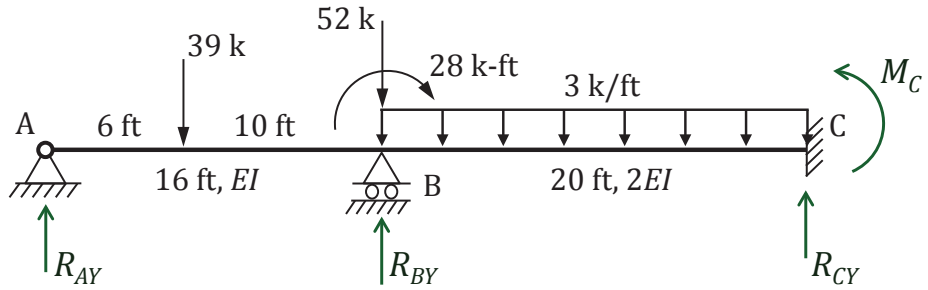
CE 411

STRUCTURAL ANALYSIS AND DESIGN - II

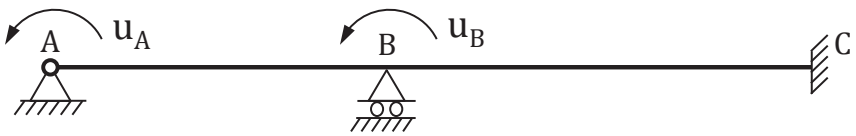
FORMALIZATION OF STIFFNESS METHOD AND DIRECT STIFFNESS METHOD

Dr. Khan Mahmud Amanat
Professor
Dept. of Civil Engineering, BUET

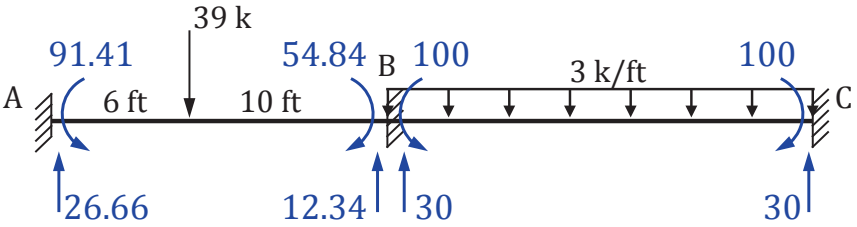
TWO-SPAN BEAM EXAMPLE: Hand calculation process



Superposition Eqn.
$\mathbf{P_j = P_m + KU}$



d.o.f = 2



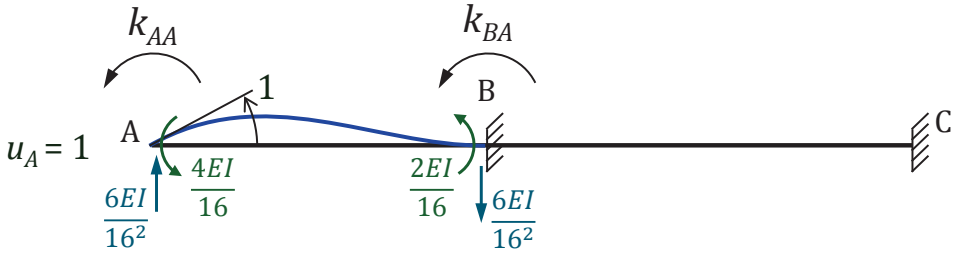
Restrained structure
Fixed end moments and shears

Joint load vector, $\mathbf{P_j} = \begin{bmatrix} 0 \\ -28 \end{bmatrix}$

Member load vector, $\mathbf{P_m} = \begin{bmatrix} 91.41 \\ 100 - 54.84 \end{bmatrix} = \begin{bmatrix} 91.41 \\ 45.16 \end{bmatrix}$

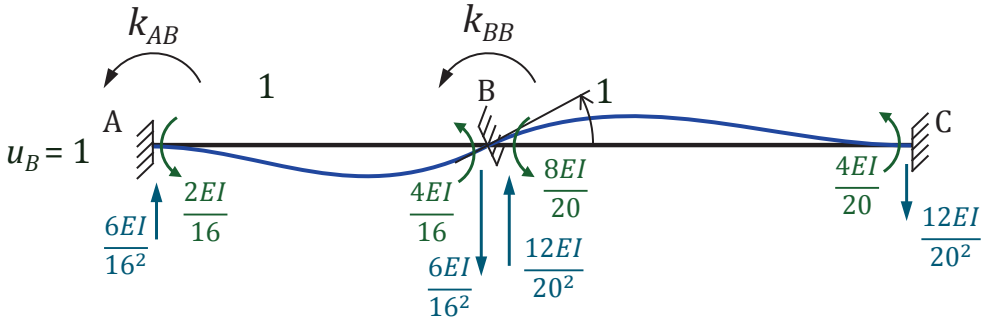
TWO-SPAN BEAM EXAMPLE: Hand calculation process

Superposition Eqn.
$\mathbf{P_j = P_m + KU}$



$$k_{AA} = \frac{4EI}{16}$$

$$k_{BA} = \frac{2EI}{16}$$



$$k_{AB} = \frac{2EI}{16}$$

$$k_{BB} = \frac{4EI}{16} + \frac{8EI}{20}$$

$$= \frac{13EI}{20}$$

TWO-SPAN BEAM EXAMPLE: Hand calculation process

Superposition equation: $\mathbf{P_j = P_m + KU}$

$$\begin{bmatrix} 0 \\ -28 \end{bmatrix} = \begin{bmatrix} 91.41 \\ 45.16 \end{bmatrix} + \begin{bmatrix} k_{AA} & k_{AB} \\ k_{BA} & k_{BB} \end{bmatrix} \begin{bmatrix} u_A \\ u_B \end{bmatrix} \quad \Rightarrow \begin{bmatrix} 0 \\ -28 \end{bmatrix} = \begin{bmatrix} 91.41 \\ 45.16 \end{bmatrix} + \begin{bmatrix} \frac{4}{16} & \frac{2}{16} \\ \frac{2}{16} & \frac{13}{20} \end{bmatrix} EI \begin{bmatrix} u_A \\ u_B \end{bmatrix} \quad \text{..... (a)}$$

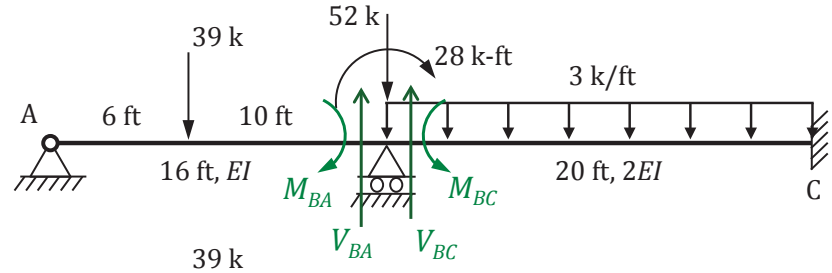
$$\Rightarrow \begin{bmatrix} -91.41 \\ -73.16 \end{bmatrix} = \begin{bmatrix} \frac{4}{16} & \frac{2}{16} \\ \frac{2}{16} & \frac{13}{20} \end{bmatrix} EI \begin{bmatrix} u_A \\ u_B \end{bmatrix} \Rightarrow \begin{bmatrix} u_A \\ u_B \end{bmatrix} = \begin{bmatrix} -342.27 \\ -46.73 \end{bmatrix} \frac{1}{EI}$$

