EI\delta=\frac{4}{9}\left(\frac{3}{2}EI\theta_B+3EI\theta_C\right)=-16.55$$

Now

$$EI\theta_B=31.03,\ EI\theta_C=-27.3,\ EI\delta=-16.55$$

Substituting these values in slope deflection equations,
The final moments are:

$$M_{AB} = \frac{1}{2}(31.03) - \frac{3}{8}(-16.55) = 21.72 \text{ KNM}$$

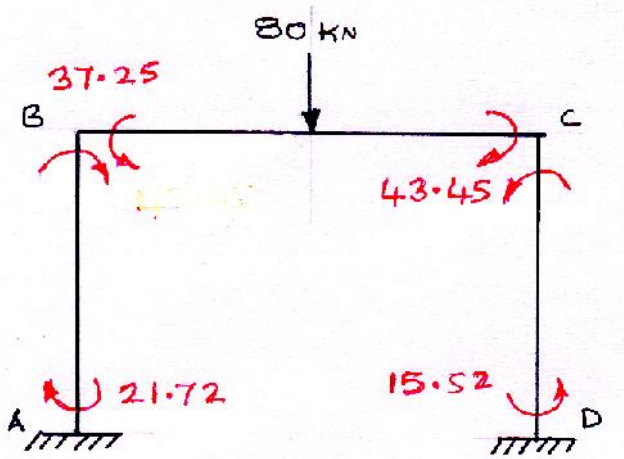
$$M_{BA} = 31.03 - \frac{3}{8}(-16.55) = 37.24 \text{ KNM}$$

$$M_{BC} = -60 + \frac{4}{3}(31.03) + \frac{2}{3}(-27.93) = -37.25 \text{ KNM}$$

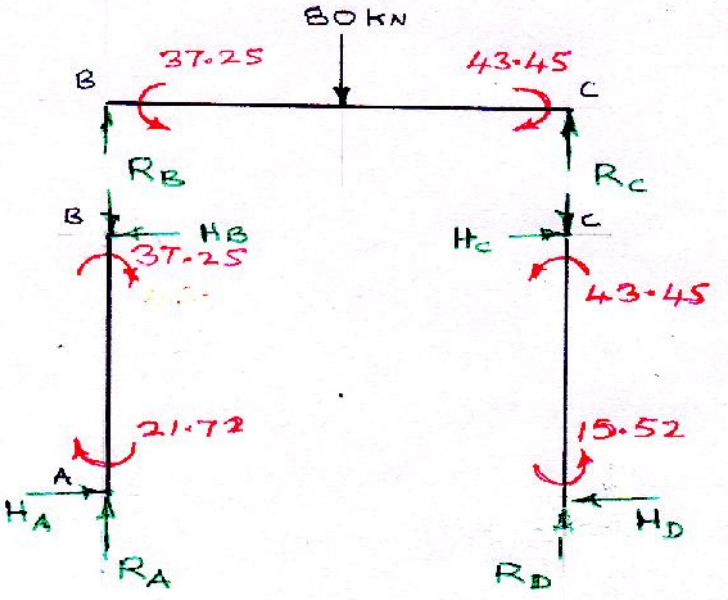
$$M_{CB} = +60 + \frac{2}{3}(31.03) + \frac{4}{3}(-27.93) = +43.43 \text{ KNM}$$

$$M_{CD} = 2(-27.93) - \frac{3}{4}(-16.55) = -43.45 \text{ KNM}$$

$$M_{DC} = -27.93 - \frac{3}{4}(-16.55) = -15.52 \text{ KNM}$$



Reactions: consider the free body diagram of beam and columns



Column AB:

$$H_A = \frac{37.25 + 21.72}{4} = 14.74 \text{ KN}$$

Beam BC:

$$R_B = \frac{37.25 - 43.45 + 80 \times 3}{6} = 38.97 \text{ KN}$$

$$\therefore R_C = 80 - R_B = 41.03$$

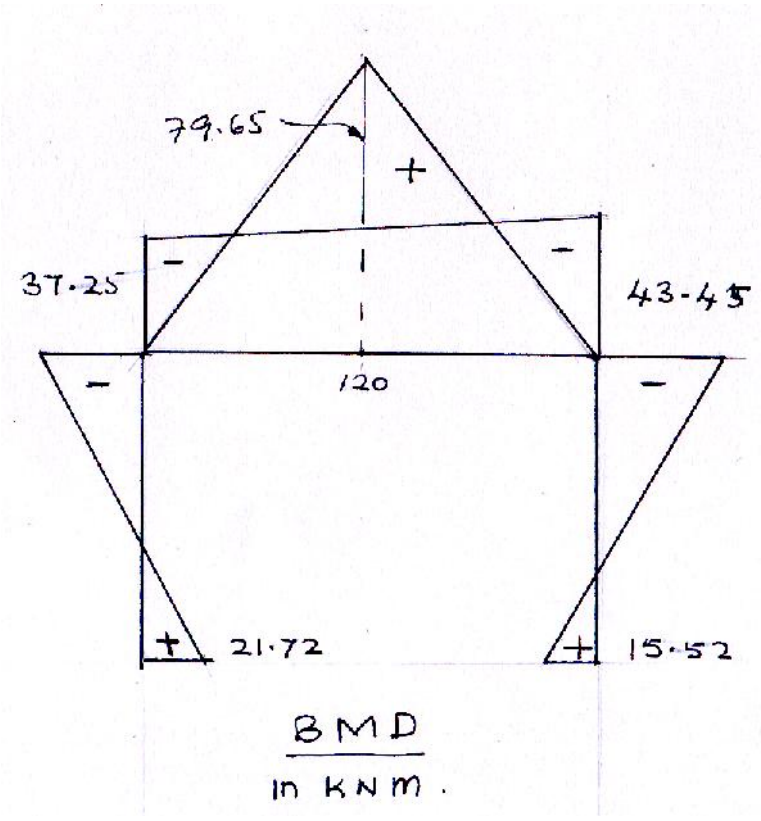
Column CD:

$$H_D = \frac{43.45 + 15.52}{4} = 14.74 \text{ KN}$$

Check:

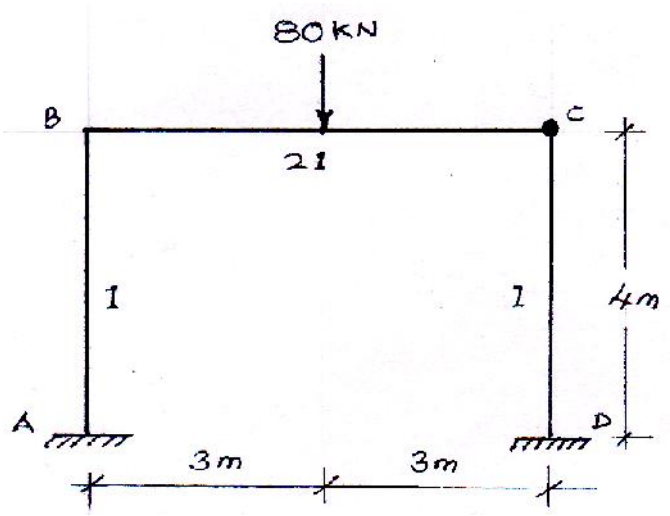
$\Sigma H = 0$
$H_A + H_D = 0$
14.74-14.74=0

Hence okay



BMD
in KNm.

Ex: Portal frame shown is fixed at ends A and D, the joint B is rigid and joint C is hinged. Analyse the frame and draw BMD.



Solution:

FEM's:

$$F_{BC} = -\frac{WL}{8} = \frac{-80\times 6}{8} = -60\text{KNM}$$

$$F_{CB} = +\frac{WL}{8} = \frac{80\times 6}{8} = +60\text{KNM}$$

Here $\theta_A = 0,\ \theta_D = 0,\ \theta_B \neq 0,\ \theta_{CB} \neq 0,\ \theta_{CD} \neq 0$

Since C is hinged member CB and CD will rotate independently. Also the frame is unsymmetrical, will also have sway. Let the sway be towards right.

The slope deflections are:

$$M_{AB} = F_{AB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(0 + \theta_B - \frac{3\delta}{4}\right)$$

$$= \frac{1}{2}EI\theta_B - \frac{3}{8}EI\delta \qquad ----->(1)$$

$$M_{BA} = F_{BA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(2\theta_B + 0 - \frac{3\delta}{4}\right)$$

$$= EI\theta_B - \frac{3}{8}EI\delta \qquad ----> (2)$$

$$M_{BC} = F_{BC} + \frac{2EI}{L}\left(2\theta_B + \theta_C - \frac{3\delta}{L}\right)$$

$$= -60 + \frac{2E.2I}{6}\left(2\theta_B + \theta_{CB}\right)$$

$$= -60 + \frac{4}{3}EI\theta_B + \frac{2}{3}EI\theta_{CB} \qquad ----> (3)$$

$$M_{CB} = F_{CB} + \frac{2EI}{L}\left(2\theta_{CB} + \theta_B\right)$$

$$= +60 + \frac{2E.2I}{6}\left(2\theta_{CB} + \theta_B\right)$$

$$= +60 + \frac{4}{3}EI\theta_{CB} + \frac{2}{3}EI\theta_B \qquad ----> (4)$$

$$M_{CD} = F_{CD} + \frac{2EI}{L}\left(2\theta_{CD} + \theta_D - \frac{3\delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(2\theta_{CD} + 0 - \frac{3\delta}{4}\right)$$

$$= EI\theta_{CD} - \frac{3}{8}EI\delta \qquad ----> (5)$$

$$M_{DC} = F_{DC} + \frac{2EI}{L}\left(2\theta_D + \theta_{CD} - \frac{3\delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(0 + \theta_{CD} - \frac{3\delta}{4}\right)$$

$$= \frac{1}{2}EI\theta_{CD} - \frac{3}{8}EI\delta \qquad ----> (6)$$

In the above equations $\theta_B, \theta_{CB}, \theta_{CD}$ and δ are unknowns. According the boundary conditions are

I. $M_{BA}+M_{BC} = 0$,

II. $M_{CB} = 0$,

III. $M_{CD} = 0$,

IV. $H_A+H_D = 0$

i..e, $\frac{M_{AB}+M_{BA}}{4}+\frac{M_{CD}+M_{DC}}{4}=0$

$\therefore \quad M_{AB}+M_{BA}+M_{CD}+M_{DC}=0$

Now using the boundary conditions:

$$M_{BA}+M_{BC}=EI\theta_B-\frac{3}{8}EI\delta-60+\frac{4}{3}EI\theta_B+\frac{2}{3}EI\theta_{CB}$$

$$=\frac{7}{3}EI\theta_B+\frac{2}{3}EI\theta_{CB}-\frac{3}{8}EI\delta-60=0 \qquad ----->(7)$$

$$M_{CB}=\frac{2}{3}EI\theta_B+\frac{4}{3}EI\theta_{CB}+60=0 \qquad ----->(8)$$

$$M_{CD}=EI\theta_{CD}-\frac{3}{8}EI\delta=0 \qquad ----->(9)$$

$$M_{AB}+M_{BA}+M_{CD}+M_{DC}=\frac{1}{2}EI\theta_B-\frac{3}{8}EI\delta+EI\theta_B-\frac{3}{8}EI\delta+EI\theta_{CD}-\frac{3}{8}EI\delta+\frac{1}{2}EI\theta_{CD}-\frac{3}{8}EI\delta=0$$

$$=\frac{3}{2}EI\theta_B+\frac{3}{2}EI\theta_{CD}-\frac{3}{2}EI\delta=0 \qquad ----->(10)$$

From (9) $EI\theta_{cD}=\frac{3}{8}EI\delta \qquad ----->(11)$

Sub in (10)

$$\therefore M_{AB}+M_{BA}+M_{CD}+M_{DC}=\frac{3}{2}EI\theta_B+\frac{3}{2}\left(\frac{3}{8}EI\delta\right)-\frac{3}{2}EI\delta=0$$

$$=\frac{3}{2}EI\theta_B-\frac{15}{16}EI\delta=0 \qquad ----->(12)$$

Equation (12) gives $EI\delta=\frac{16}{15}\times\frac{3}{2}EI\theta_B=\frac{8}{5}EI\theta_B \qquad ----->(13)$

Substituting in Equation (7)

$$M_{BA}+M_{BC}=\frac{7}{3}EI\theta_B+\frac{2}{3}EI\theta_{CB}-\frac{3}{8}\left(\frac{8}{5}EI\theta_B\right)-60=0$$

$$=\left(\frac{7}{3}-\frac{3}{5}\right)EI\theta_B+\frac{2}{3}EI\theta_{CB}-60$$

$$=\frac{26}{15}EI\theta_B+\frac{2}{3}EI\theta_{CB}-60=0 \qquad ----->(14)$$

Substituting in Equation (8) and multiplying equation (14) by 2 we have

$$\frac{2}{3}EI\theta_B + \frac{4}{3}EI\theta_{CB} + 60 = 0$$

$$\frac{52}{15}EI\theta_B + \frac{4}{3}EI\theta_{CB} - 120 = 0$$

$$-\frac{42}{15}EI\theta_B + 180 = 0$$

$$\therefore EI\theta_B = 180 \times \frac{15}{42} = 64.29$$

From(13) $EI\delta = \frac{8}{5}EI\theta_B = \frac{8}{5} \times 64.29 = 102.864$

From (11) $EI\theta_{CD} = \frac{3}{8}EI\delta = 38.574$

From (7) $EI\theta_{CB} = -\frac{3}{2}\left(\frac{7}{3}EI\theta_B - \frac{3}{8}EI\delta - 60\right)$

$$= -\frac{3}{2}\left(\frac{7}{3} \times 64.29 - \frac{3}{8} \times 102.864 - 60\right)$$

$$= -77.165$$

$EI\theta_B = 64.29$, $EI\theta_{CB} = -77.165$, $EI\theta_{CD} = 38.57$, $EI\delta = 102.864$

$\therefore$ Final Moments are

$$M_{AB} = \frac{1}{2}(64.29) - \frac{3}{8}(102.864) = -6.42 \text{ KNM}$$

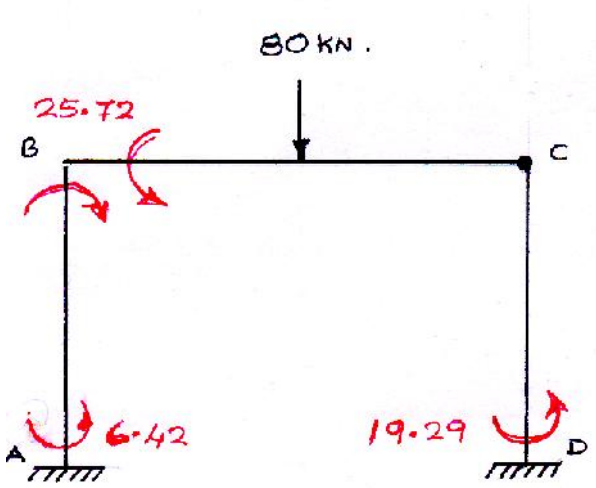
$$M_{BA} = 64.29 - \frac{3}{8}(102.864) = 25.72 \text{ KNM}$$

$$M_{BC} = -60 + \frac{4}{3}(64.29) + \frac{2}{3}(-77.165) = -25.72 \text{ KNM}$$

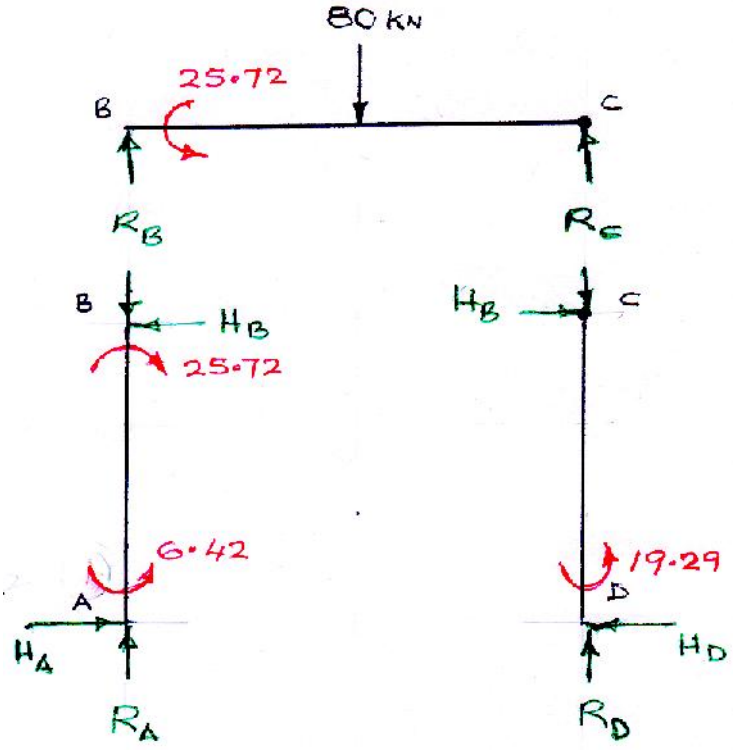
$$M_{CB} = +60 + \frac{4}{3}(-77.165) + \frac{2}{3}(64.29) = 0$$

$$M_{CD} = 38.574 - \frac{3}{8}(102.864) = 0$$

$$M_{DC} = \frac{1}{2}(38.574) - \frac{3}{8}(102.864) = -19.29 \text{ KNM}$$



Reactions: Consider the free body diagram of various members



Column AB:

$$H_A = \frac{25.72 - 6.42}{4} = 4.825\ \text{KN}$$

Beam BC:

$$R_B = \frac{25.72 + (80 \times 3)}{6} = 44.29\ \text{KN}$$

$$R_C = 80 - 44.29 = 35.71\,\text{KN}$$

Column CD:

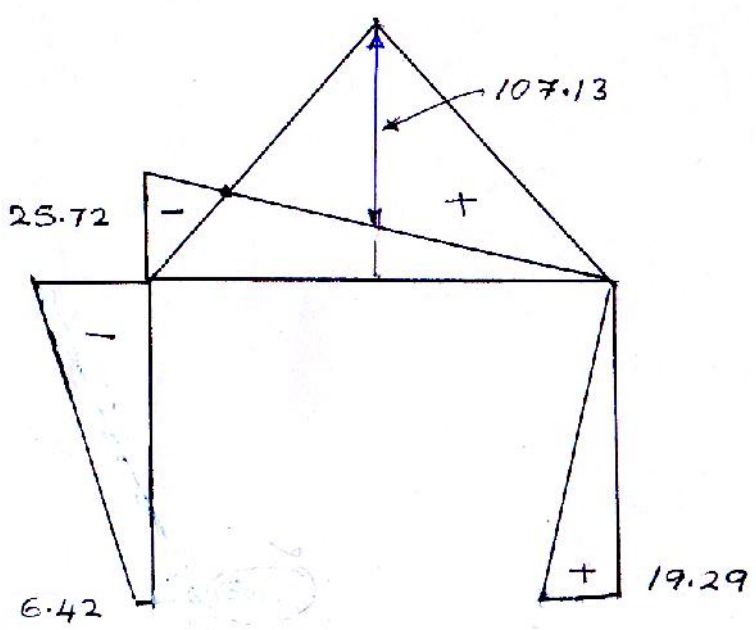
$$H_D = \frac{19.28}{4} = 4.82 \text{ KN}$$

Check:

$\Sigma H = 0$

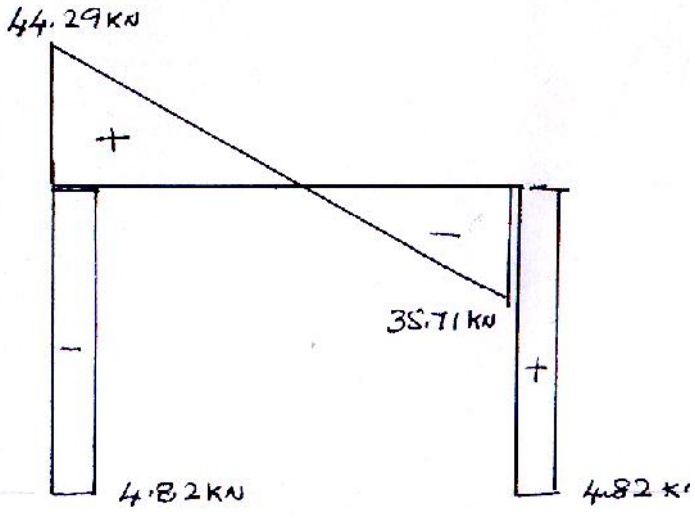
$H_A + H_D = 0$

Hence okay.



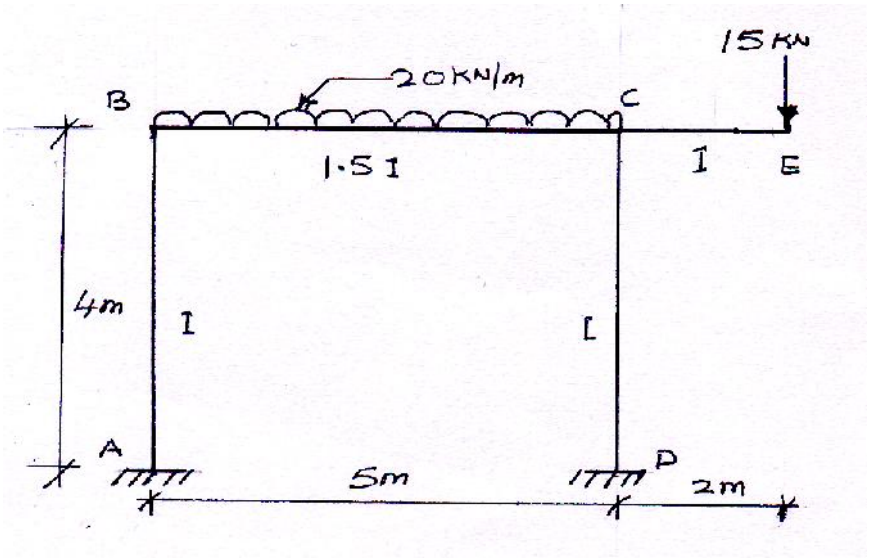
BMD

in kN-m.



SFD

Example: Analyse the portal frame shown in figure the deflection method and then draw the bending moment diagram



Fig

Solution:

The frame is unsymmetrical, hence there is a sway. Let the sway be towards right.

$$\theta_A = 0, \theta_B \neq 0, \theta_C \neq 0, \theta_D = 0$$

FEMS:

$$F_{BC} = -\frac{20 \times 5^2}{12} = -41.67 \quad \text{KNM}$$

$$F_{CB} = +\frac{20 \times 5^2}{12} = +41.67 \quad \text{KNM}$$

$$F_{CE} = -15 \times 2 = -30 \, \text{KNM}$$

Slope deflection equations

$$M_{AB} = F_{AB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(0 + \theta_B - \frac{3\delta}{4}\right) = 0.5EI\theta_B - 0.375EI\delta \quad \text{-------->} (1)$$

$$M_{BA} = F_{BA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(2\theta_B + 0 - \frac{3\delta}{4}\right) = EI\theta_B - 0.375EI\delta \quad \text{-------->} (2)$$

$$M_{BC} = F_{BC} + \frac{2EI}{L}(2\theta_B + \theta_C)$$

$$= -41.67 + \frac{2E \times 1.5I}{5}(2\theta_B + \theta_C) = -41.67 + 1.2EI\theta_B + 0.6EI\theta_C \quad \text{--------->} (3)$$

$$M_{CB} = F_{CB} + \frac{2EI}{L}(2\theta_C + \theta_B)$$

$$= +41.67 + \frac{2E \times 1.5I}{5}(2\theta_C + \theta_B) = 41.67 + 1.2EI\theta_C + 0.6EI\theta_B \quad \text{--------->} (4)$$

$$M_{CD} = F_{CD} + \frac{2EI}{L}\left(2\theta_C + \theta_D - \frac{3\delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(2\theta_C + 0 - \frac{3\delta}{4}\right) = EI\theta_C - 0.375EI\delta \quad \text{--------->} (5)$$

$$M_{DC} = F_{DC} + \frac{2EI}{L}\left(2\theta_D + \theta_C - \frac{3\delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(0 + \theta_C - \frac{3\delta}{4}\right) = 0.5EI\theta_C - 0.375EI\delta \quad \text{--------->} (6)$$

In the above equation there are three unknowns θ_B, θ_C and δ, accordingly the boundary conditions are,

$$M_{BA} + M_{BC} = 0$$

$$M_{CB} + M_{CD} + M_{CE} = 0$$

$$H_A + H_D = 0$$

i.e, $M_{AB} + M_{BA} + M_{CD} + M_{DC} = 0$

Now,

$$M_{BA}+M_{BC}=0$$

$$EI\theta_B-0.375EI\delta-141.67+1.2EI\theta_B+0.6EI\theta_C=0$$

$$=2.2EI\theta_B+0.6EI\theta_C-0.375EI\delta-41.67=0 \qquad ------>(7)$$

And, $M_{CB}+M_{CD}=41.67+1.2EI\theta_C+0.6EI\theta_B+EI\theta_C-0.375EI\delta-30=0$

$$=0.6EI\theta_B+2.2EI\theta_C-0.375EI\delta+11.67=0 \qquad ------>(8)$$

$\therefore\ M_{AB}+M_{BC}+M_{CD}+M_{DC}=0$

$$0.5EI\theta_B-0.375EI\delta+EI\theta_B-0.375EI\delta+2EI\theta_C-0.375EI\delta+0.5EI\theta_C-0.375EI\delta=0$$

$$1.5EI\theta_B+1.5EI\theta_C-1.5EI\delta=0 \qquad ------>(9)$$

Solving the above equations
we get, $EI\,\theta_B=23.98$, $EI\,\theta_C=-9.36$, $EI\delta=14.62$