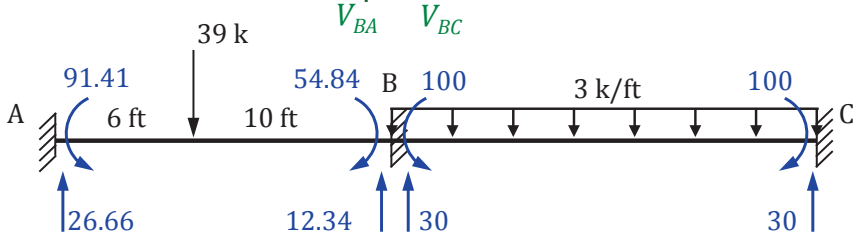
Support reactions

$$\begin{bmatrix} R_{AY} \\ R_{BY} - 52 \\ R_{CY} \\ M_C \end{bmatrix} = \begin{bmatrix} 26.66 \\ 12.34 + 30 \\ 30 \\ -100 \end{bmatrix} + \begin{bmatrix} \frac{6EI}{16^2} & \frac{6EI}{16^2} \\ -\frac{6EI}{16^2} & -\frac{6EI}{16^2} + \frac{12EI}{20^2} \\ 0 & -\frac{12EI}{20^2} \\ 0 & \frac{4EI}{20} \end{bmatrix} \begin{bmatrix} -342.27 \\ -46.73 \end{bmatrix} \frac{1}{EI} \Rightarrow \begin{bmatrix} R_{AY} \\ R_{BY} \\ R_{CY} \\ M_C \end{bmatrix} = \begin{bmatrix} 17.54 \\ 102.06 \\ 31.40 \\ -109.35 \end{bmatrix} \quad \text{..... (b)}$$

TWO-SPAN BEAM EXAMPLE:

Calculation of member end shear and moments by superposition

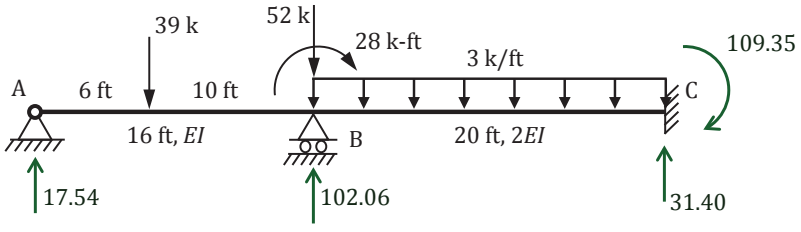


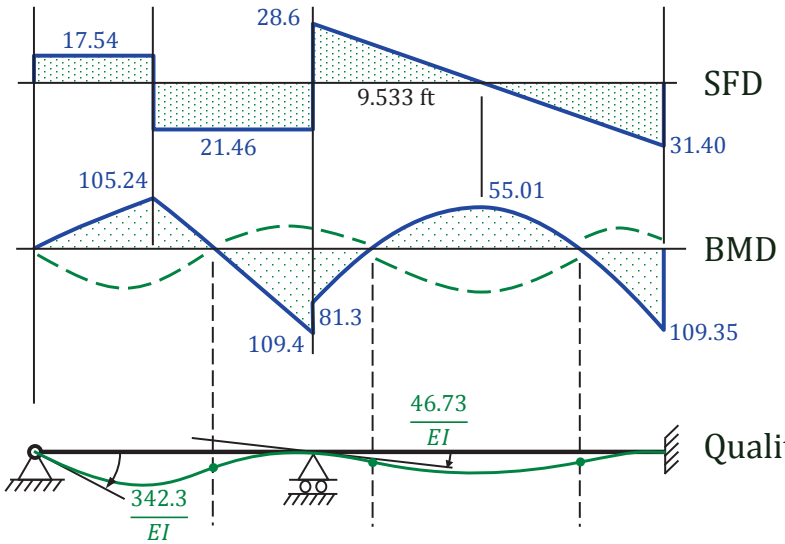


Restrained structure
Fixed end moments and shears

$$\begin{bmatrix} M_{BA} \\ V_{BA} \\ M_{BC} \\ V_{BC} \end{bmatrix} = \begin{bmatrix} 54.84 \\ 12.34 \\ 100 \\ 30 \end{bmatrix} + \begin{bmatrix} -\frac{2EI}{16} & -\frac{4EI}{16} \\ -\frac{6EI}{16^2} & -\frac{6EI}{16^2} \\ 0 & \frac{8EI}{20} \\ 0 & \frac{12EI}{20^2} \end{bmatrix} \begin{bmatrix} -342.27 \\ -46.73 \end{bmatrix} \frac{1}{EI} \Rightarrow \begin{bmatrix} M_{BA} \\ V_{BA} \\ M_{BC} \\ V_{BC} \end{bmatrix} = \begin{bmatrix} 109.31 \\ 21.46 \\ 81.31 \\ 28.6 \end{bmatrix}$$

TWO-SPAN BEAM EXAMPLE: Shear force and bending moment diagrams



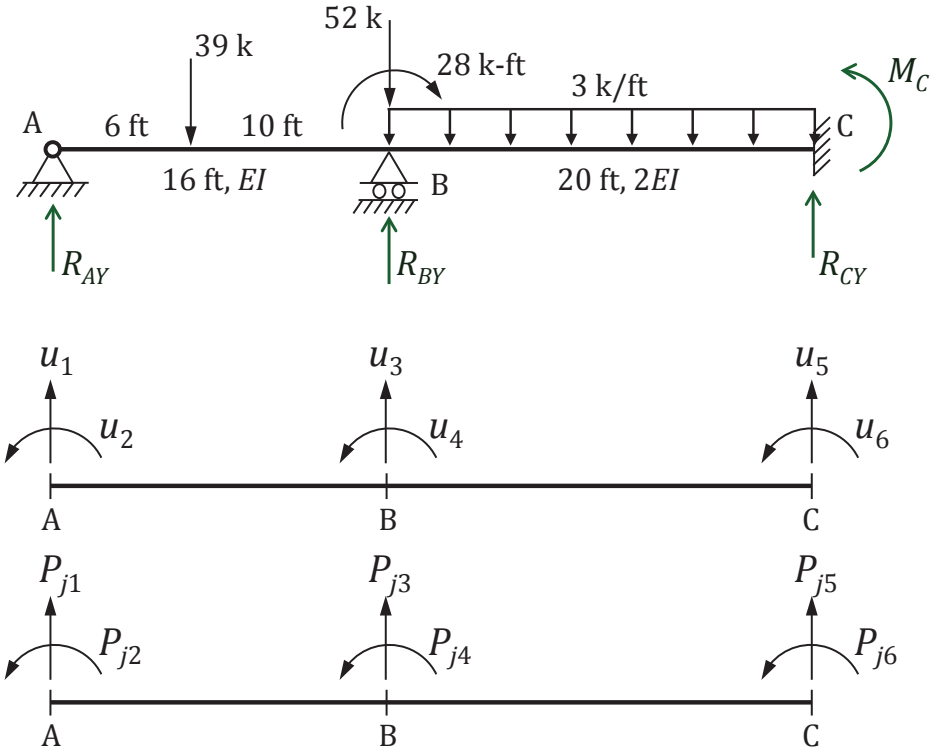


Qualitative deflected shape

$$\begin{bmatrix} u_A \\ u_B \end{bmatrix} = \begin{bmatrix} -342.27 \\ -46.73 \end{bmatrix} \frac{1}{EI}$$