Substituting these values in slope deflection equations, we have

$M_{AB}=0.5(23.98)-0.375(14.62)=6.50$ KNM

$M_{BA}=23.98-0.375(14.62)=+18.50$ KNM

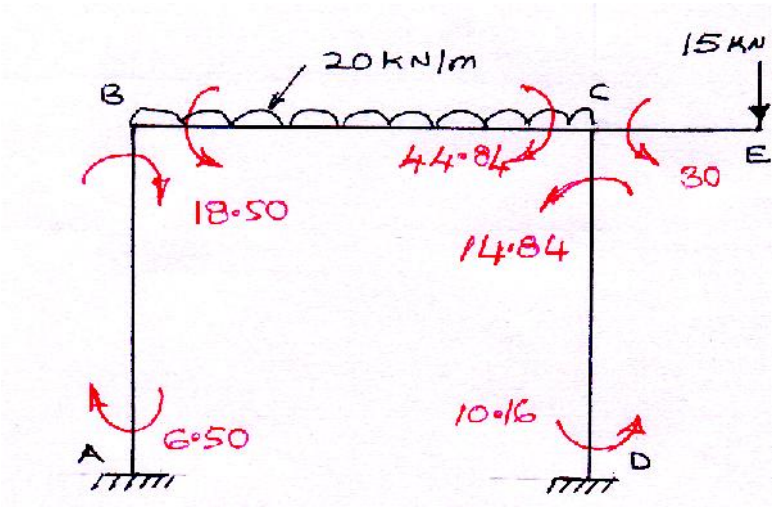
$M_{BC}=-41.67+1.2(23.98)+0.6(-9.36)=-18.51$ KNM

$M_{CB}=+41.67+1.2(-9.36)+0.6(23.98)=+44.83$ KNM

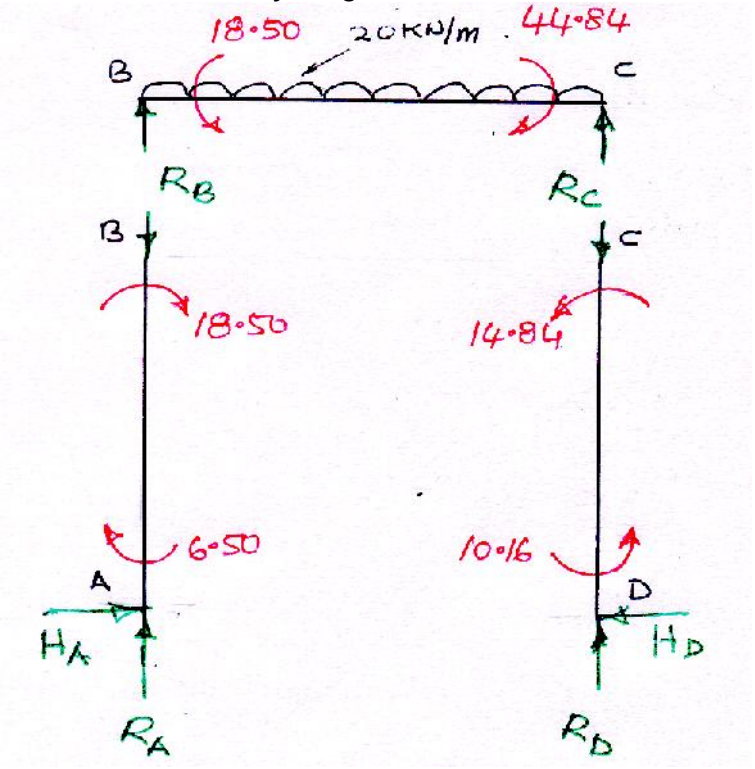
$M_{CD}=-9.36-0.375(+14.62)=-14.84$ KNM

$M_{DC}=0.5\times-9.36-0.375\times14.62=-10.16$ KNM

$M_{CE}=-30$ KNM



Reactions: consider the free body diagram of beam and columns



Column AB:

$$H_A = \frac{18.5 + 6.5}{4} = 6.25 \text{ KN}$$

Span BC:

$$R_C = \frac{44.83 - 18.5 + 20 \times 5 \times 2.5}{5} = 55.27 \text{ KN}$$

$$\therefore R_B = 20 \times 5 - R_C = 44.73$$

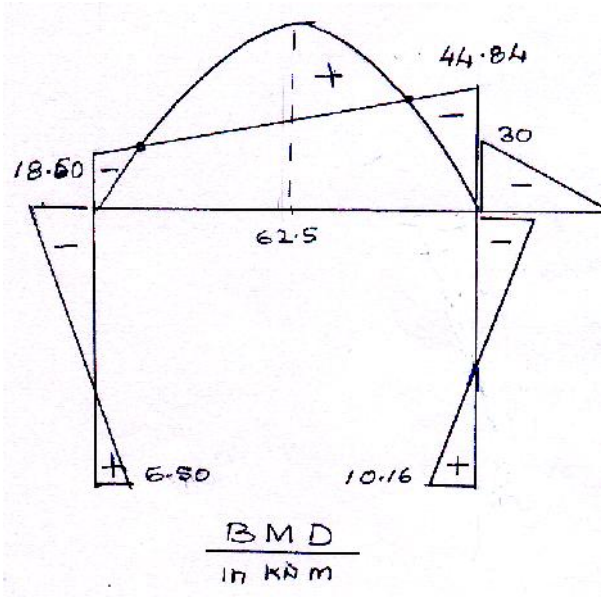
Column CD:

$$H_D = \frac{10.16 + 14.84}{4} = 6.25$$

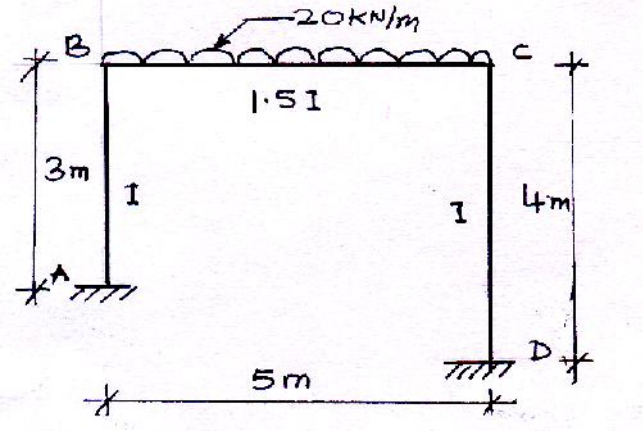
Check:

$\Sigma H = 0$
$H_A + H_D = 0$
$\Sigma = 0$

Hence okay



Example: Analyse the portal frame shown and then draw bending moment diagram.



Solution:

It is an unsymmetrical problem hence there is a sway be towards right

$\theta_A = 0, \theta_B \neq 0, \theta_C \neq 0, \theta_D = 0$

FEMs:

$$F_{BC} = -\frac{wl^2}{12} = -\frac{20 \times 5^2}{12} = \text{ - } 41.67 \quad \text{KNM}$$

$$F_{CB} = \frac{wl^2}{12} = \frac{20 \times 5^2}{12} = \text{ + } 41.67 \quad \text{KNM}$$

Slope deflection equations:

$$M_{AB} = F_{AB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\delta}{L}\right)$$

$$= 0 + \frac{2EI}{3}\left(0 + \theta_B - \frac{3\delta}{3}\right)$$

$$= \frac{2}{3}EI\theta_B - \frac{2}{3}EI\delta \qquad \text{--------->} (1)$$

$$M_{BA} = F_{BA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\delta}{L}\right)$$

$$= 0 + \frac{2EI}{3}\left(2\theta_B + 0 - \frac{3\delta}{3}\right)$$

$$= \frac{4}{3}EI\theta_B - \frac{2}{3}EI\delta \qquad \text{--------->} (2)$$

$$M_{BC} = F_{BC} + \frac{2EI}{L}\left(2\theta_B + \theta_C - \frac{3\delta}{L}\right)$$

$$= -41.67 + \frac{2E \times 1.5I}{5}(2\theta_B + \theta_C)$$

$$= -41.67 + \frac{6}{5}EI\theta_B + \frac{3}{5}EI\theta_C \quad \text{---------->} (3)$$

$$M_{CB} = F_{CB} + \frac{2EI}{L}\left(2\theta_C + \theta_B - \frac{3\delta}{L}\right)$$

$$= 41.67 + \frac{2E \times 1.5I}{5}(2\theta_C + \theta_B - 0)$$

$$= 41.67 + 1.2EI\theta_C + 0.6\ EI\theta_B \quad \text{---------->} (4)$$

$$M_{CD} = F_{CD} + \frac{2EI}{L}\left(2\theta_C + \theta_D - \frac{3\delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(2\theta_C + 0 - \frac{3\delta}{4}\right)$$

$$= EI\,\theta_C - 0.375\,EI\delta \quad \text{---------->} (5)$$

$$M_{DC} = F_{DC} + \frac{2EI}{L}\left(2\theta_D + \theta_C - \frac{3\delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(0 + \theta_C - \frac{3\delta}{4}\right)$$

$$= 0.5EI\,\theta_C - 0.375\,EI\delta \quad \text{---------->} (6)$$

In the above equations there are three unknown θ_B, θ_C and δ and accordingly the Boundary conditions are:

$$M_{BA} + M_{BC} = 0$$

$$M_{CB} + M_{CD} = 0$$

$$H_A + H_D = 0$$

i.e $$\frac{M_{AB} + M_{BA}}{3} + \frac{M_{CD} + M_{DC}}{4} = 0$$

$$\therefore \quad 4(M_{AB} + M_{BA}) + 3(M_{CD} + M_{DC}) = 0$$

Now

$$M_{BA}+M_{BC}=0$$

$$\frac{4}{3}EI\theta_B-\frac{2}{3}EI\delta+\frac{6}{5}EI\theta_B+\frac{3}{5}EI\theta_C-41.67$$

$$2.53EI\theta_B+\frac{3}{5}EI\theta_C-\frac{2}{3}EI\delta-41.67=0 \quad ---------> (7)$$

$$M_{CB}+M_{CD}=0$$

$$41.67+1.2EI\theta_C+0.6EI\theta_B+EI\theta_C-0.375EI\delta=0$$

$$41.67+2.2EI\theta_C+0.6EI\theta_B-0.375EI\delta=0 \quad -------> (8)$$

$$\frac{M_{AB}+M_{BA}}{3}+\frac{M_{CD}+M_{DC}}{4}=0$$

$$4\left[\frac{2}{3}EI\theta_B-\frac{2}{3}EI\delta+\frac{4}{3}EI\theta_B-\frac{2}{3}EI\delta\right]+$$

$$3\left[EI\theta_C-0.375EI\delta+0.5EI\theta_C-0.375EI\delta\right]=0$$

$$\frac{8}{3}EI\theta_B-\frac{8}{3}EI\delta+\frac{16}{3}EI\theta_B-\frac{8}{3}EI\delta+4.5EI\theta_C-2.25EI\delta=0$$

$$8EI\theta_B+4.5EI\theta_C-7.53EI\delta=0 \quad -------> (9)$$

By solving (7), (8) and (9) we get

$$EI\theta_B=+25.46$$

$$EI\theta_C=-23.17$$

$$EI\delta=+12.8$$

Final moments:

$$M_{AB}=\frac{2}{3}\times25.46-\frac{2}{3}\times12.8=8.44 \quad \text{KNM}$$

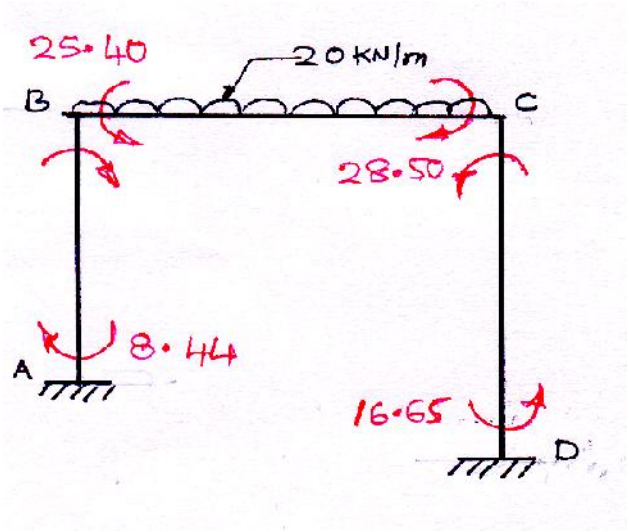
$$M_{BA}=\frac{4}{3}\times25.46-\frac{2}{3}\times12.8=25.40 \quad \text{KNM}$$

$$M_{BC}=\frac{6}{5}\times25.46+\frac{3}{5}\times(-23.17)-41.67=-25.40 \quad \text{KNM}$$

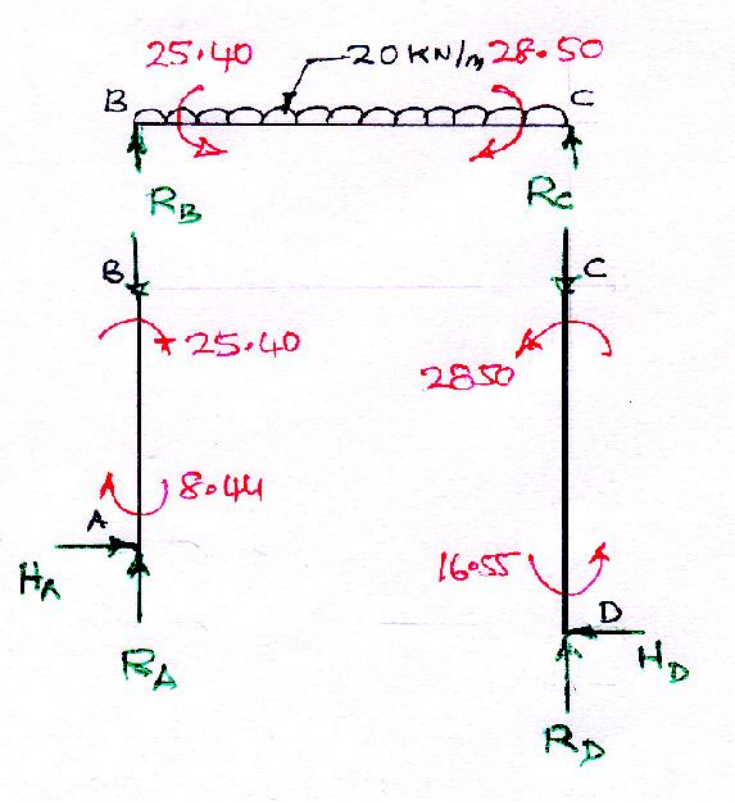
$$M_{CB}=41.67+1.2(-23.17)+0.60(20.46)=28.50 \quad \text{KNM}$$

$$M_{CD}=-23.70-0.375\,(12.80)=-28.50 \quad \text{KNM}$$

$$M_{DC}=0.5\times-23.70-0.375\times12.80=-16.65 \quad \text{KNM}$$



Reactions: Consider the free body diagram



Member AB:

$$H_A = \frac{25.40 + 8.44}{3} = 11.28\ KN$$

Member BC:

$$R_C = \frac{28.5 - 20.30 + 20 \times 5 \times \frac{5}{2}}{2} = 51.64 \text{ KN}$$

$$\therefore R_B = 20 \times 5 - 51.64 = 48.36 \text{ KN}$$

Member CD:

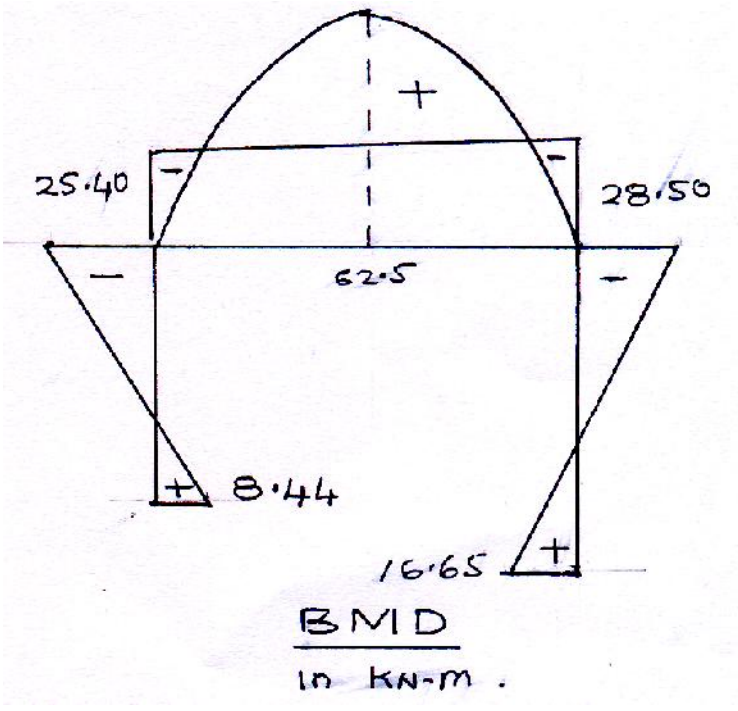
$$H_D = \frac{28.5 + 16.65}{4} = 11.28 \text{ KN}$$

Check:

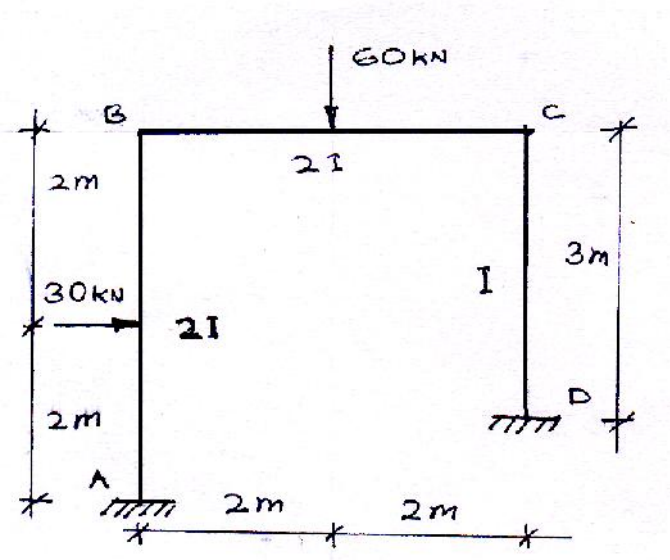
$\Sigma H = 0$

$H_A + H_D = 0$

Satisfied, hence okay



Example: A portal frame having different column heights are subjected for forces as shown in figure. Analyse the frame and draw bending moment diagram.