TWO-SPAN BEAM EXAMPLE: Formalization

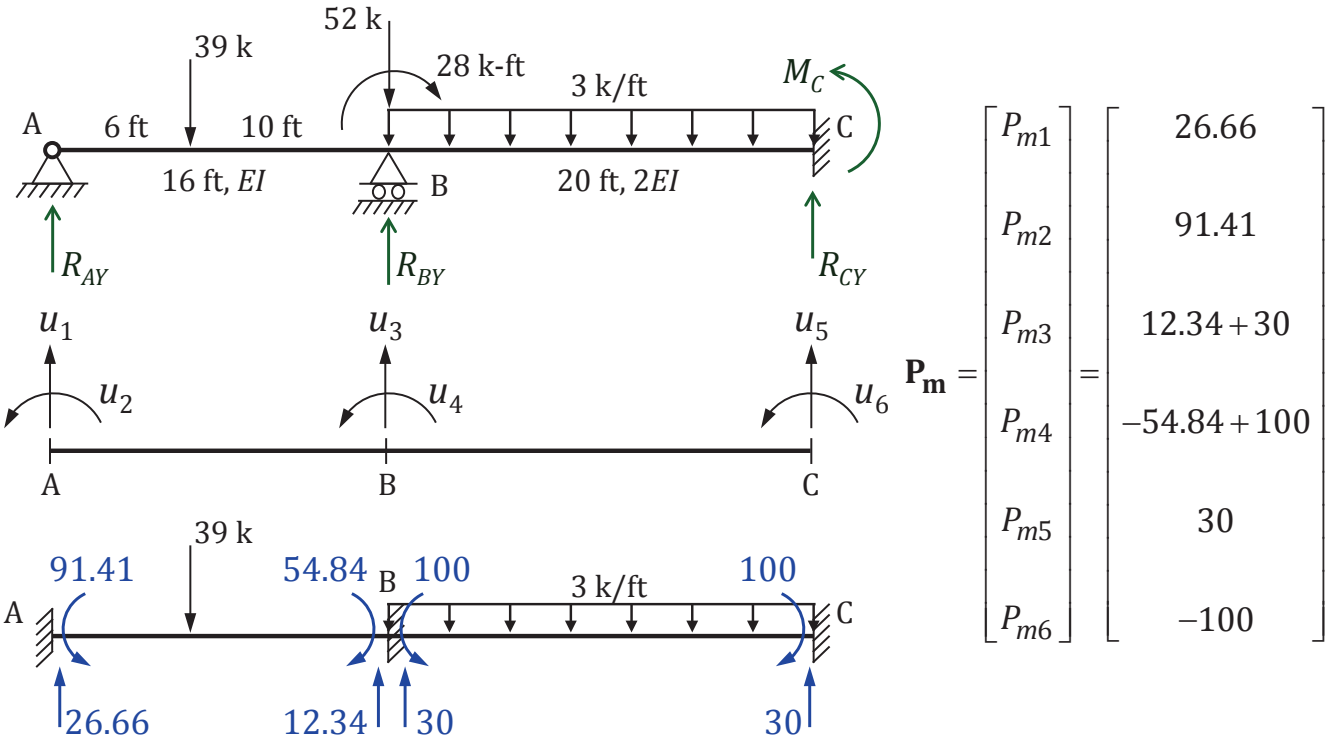
Joint Load Vector, $\mathbf{P_j}$



$$\mathbf{P_j} = \begin{bmatrix} P_{j1} \\ P_{j2} \\ P_{j3} \\ P_{j4} \\ P_{j5} \\ P_{j6} \end{bmatrix} = \begin{bmatrix} R_{AY} \\ 0 \\ R_{BY} - 52 \\ -28 \\ R_{CY} \\ M_C \end{bmatrix}$$

TWO-SPAN BEAM EXAMPLE: Formalization

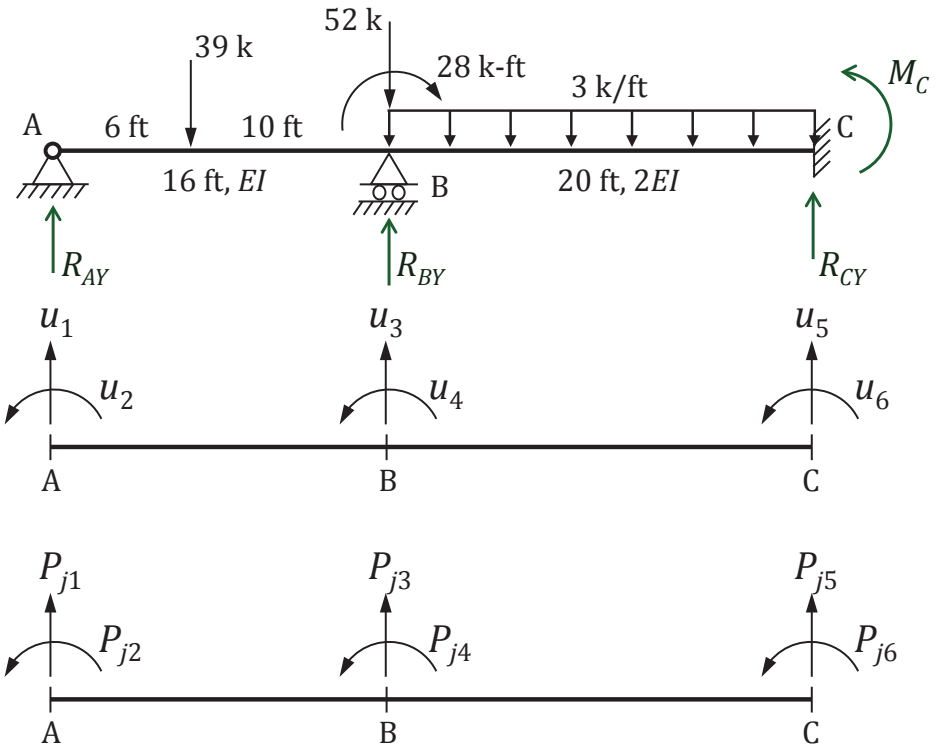
Member Load Vector, $\mathbf{P_m}$



$$\mathbf{P_m} = \begin{bmatrix} P_{m1} \\ P_{m2} \\ P_{m3} \\ P_{m4} \\ P_{m5} \\ P_{m6} \end{bmatrix} = \begin{bmatrix} 26.66 \\ 91.41 \\ 12.34 + 30 \\ -54.84 + 100 \\ 30 \\ -100 \end{bmatrix}$$

TWO-SPAN BEAM EXAMPLE: Formalization

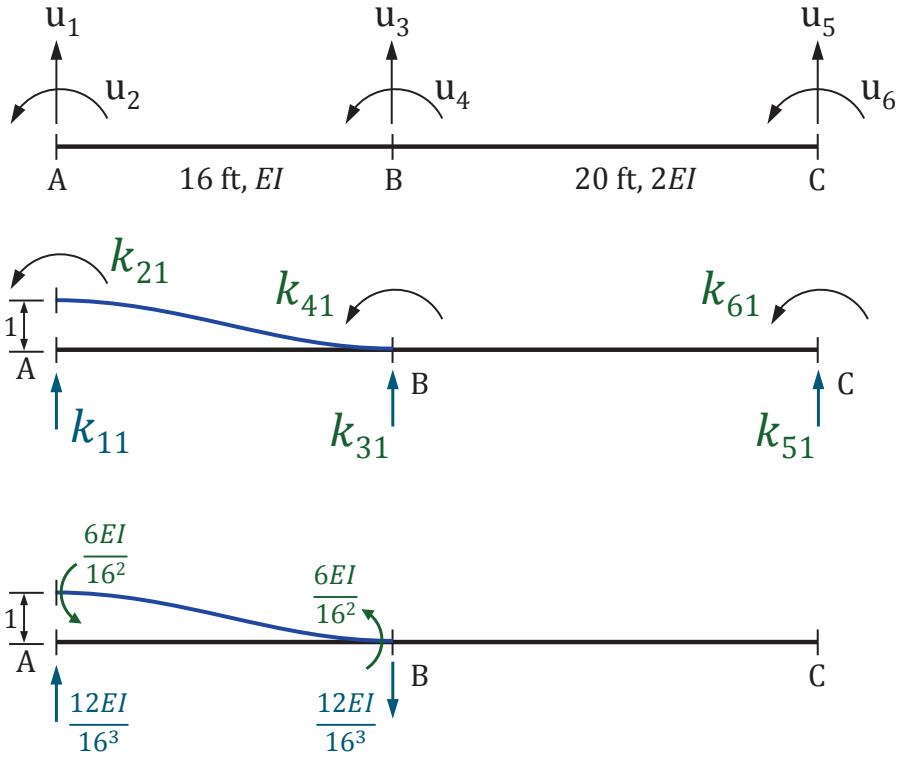
Displacement Vector, U



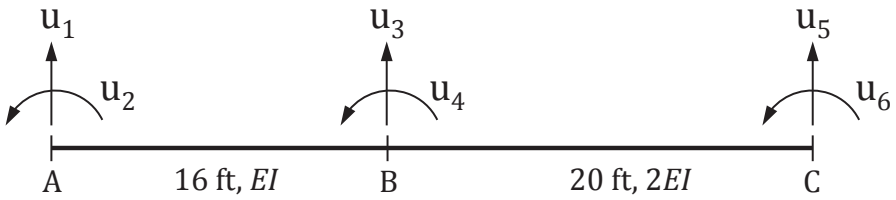
$$\mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} 0 \\ u_2 \\ 0 \\ u_4 \\ 0 \\ 0 \end{bmatrix}$$