Solution:-

It is an unsymmetrical problem

$\theta_A = 0, \theta_B \neq 0, \theta_C \neq 0, \theta_D = 0$, hence there is a sway be towards right.

FEMs:

$$F_{AB} = -\frac{Wl}{8} = -\frac{30 \times 4}{8} = -15 \text{ KNM}$$

$$F_{BA} = +\frac{Wl}{8} = +\frac{30 \times 4}{8} = +15 \text{ KNM}$$

$$F_{BC} = -\frac{Wl}{8} = -\frac{60 \times 4}{8} = -30 \text{ KNM}$$

$$F_{CB} = +\frac{Wl}{8} = +\frac{60 \times 4}{8} = +30 \text{ KNM}$$

$$F_{CD} = F_{DC} = 0$$

Slope deflection equations:

$$M_{AB} = F_{AB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\delta}{L}\right)$$

$$= -15 + \frac{2E\,2I}{4}\left(0 + \theta_B - \frac{3\delta}{4}\right)$$

$$= -15 + EI\theta_B - 0.75\,EI\delta \qquad \text{--------} > (1)$$

$$M_{BA} = F_{BA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\delta}{L}\right)$$

$$= 15 + \frac{2E\ 2I}{4}\left(2\theta_B + 0 - \frac{3\delta}{4}\right)$$

$$= +15 + 2EI\theta_B - 0.75EI\delta \qquad \text{--------} > (2)$$

$$M_{BC} = F_{BC} + \frac{2EI}{L}(2\theta_B + \theta_C)$$

$$= -30 + \frac{2E \times 2I}{4}(2\theta_B + \theta_C)$$

$$= -30 + 2EI\theta_B + EI\theta_C \qquad \text{--------} > (3)$$

$$M_{CB} = F_{CB} + \frac{2EI}{L}(2\theta_C + \theta_B)$$

$$= 30 + \frac{2E \times 2I}{4}(2\theta_C + \theta_B)$$

$$= 30 + 2EI\theta_C + EI\theta_B \qquad \text{--------} > (4)$$

$$M_{CD} = F_{CD} + \frac{2EI}{L}\left(2\theta_C + \theta_D - \frac{3\delta}{L}\right)$$

$$= 0 + \frac{2EI}{3}\left(2\theta_C + 0 - \frac{3\delta}{3}\right)$$

$$= \frac{4}{3}EI\,\theta_C - \frac{2}{3}EI\delta \qquad \text{--------} > (5)$$

$$M_{DC} = F_{DC} + \frac{2EI}{L}\left(2\theta_D + \theta_C - \frac{3\delta}{L}\right)$$

$$= 0 + \frac{2EI}{3}\left(0 + \theta_C - \frac{3\delta}{3}\right)$$

$$= \frac{2}{3}EI\,\theta_C - \frac{2}{3}EI\delta \qquad \text{--------} > (6)$$

There are three unknowns, $EI, \theta_B, EI\theta_C$ & $EI\delta$, accordingly the Boundary conditions are

$$M_{BA}+M_{BC}=0$$
$$M_{CB}+M_{CD}=0$$
$$H_A+H_D+30=0$$

i.e, $$\frac{M_{AB}+M_{BA}-60}{4}+\frac{M_{CD}+M_{DC}}{3}+30=0$$

$$3(M_{AB}+M_{BA})+4(M_{CD}+M_{DC})+180=0$$

Now

$$M_{BA}+M_{BC}=15+2EI\theta_B-0.75EI\delta-30+2EI\theta_B+EI\theta_C$$
$$=4EI\theta_B+EI\theta_C-0.75EI\delta-15=0 \qquad ------>(7)$$

$$M_{CB}+M_{CD}=+30+2EI\theta_C+EI\theta_B+\frac{4}{3}EI\theta_C-\frac{2}{3}EI\delta$$
$$=EI\theta_B+\frac{10}{3}EI\theta_C-\frac{2}{3}EI\delta+30=0 \qquad ------>(8)$$

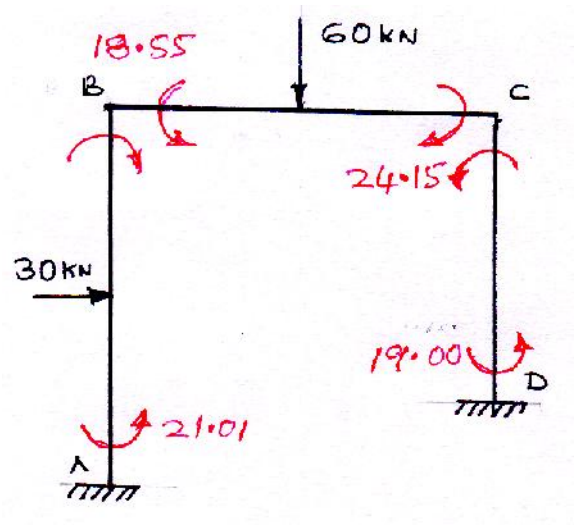
$$3(M_{AB}+M_{BA})+4(M_{CD}+M_{DC})+180=3\left(-15+EI\theta_B-0.75EI\delta+15+2EI\theta_B-0.75EI\delta\right)$$
$$+4\left(\frac{4}{3}EI\theta_C-\frac{2}{3}EI\delta+\frac{2}{3}EI\theta_C-\frac{2}{3}EI\delta\right)+180$$
$$=9EI\theta_B+8EI\theta_C-9.833EI\delta+180=0 \qquad ------>(9)$$

By solving (7), (8) & (9) we get

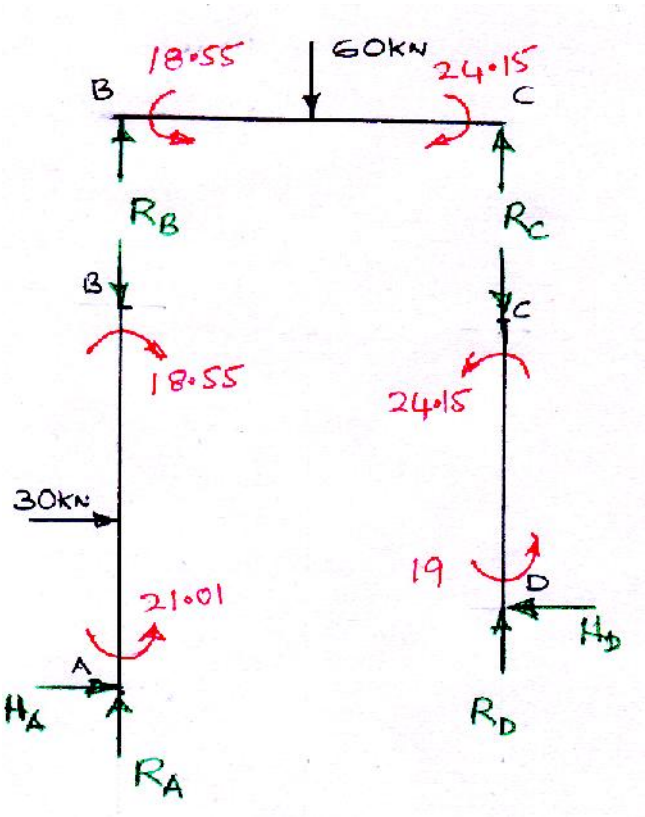
$$EI\theta_B=+9.577$$
$$EI\theta_C=-7.714$$
$$EI\delta=+20.795$$

Substituting these values in the slope deflection equations we get

$$M_{AB}=-15+9.577-0.75\,(20.795)=-21.01 \text{ KNM}$$
$$M_{BA}=+15+2\,(9.577)-0.75\,(20.795)=18.55 \text{ KNM}$$
$$M_{BC}=-30+2\,(9.577)-7.714=-18.55 \text{ KNM}$$
$$M_{CB}=30+2\,(-7.714)+9.577=24.15 \text{ KNM}$$
$$M_{CD}=\frac{4}{3}(-7.714)-\frac{2}{3}(20.795)=-24.15 \text{ KNM}$$
$$M_{DC}=\frac{2}{3}(-7.714)-\frac{2}{3}(20.795)=-19.00 \text{ KNM}$$



Reactions: Consider free body diagrams of the members



Member AB:

$$H_A = \frac{18.55 - 21.01 - 30 \times 2}{4} = -15.615 \text{ KN}$$

-ve sign indicates the direction of H_A is from right to left.