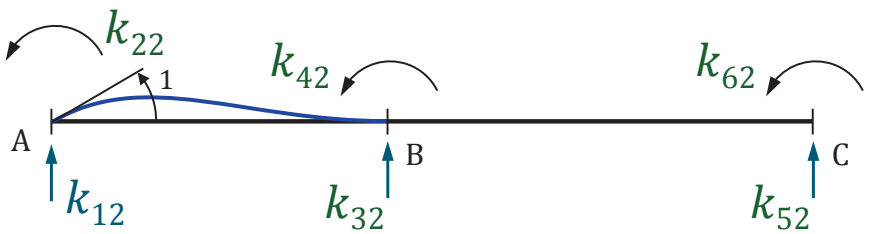
TWO-SPAN BEAM EXAMPLE: Formalization

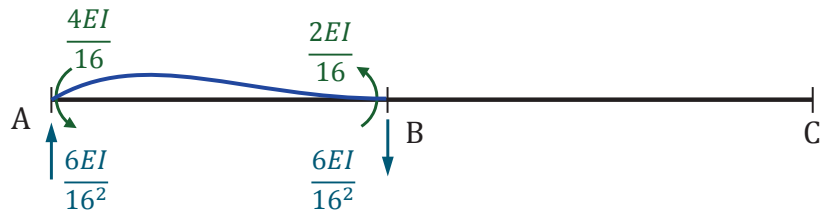
Determination of stiffness co-efficients, k_{ij}



$$\begin{bmatrix} k_{11} \\ k_{21} \\ k_{31} \\ k_{41} \\ k_{51} \\ k_{61} \end{bmatrix} = \begin{bmatrix} \frac{12EI}{16^3} \\ \frac{6EI}{16^2} \\ -\frac{12EI}{16^3} \\ \frac{6EI}{16^2} \\ 0 \\ 0 \end{bmatrix}$$



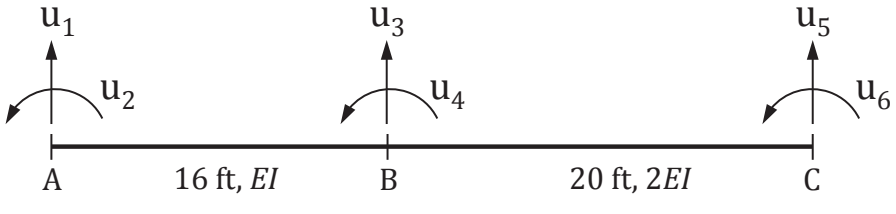


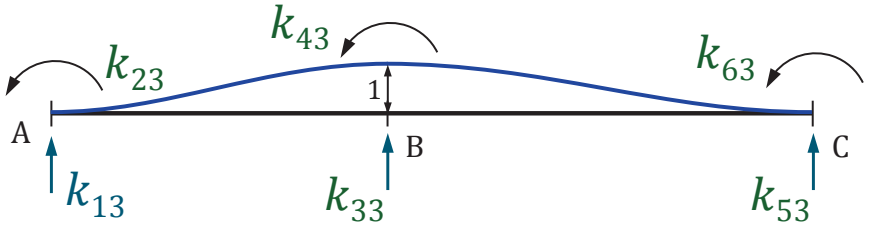


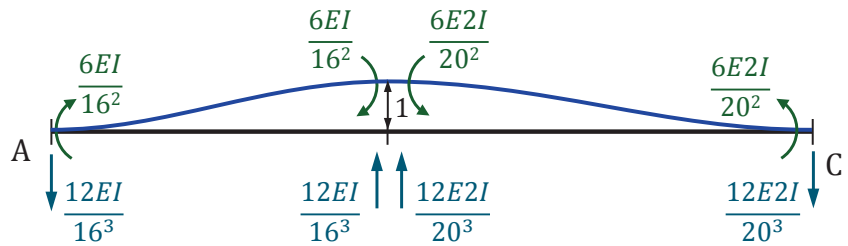
$$\begin{bmatrix} k_{12} \\ k_{22} \\ k_{32} \\ k_{42} \\ k_{52} \\ k_{62} \end{bmatrix} = \begin{bmatrix} \frac{6EI}{16^2} \\ \frac{4EI}{16} \\ -\frac{6EI}{16^2} \\ \frac{2EI}{16} \\ 0 \\ 0 \end{bmatrix}$$

TWO-SPAN BEAM EXAMPLE: Formalization

Determination of stiffness co-efficients, k_{ij}



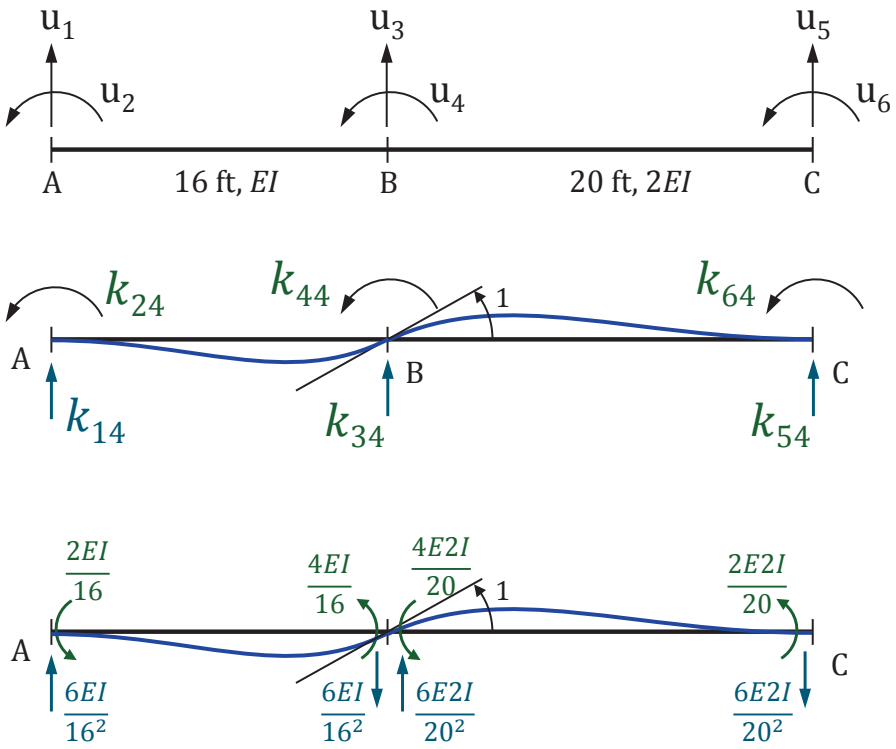




$$\begin{bmatrix} k_{13} \\ k_{23} \\ k_{33} \\ k_{43} \\ k_{53} \\ k_{63} \end{bmatrix} = \begin{bmatrix} -\frac{12EI}{16^3} \\ -\frac{6EI}{16^2} \\ \frac{12EI}{16^3} + \frac{12E2I}{20^3} \\ -\frac{6EI}{16^2} + \frac{6E2I}{20^2} \\ -\frac{12E2I}{20^3} \\ \frac{6E2I}{20^2} \end{bmatrix}$$

TWO-SPAN BEAM EXAMPLE: Formalization

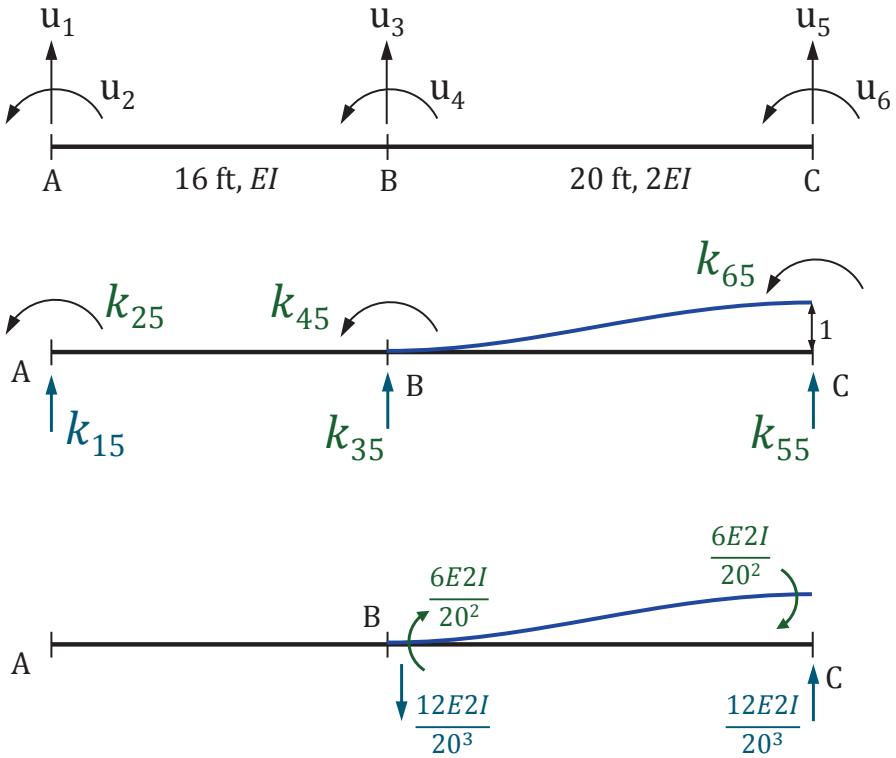
Determination of stiffness co-efficients, k_{ij}



$$\begin{bmatrix} k_{14} \\ k_{24} \\ k_{34} \\ k_{44} \\ k_{54} \\ k_{64} \end{bmatrix} = \begin{bmatrix} \frac{6EI}{16^2} \\ \frac{2EI}{16} \\ -\frac{6EI}{16^2}+\frac{6E2I}{20^2} \\ \frac{4EI}{16}+\frac{4E2I}{20} \\ -\frac{6E2I}{20^2} \\ \frac{2E2I}{20} \end{bmatrix}$$

TWO-SPAN BEAM EXAMPLE: Formalization

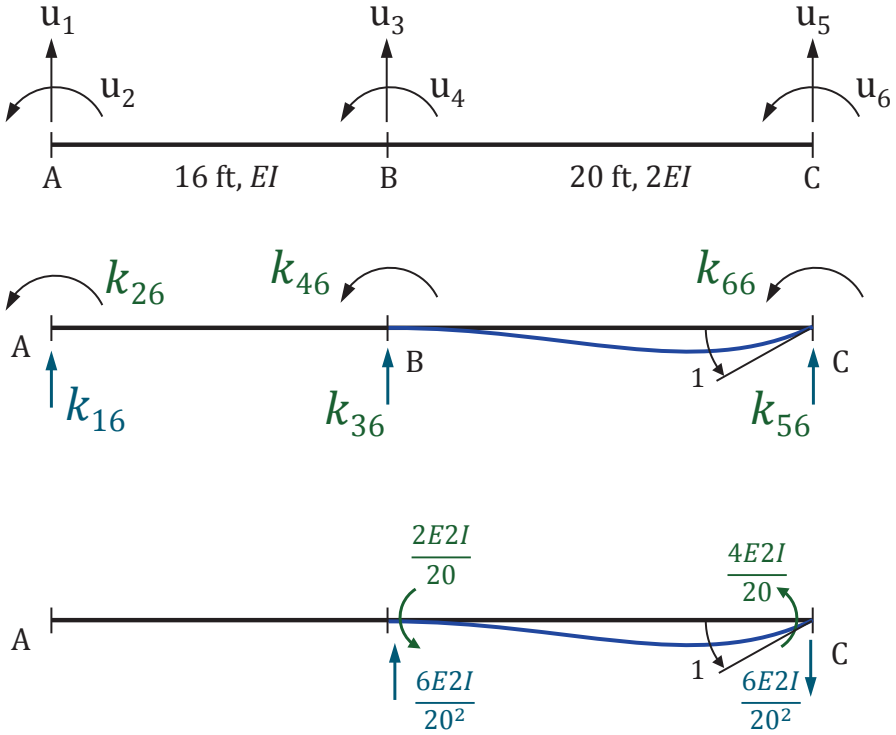
Determination of stiffness co-efficients, k_{ij}



$$\begin{bmatrix} k_{15} \\ k_{25} \\ k_{35} \\ k_{45} \\ k_{55} \\ k_{65} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{12E2I}{20^3} \\ -\frac{6E2I}{20^2} \\ \frac{12E2I}{20^3} \\ -\frac{6E2I}{20^2} \end{bmatrix}$$

TWO-SPAN BEAM EXAMPLE: Formalization

Determination of stiffness co-efficients, k_{ij}



$$\begin{bmatrix} k_{16} \\ k_{26} \\ k_{36} \\ k_{46} \\ k_{56} \\ k_{66} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ +\frac{6E2I}{20^2} \\ \frac{2E2I}{20} \\ -\frac{6E2I}{20^2} \\ \frac{4E2I}{20} \end{bmatrix}$$

TWO-SPAN BEAM EXAMPLE: Formalization

Superposition equation

$$\mathbf{P_j = P_m + KU}$$

$$\begin{bmatrix} P_{j1} \\ P_{j2} \\ P_{j3} \\ P_{j4} \\ P_{j5} \\ P_{j6} \end{bmatrix} = \begin{bmatrix} P_{m1} \\ P_{m2} \\ P_{m3} \\ P_{m4} \\ P_{m5} \\ P_{m6} \end{bmatrix} + \underbrace{\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \end{bmatrix}}_{\text{Global Stiffness Matrix, K}} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}$$

TWO-SPAN BEAM EXAMPLE: Formalization

Superposition equation

$$\mathbf{P_j = P_m + KU}$$

After putting values of the elements of matrices..

$$\begin{bmatrix} R_{AY} \\ 0 \\ R_{BY}-52 \\ -28 \\ R_{CY} \\ M_C \end{bmatrix} = \begin{bmatrix} 26.66 \\ 91.41 \\ 42.3 \\ 45.16 \\ 30 \\ -100 \end{bmatrix} + \underbrace{\begin{matrix} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \begin{matrix} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} \frac{12}{16^3} & \frac{6}{16^2} & -\frac{12}{16^3} & \frac{6}{16^2} & 0 & 0 \\ \frac{6}{16^2} & \frac{4}{16} & -\frac{6}{16^2} & \frac{2}{16} & 0 & 0 \\ -\frac{12}{16^3} & -\frac{6}{16^2} & \frac{12}{16^3}+\frac{24}{20^3} & -\frac{6}{16^2}+\frac{12}{20^2} & -\frac{24}{20^3} & +\frac{12}{20^2} \\ \frac{6}{16^2} & \frac{2}{16} & -\frac{6}{16^2}+\frac{12}{20^2} & \frac{4}{16}+\frac{8}{20} & -\frac{12}{20^2} & \frac{4}{20} \\ 0 & 0 & -\frac{24}{20^3} & -\frac{12}{20^2} & \frac{24}{20^3} & -\frac{12}{20^2} \\ 0 & 0 & \frac{12}{20^2} & \frac{4}{20} & -\frac{12}{20^2} & \frac{8}{20} \end{bmatrix} \end{matrix}}_{\textbf{Global Stiffness Matrix, K}} EI \begin{bmatrix} 0 \\ u_2 \\ 0 \\ u_4 \\ 0 \\ 0 \end{bmatrix}$$

Global Stiffness Matrix, K

TWO-SPAN BEAM EXAMPLE: Formalization

Row interchange: $R'_1 = R_2$; $R'_2 = R_4$; $R'_3 = R_1$; $R'_4 = R_3$
Rows of $\mathbf{P_j}$, $\mathbf{P_m}$ and $\mathbf{K}$ are moved/interchanged. Sequence of $\boldsymbol{U}$ do not change.

$$\begin{bmatrix} 0 \\ -28 \\ R_{AY} \\ R_{BY}-52 \\ R_{CY} \\ M_C \end{bmatrix} = \begin{bmatrix} 91.41 \\ 45.16 \\ 26.66 \\ 42.3 \\ 30 \\ -100 \end{bmatrix} + \begin{matrix} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} \frac{6}{16^2} & \frac{4}{16} & -\frac{6}{16^2} & \frac{2}{16} & 0 & 0 \\ \frac{6}{16^2} & \frac{2}{16} & -\frac{6}{16^2}+\frac{12}{20^2} & \frac{4}{16}+\frac{8}{20} & -\frac{12}{20^2} & \frac{4}{20} \\ \frac{12}{16^3} & \frac{6}{16^2} & -\frac{12}{16^3} & \frac{6}{16^2} & 0 & 0 \\ -\frac{12}{16^3} & -\frac{6}{16^2} & \frac{12}{16^3}+\frac{24}{20^3} & -\frac{6}{16^2}+\frac{12}{20^2} & -\frac{24}{20^3} & +\frac{12}{20^2} \\ 0 & 0 & -\frac{24}{20^3} & -\frac{12}{20^2} & \frac{24}{20^3} & -\frac{12}{20^2} \\ 0 & 0 & \frac{12}{20^2} & \frac{4}{20} & -\frac{12}{20^2} & \frac{8}{20} \end{bmatrix} \end{matrix} EI \begin{matrix} \begin{bmatrix} 0 \\ u_2 \\ 0 \\ u_4 \\ 0 \\ 0 \end{bmatrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \end{matrix}$$