Member BC:

$$R_B = \frac{18.55 + 60 \times 2 - 24.15}{4} = 28.60 \text{ KN}$$

$$R_C = 60 - R_B = 28.60 = 31.40 \text{ KN}$$

Member CD:

$$H_D = \frac{19 + 24.15}{3} = 14.38 \text{ KN}$$

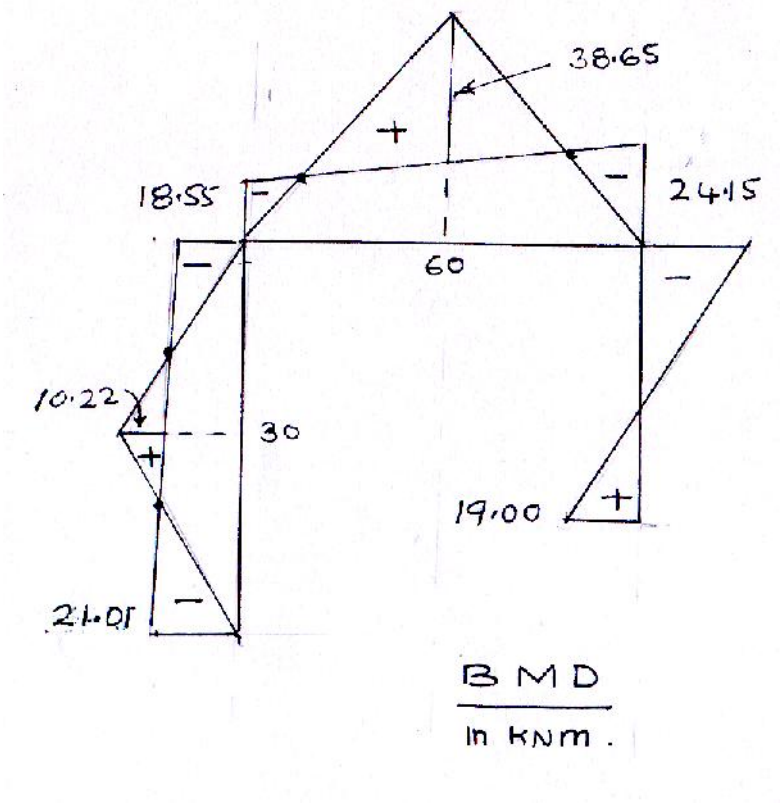
Check:

$\Sigma H = 0$

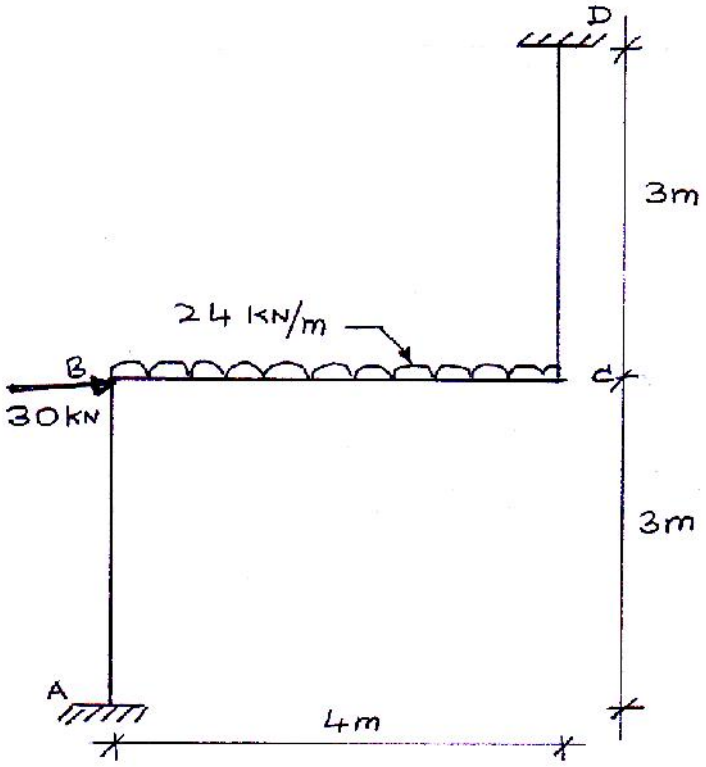
$H_A + H_D + 30 = 0$

$-15.62 - 14.38 + 30 = 0$

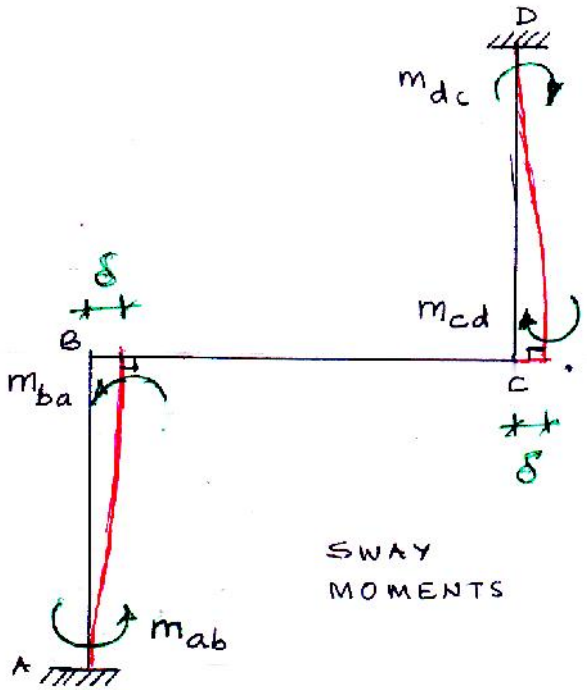
Hence okay



Example: Analyse the frame using slope deflection method and draw the Bending Moment Diagram.



Solution: Assume sway towards right



It can be observed from figure in that direction of moments due to sway in member AB are anticlockwise and that for member CD are clockwise. Wise shall be taken to incorporate the same in the slope deflection equation.

FEMS

$$F_{BC} = -\frac{wl^2}{12}$$

$$= -\frac{24 \times 4^2}{12} = -32 \text{ KNM}$$

$$F_{CB} = +\frac{wl^2}{12} = +32 \text{ KNM}$$

Here $\theta_A = \theta_D = 0$

Slope deflection equations are:

$$M_{AB} = F_{AB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\delta}{L}\right)$$

$$= \frac{2EI}{3}\left(\theta_B - \frac{3\delta}{3}\right)$$

$$= \frac{2}{3}EI\theta_B - \frac{2}{3}EI\delta \qquad \text{------->(1)}$$

$$M_{BA} = F_{BA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\delta}{L}\right)$$

$$= \frac{2EI}{3}\left(2\theta_B - \frac{3\delta}{3}\right)$$

$$= \frac{4}{3}EI\theta_B - \frac{2}{3}EI\delta \qquad \text{-----}>(2)$$

$$M_{BC}=F_{BC}+\frac{2EI}{L}(2\theta_B+\theta_C)$$

$$=-32+\frac{2E2I}{4}(2\theta_B+\theta_C)$$

$$=-32+2EI\theta_B+EI\theta_C \qquad \text{---------->}(3)$$

$$M_{CB}=F_{CB}+\frac{2EI}{L}(2\theta_C+\theta_B)$$

$$=+32+\frac{2E2I}{4}(2\theta_C+\theta_B)$$

$$=+32+2EI\theta_C+EI\theta_B \qquad \text{-------->}(4)$$

$$M_{CD}=F_{CD}+\frac{2EI}{L}\left(2\theta_C+\theta_D+\frac{3\delta}{L}\right)$$

$$=\frac{2EI}{3}\left[2\theta_C+\frac{3\delta}{3}\right]$$

$$=\frac{4}{3}EI\theta_C+\frac{2}{3}EI\delta \qquad \text{----->}(5)$$

$$M_{DC}=F_{DC}+\frac{2EI}{L}\left[2\theta_D+\theta_C+\frac{3\delta}{L}\right]$$

$$=\frac{2EI}{3}\left(\theta_C+\frac{3\delta}{3}\right)$$

$$=\frac{2}{3}EI\theta_C+\frac{2}{3}EI\delta \qquad \text{-------->}(6)$$

The unknown are $\theta_{B,}\theta_C \& \delta$ Accordingly the boundary conditions are

$$M_{BA}+M_{BC}=0$$

$$M_{CB}+M_{CD}=0$$

$$H_A-H_D+30=0$$

$$\text{i.e, } \frac{M_{AB}+M_{BA}}{3}-\frac{M_{CD}+M_{DC}}{3}+30=0$$

$$M_{AB}+M_{BA}-M_{CD}-M_{DC}+90=0$$

$$\text{Now, } M_{BA}+M_{BC}=\frac{4}{3}EI\theta_B-\frac{2}{3}EI\delta-32+2EI\theta_B+EI\theta_C$$

$$=\frac{10}{3}EI\theta_B+EI\theta_C-\frac{2}{3}EI\delta-32=0 \qquad \text{----->}(7)$$

$$M_{CB}+M_{CD}=+32+2EI\theta_C+EI\theta_B+\frac{4}{3}EI\theta_C+\frac{2}{3}EI\delta$$

$$=EI\theta_B+\frac{10}{3}EI\theta_C+\frac{2}{3}EI\delta+32=0 \qquad ------>(8)$$