TWO-SPAN BEAM EXAMPLE: Formalization

Column interchange: $C'_1 = C_2$; $C'_2 = C_4$; $C'_3 = C_1$; $C'_4 = C_3$
Only columns of ***K*** are moved/interchanged. Sequence of ***U*** is **changed.**
Then we do matrix partitioning.

$$\begin{bmatrix} 0 \\ -28 \\ \hdashline R_{AY} \\ R_{BY}-52 \\ R_{CY} \\ M_C \end{bmatrix} = \begin{bmatrix} 91.41 \\ 45.16 \\ \hdashline 26.66 \\ 42.3 \\ 30 \\ -100 \end{bmatrix} + \begin{bmatrix} \frac{4}{16} & \frac{2}{16} & \frac{6}{16^2} & -\frac{6}{16^2} & 0 & 0 \\ \frac{2}{16} & \frac{4}{16}+\frac{8}{20} & \frac{6}{16^2} & -\frac{6}{16^2}+\frac{12}{20^2} & -\frac{12}{20^2} & \frac{4}{20} \\ \hdashline \frac{6}{16^2} & \frac{6}{16^2} & \frac{12}{16^3} & -\frac{12}{16^3} & 0 & 0 \\ -\frac{6}{16^2} & -\frac{6}{16^2}+\frac{12}{20^2} & -\frac{12}{16^3} & \frac{12}{16^3}+\frac{24}{20^3} & -\frac{24}{20^3} & +\frac{12}{20^2} \\ 0 & -\frac{12}{20^2} & 0 & -\frac{24}{20^3} & \frac{24}{20^3} & -\frac{12}{20^2} \\ 0 & \frac{4}{20} & 0 & \frac{12}{20^2} & -\frac{12}{20^2} & \frac{8}{20} \end{bmatrix} EI \begin{bmatrix} u_2 \\ u_4 \\ \hdashline 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(2, 4, 1, 3, 5, 6 — 2, 4, 1, 3, 5, 6)

TWO-SPAN BEAM EXAMPLE: Formalization

Known nodal forces

Unknown nodal d.o.f

$$\begin{bmatrix} 0 \\ -28 \\ \hdashline R_{AY} \\ R_{BY}-52 \\ R_{CY} \\ M_C \end{bmatrix} = \begin{bmatrix} 91.41 \\ 45.16 \\ \hdashline 26.66 \\ 42.3 \\ 30 \\ -100 \end{bmatrix} + \begin{bmatrix} \frac{4}{16} & \frac{2}{16} & \frac{6}{16^2} & -\frac{6}{16^2} & 0 & 0 \\ \frac{2}{16} & \frac{4}{16}+\frac{8}{20} & \frac{6}{16^2} & -\frac{6}{16^2}+\frac{12}{20^2} & -\frac{12}{20^2} & \frac{4}{20} \\ \hdashline \frac{6}{16^2} & \frac{6}{16^2} & \frac{12}{16^3} & -\frac{12}{16^3} & 0 & 0 \\ -\frac{6}{16^2} & -\frac{6}{16^2}+\frac{12}{20^2} & -\frac{12}{16^3} & \frac{12}{16^3}+\frac{24}{20^3} & -\frac{24}{20^3} & +\frac{12}{20^2} \\ 0 & -\frac{12}{20^2} & 0 & -\frac{24}{20^3} & \frac{24}{20^3} & -\frac{12}{20^2} \\ 0 & \frac{4}{20} & 0 & \frac{12}{20^2} & -\frac{12}{20^2} & \frac{8}{20} \end{bmatrix} EI \begin{bmatrix} u_2 \\ u_4 \\ \hdashline 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(2, 4, 1, 3, 5, 6 — 2, 4, 1, 3, 5, 6)

Unknown nodal forces

known nodal d.o.f

The above equation can be written in short as follows

$$\begin{bmatrix} \mathbf{P_{j,u}} \\ \hdashline \mathbf{P_{j,r}} \end{bmatrix} = \begin{bmatrix} \mathbf{P_{m,u}} \\ \hdashline \mathbf{P_{m,r}} \end{bmatrix} + \begin{bmatrix} \mathbf{K_{uu}} & \mathbf{K_{ur}} \\ \hdashline \mathbf{K_{ru}} & \mathbf{K_{rr}} \end{bmatrix} \begin{bmatrix} \mathbf{u_u} \\ \hdashline \mathbf{u_r} \end{bmatrix}$$

TWO-SPAN BEAM EXAMPLE: Formalization

Submatrices of the partitioned matrices

$$\mathbf{P}_{j,u} = \begin{bmatrix} 0 \\ -28 \end{bmatrix} \qquad \mathbf{P}_{m,u} = \begin{bmatrix} 91.41 \\ 45.16 \end{bmatrix} \qquad \mathbf{K}_{uu} = \begin{bmatrix} \frac{4}{16} & \frac{2}{16} \\ \frac{2}{16} & \frac{13}{20} \end{bmatrix} EI \qquad \mathbf{K}_{ur} = \begin{bmatrix} \frac{6}{16^2} & -\frac{6}{16^2} & 0 & 0 \\ \frac{6}{16^2} & -\frac{6}{16^2}+\frac{12}{20^2} & -\frac{12}{20^2} & \frac{4}{20} \end{bmatrix} EI$$

$$\mathbf{P}_{j,r} = \begin{bmatrix} R_{AY} \\ R_{BY}-52 \\ R_{CY} \\ M_C \end{bmatrix} \qquad \mathbf{P}_{m,r} = \begin{bmatrix} 26.66 \\ 42.3 \\ 30 \\ -100 \end{bmatrix} \qquad \mathbf{K}_{ru} = \begin{bmatrix} \frac{6}{16^2} & \frac{6}{16^2} \\ -\frac{6}{16^2} & -\frac{6}{16^2}+\frac{12}{20^2} \\ 0 & -\frac{12}{20^2} \\ 0 & \frac{4}{20} \end{bmatrix} EI \qquad \mathbf{K}_{rr} = \begin{bmatrix} \frac{12}{16^3} & -\frac{12}{16^3} & 0 & 0 \\ -\frac{12}{16^3} & \frac{12}{16^3}+\frac{24}{20^3} & -\frac{24}{20^3} & +\frac{12}{20^2} \\ 0 & -\frac{24}{20^3} & \frac{24}{20^3} & -\frac{12}{20^2} \\ 0 & \frac{12}{20^2} & -\frac{12}{20^2} & \frac{8}{20} \end{bmatrix} EI$$

$$\mathbf{u}_u = \begin{bmatrix} u_2 \\ u_4 \end{bmatrix} \qquad \mathbf{u}_r = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

TWO-SPAN BEAM EXAMPLE: Formalization

$$\begin{bmatrix} \mathbf{P}_{j,u} \\ -- \\ \mathbf{P}_{j,r} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{m,u} \\ -- \\ \mathbf{P}_{m,r} \end{bmatrix} + \left[\begin{array}{c|c} \mathbf{K}_{uu} & \mathbf{K}_{ur} \\ \hline \mathbf{K}_{ru} & \mathbf{K}_{rr} \end{array}\right] \begin{bmatrix} \mathbf{u}_u \\ -- \\ \mathbf{u}_r \end{bmatrix}$$

We can write,

$$\mathbf{P}_{j,u} = \mathbf{P}_{m,u} + \mathbf{K}_{uu}\mathbf{u}_u + \mathbf{K}_{ur}\mathbf{u}_r \qquad (1)$$

$$\mathbf{P}_{j,r} = \mathbf{P}_{m,r} + \mathbf{K}_{ru}\mathbf{u}_u + \mathbf{K}_{rr}\mathbf{u}_r \qquad (2)$$

From (1) we have

$$\mathbf{u}_u = \mathbf{K}_{uu}^{-1}[\mathbf{P}_{j,u} - \mathbf{P}_{m,u} - \mathbf{K}_{rr}\mathbf{u}_r]$$

For the present case, $\mathbf{u}_r = [0]$, therefore, eq.(1) becomes

$$\mathbf{P}_{j,u} = \mathbf{P}_{m,u} + \mathbf{K}_{uu}\mathbf{u}_u$$

Putting values of the matrices,

$$\begin{bmatrix} 0 \\ -28 \end{bmatrix} = \begin{bmatrix} 91.41 \\ 45.16 \end{bmatrix} + \begin{bmatrix} \frac{4}{16} & \frac{2}{16} \\ \frac{2}{16} & \frac{13}{20} \end{bmatrix} EI \begin{bmatrix} u_2 \\ u_4 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} u_2 \\ u_4 \end{bmatrix} = \begin{bmatrix} -342.3 \\ -46.73 \end{bmatrix} \frac{1}{EI}$$

TWO-SPAN BEAM EXAMPLE: Formalization

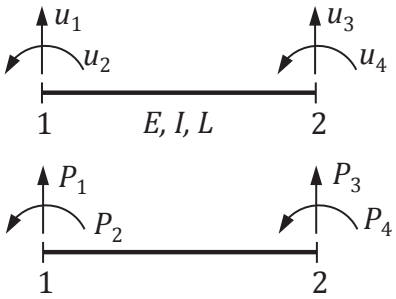
Support reactions:

Since $\mathbf{u_r = [0]}$, therefore, eq.(2) becomes
$\mathbf{P_{j,r} = P_{m,r} + K_{ru}u_u}$

Putting values of the matrices,

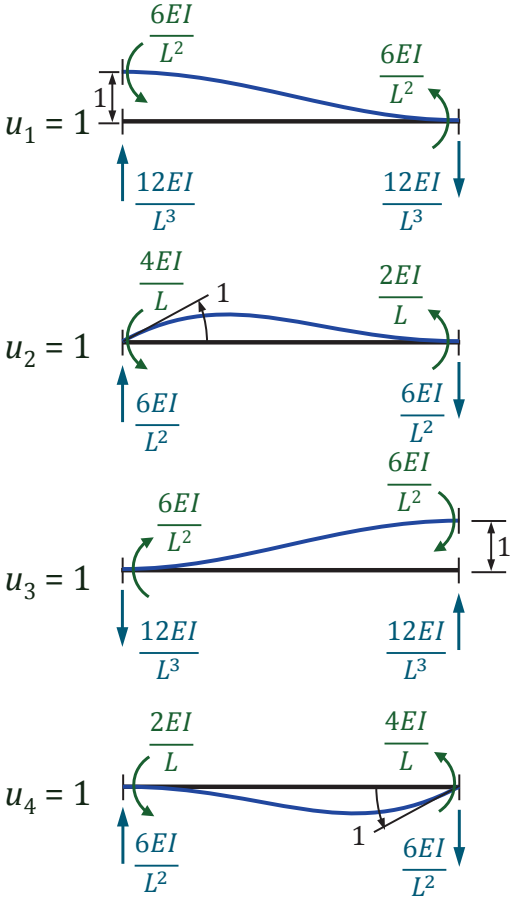
$$\begin{bmatrix} R_{AY} \\ R_{BY} - 52 \\ R_{CY} \\ M_C \end{bmatrix} = \begin{bmatrix} 26.66 \\ 42.3 \\ 30 \\ -100 \end{bmatrix} + \begin{bmatrix} \frac{6}{16^2} & \frac{6}{16^2} \\ -\frac{6}{16^2} & -\frac{6}{16^2}+\frac{12}{20^2} \\ 0 & -\frac{12}{20^2} \\ 0 & \frac{4}{20} \end{bmatrix} EI \begin{bmatrix} -342.3 \\ -46.73 \end{bmatrix} \frac{1}{EI} \quad \Rightarrow \quad \begin{bmatrix} R_{AY} \\ R_{BY} \\ R_{CY} \\ M_C \end{bmatrix} = \begin{bmatrix} 17.54 \\ 102.06 \\ 31.40 \\ -109.35 \end{bmatrix}$$

STIFFNESS OF A BEAM ELEMENT

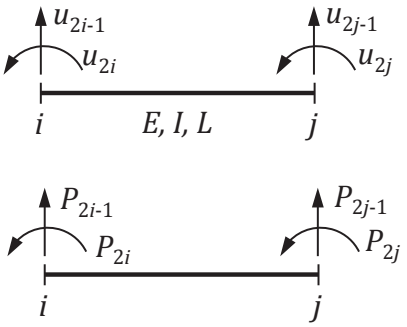


P = KU (Since there is no span load)

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$



STIFFNESS OF A BEAM ELEMENT



$$\begin{matrix} & 2i\text{-}1 & 2i & 2j\text{-}1 & 2j \\ \begin{bmatrix} P_{2i-1} \\ P_{2i} \\ P_{2j-1} \\ P_{2j} \end{bmatrix} = & \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} & \begin{bmatrix} u_{2i-1} \\ u_{2i} \\ u_{2j-1} \\ u_{2j} \end{bmatrix} \end{matrix}$$

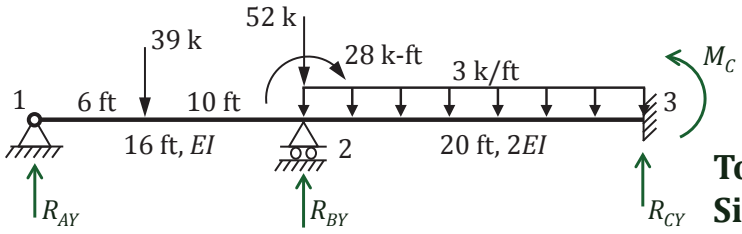
Element Stiffness Matrix

$$K = \begin{matrix} \\ \\ \\ \\ \end{matrix} \begin{matrix} 2i\text{-}1 & 2i & 2j\text{-}1 & 2j \\ \end{matrix} \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{matrix} 2i\text{-}1 \\ 2i \\ 2j\text{-}1 \\ 2j \end{matrix}$$

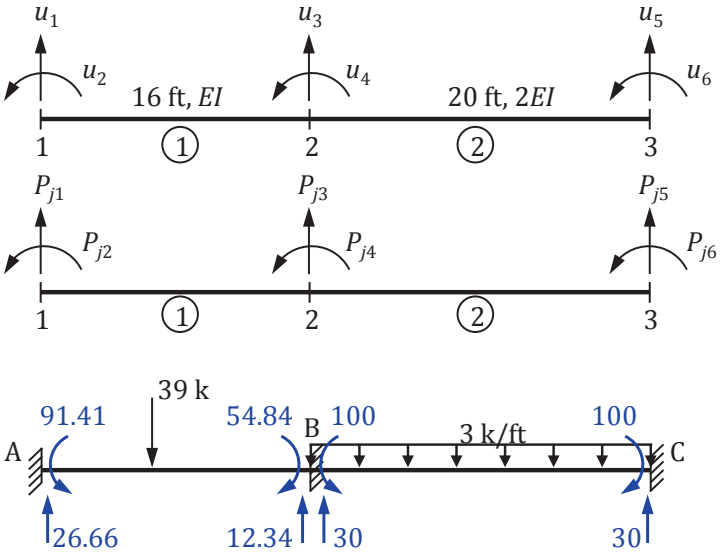
Element Destination Vector

$$\begin{bmatrix} 2i-1 \\ 2i \\ 2j-1 \\ 2j \end{bmatrix}$$

TWO-SPAN BEAM EXAMPLE: Direct Stiffness Formulation



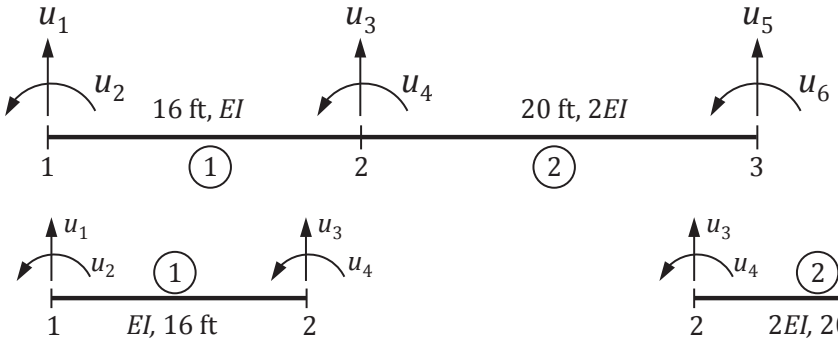
Total d.o.f. = 6
Size of Global Stiffness Matrix = 6×6