$$M_{AB}+M_{BA}-M_{CD}-M_{DC}+90=\frac{2}{3}EI\theta_B-\frac{2}{3}EI\delta+\frac{4}{3}EI\theta_B-\frac{2}{3}EI\sigma-\frac{4}{3}EI\theta_C$$

$$-\frac{2}{3}EI\delta-\frac{2}{3}EI\theta_C-\frac{2}{3}EI\delta+90$$

$$=2EI\theta_B-2EI\theta_C-\frac{8}{3}EI\delta+90$$

$$=EI\theta_B-EI\theta_C-\frac{4}{3}EI\delta+45=0 \qquad ------>(9)$$

From (7) & (9)

$$\frac{20}{3}EI\theta_B+2EI\theta_C-\frac{4}{3}EI\delta-64=0$$

$$EI\theta_B-EI\theta_C-\frac{4}{3}EI\delta+45=0$$

$$\frac{17}{3}EI\theta_B+3EI\theta_C-109=0 \qquad -------->(10)$$

By (8) and (9)

$$2EI\theta_B+\frac{20}{3}EI\theta_C+\frac{4}{3}EI\delta+64=0$$

$$EI\theta_B-EI\theta_C-\frac{4}{3}EI\gamma+45=0$$

$$3EI\theta_B+\frac{17}{3}EI\theta_C+109=0 \qquad -------->(11)$$

By (10) & (11)

$$\frac{17}{3}EI\theta_B+3EI\theta_C-109=0$$

$$\frac{27}{17}EI\theta_B+3EI\theta_C+57.71=0$$

$$\frac{208}{17\times3}EI\theta_B-166.71=0$$

$$EI\theta_B=\frac{166.71\times17\times3}{208}=+40.88$$

From (10)

$$EI\theta_C = \frac{1}{3}\left[109 - \frac{17}{3}EI\theta_B\right] = -40.88$$

From (9)

$$EI\delta = \frac{3}{4}\left[EI\theta_B - EI\theta_C + 45\right]$$

$$= \frac{3}{4}\left[40.88 - (-40.88) + 45\right] = +95.07$$

Thus $\therefore EI\theta_B = 40.88,\ EI\theta_C = -40.88,\ EI\delta = 95.07$

Substituting these values in slope deflection equations

$$M_{AB} = \frac{2}{3}(40.88) - \frac{2}{3}(95.07) = -36.12 \text{ KNM}$$

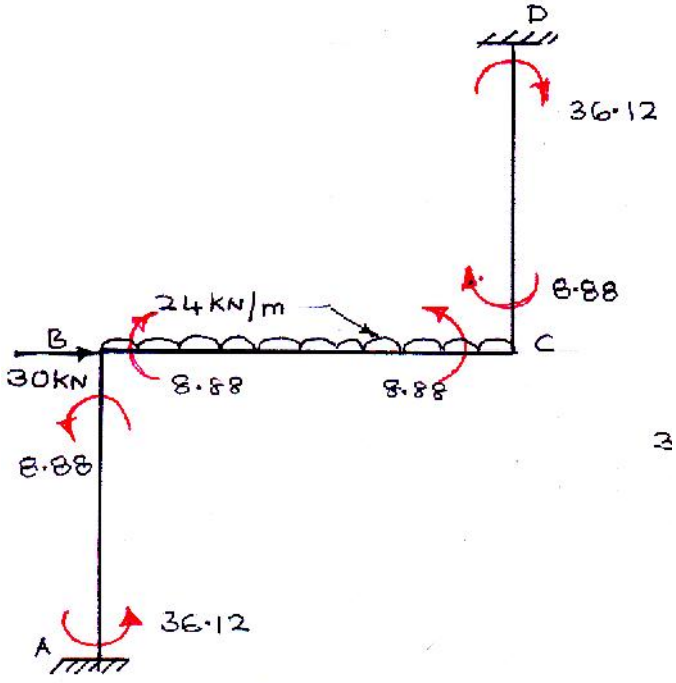
$$M_{BA} = \frac{4}{3}(40.88) - \frac{2}{3}(95.07) = -8.88 \text{ KNM}$$

$$M_{BC} = -32 + 2(40.88) + (-40.88) = 8.88 \text{ KNM}$$

$$M_{CB} = +32 + 2(-40.88) + (40.88) = -8.88 \text{ KNM}$$

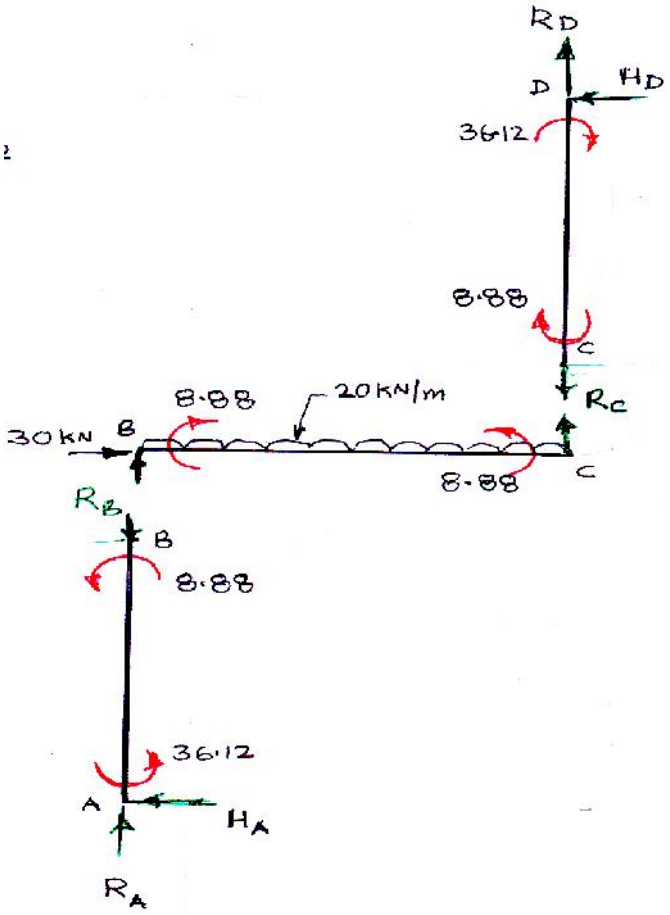
$$M_{CD} = \frac{4}{3}(-40.88) + \frac{2}{3}(95.07) = +8.88 \text{ KNM}$$

$$M_{DC} = \frac{2}{3}(-40.88) + \frac{2}{3}(95.07) = +36.12 \text{ KNM}$$



M in KNm

To find the reaction consider the free body diagram of the frame



Reactions:

Column AB

$$H_A = \frac{8.88 + 36.12}{3} = 15 \text{ KN}$$

Beam AB

$$R_B = \frac{+8.88 - 8.88 + 24 \times 4 \times \frac{4}{2}}{4} = 48 \text{ KN}$$

$$\therefore R_C = 24 \times 4 - 48 = 48 \text{ KN}$$

Column CD

$$H_D = \frac{8.88 + 36.12}{3} = 15 \text{ KN}$$

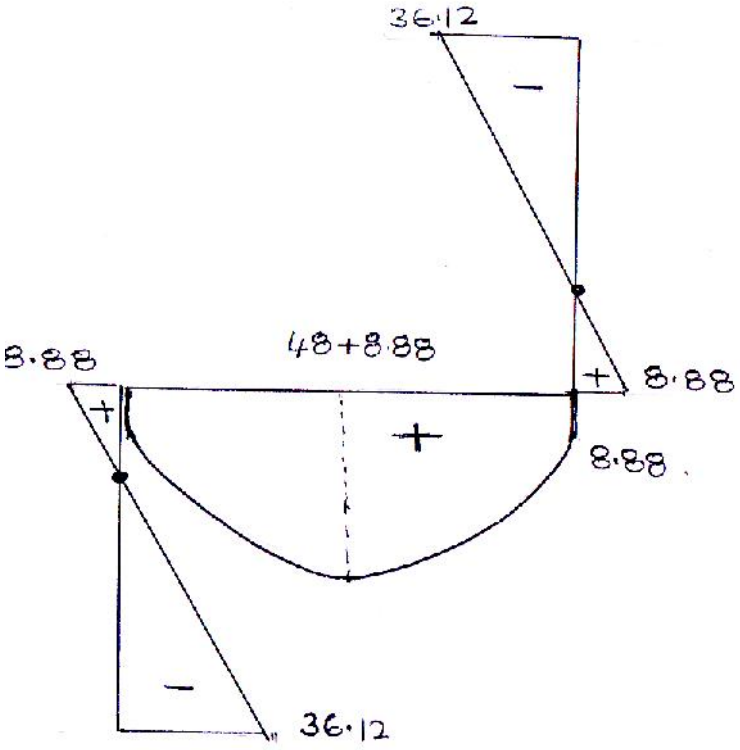
Check

$\Sigma H = 0$

$H_A + H_D + P = 0$

$-15 - 15 + 30 = 0$

Hence okay



BMD

In KN·m