$$\begin{bmatrix} P_{j1} \\ P_{j2} \\ P_{j3} \\ P_{j4} \\ P_{j5} \\ P_{j6} \end{bmatrix} = \begin{bmatrix} R_{AY} \\ 0 \\ R_{BY} - 52 \\ -28 \\ R_{CY} \\ M_C \end{bmatrix} \quad \begin{bmatrix} P_{m1} \\ P_{m2} \\ P_{m3} \\ P_{m4} \\ P_{m5} \\ P_{m6} \end{bmatrix} = \begin{bmatrix} 26.66 \\ 91.41 \\ 12.34 + 30 \\ -54.84 + 100 \\ 30 \\ -100 \end{bmatrix}$$

TWO-SPAN BEAM EXAMPLE: Direct Stiffness Formulation

Formation of element stiffness matrices



Element-1 Stiffness Matrix

$$K_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{12EI}{16^3} & \frac{6EI}{16^2} & -\frac{12EI}{16^3} & \frac{6EI}{16^2} \\ \frac{6EI}{16^2} & \frac{4EI}{16} & -\frac{6EI}{16^2} & \frac{2EI}{16} \\ -\frac{12EI}{16^3} & -\frac{6EI}{16^2} & \frac{12EI}{16^3} & -\frac{6EI}{16^2} \\ \frac{6EI}{16^2} & \frac{2EI}{16} & -\frac{6EI}{16^2} & \frac{4EI}{16} \end{bmatrix} \end{array}$$

Element-2 Stiffness Matrix

$$K_2 = \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} \frac{24EI}{20^3} & \frac{12EI}{20^2} & -\frac{24EI}{20^3} & \frac{12EI}{20^2} \\ \frac{12EI}{20^2} & \frac{8EI}{20} & -\frac{12EI}{20^2} & \frac{4EI}{20} \\ -\frac{24EI}{20^3} & -\frac{12EI}{20^2} & \frac{24EI}{20^3} & -\frac{12EI}{20^2} \\ \frac{12EI}{20^2} & \frac{4EI}{20} & -\frac{12EI}{20^2} & \frac{8EI}{20} \end{bmatrix} \end{array}$$

TWO-SPAN BEAM EXAMPLE: Direct Stiffness Formulation

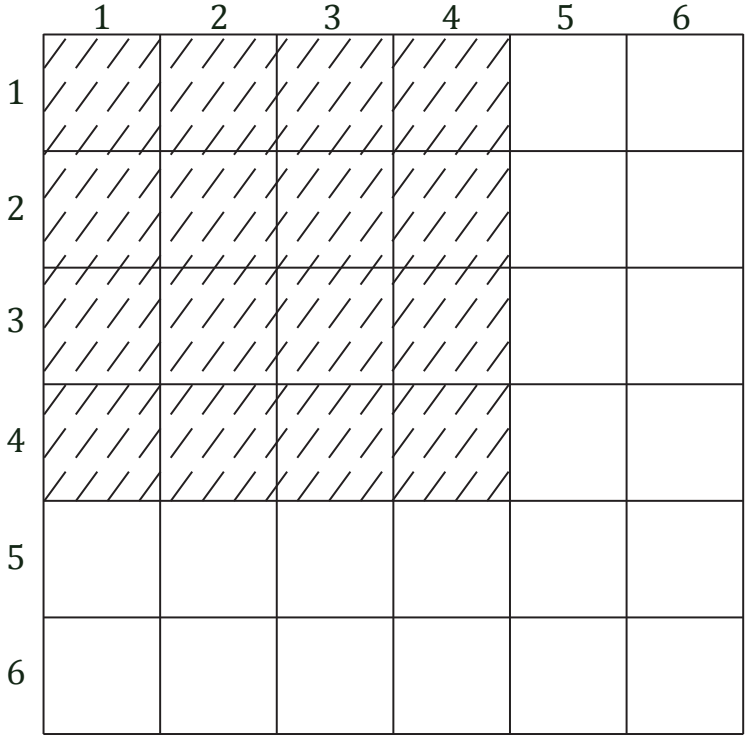
Assembly of element-1 stiffness matrix in global matrix

Element stiffness matrix is assembled into global matrix in accordance with destination vector of the element.

Element-1 Stiffness Matrix

$$K_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{12EI}{16^3} & \frac{6EI}{16^2} & -\frac{12EI}{16^3} & \frac{6EI}{16^2} \\ \frac{6EI}{16^2} & \frac{4EI}{16} & -\frac{6EI}{16^2} & \frac{2EI}{16} \\ -\frac{12EI}{16^3} & -\frac{6EI}{16^2} & \frac{12EI}{16^3} & -\frac{6EI}{16^2} \\ \frac{6EI}{16^2} & \frac{2EI}{16} & -\frac{6EI}{16^2} & \frac{4EI}{16} \end{bmatrix} \end{array}$$

Global Stiffness Matrix



TWO-SPAN BEAM EXAMPLE: Direct Stiffness Formulation

Assembly of element-1 stiffness matrix in global matrix

Element stiffness matrix is assembled into global matrix in accordance with destination vector of the element.

Element-1 Stiffness Matrix

$$K_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{12EI}{16^3} & \frac{6EI}{16^2} & -\frac{12EI}{16^3} & \frac{6EI}{16^2} \\ \frac{6EI}{16^2} & \frac{4EI}{16} & -\frac{6EI}{16^2} & \frac{2EI}{16} \\ -\frac{12EI}{16^3} & -\frac{6EI}{16^2} & \frac{12EI}{16^3} & -\frac{6EI}{16^2} \\ \frac{6EI}{16^2} & \frac{2EI}{16} & -\frac{6EI}{16^2} & \frac{4EI}{16} \end{bmatrix} \end{array}$$

Global Stiffness Matrix

	1	2	3	4	5	6
1	$\frac{12EI}{16^3}$	$\frac{6EI}{16^2}$	$-\frac{12EI}{16^3}$	$\frac{6EI}{16^2}$		
2	$\frac{6EI}{16^2}$	$\frac{4EI}{16}$	$-\frac{6EI}{16^2}$	$\frac{2EI}{16}$		
3	$-\frac{12EI}{16^3}$	$-\frac{6EI}{16^2}$	$\frac{12EI}{16^3}$	$-\frac{6EI}{16^2}$		
4	$\frac{6EI}{16^2}$	$\frac{2EI}{16}$	$-\frac{6EI}{16^2}$	$\frac{4EI}{16}$		
5						
6						

TWO-SPAN BEAM EXAMPLE: Direct Stiffness Formulation

Assembly of element-2 stiffness matrix in global matrix

Element stiffness matrix is assembled into global matrix in accordance with destination vector of the element.

Element-2 Stiffness Matrix

$$K_2 = \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} \frac{24EI}{20^3} & \frac{12EI}{20^2} & -\frac{24EI}{20^3} & \frac{12EI}{20^2} \\ \frac{12EI}{20^2} & \frac{8EI}{20} & -\frac{12EI}{20^2} & \frac{4EI}{20} \\ -\frac{24EI}{20^3} & -\frac{12EI}{20^2} & \frac{24EI}{20^3} & -\frac{12EI}{20^2} \\ \frac{12EI}{20^2} & \frac{4EI}{20} & -\frac{12EI}{20^2} & \frac{8EI}{20} \end{bmatrix} \end{array}$$

Global Stiffness Matrix

	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

TWO-SPAN BEAM EXAMPLE: Direct Stiffness Formulation

Assembly of element stiffness matrices in global matrix

Element-1 Stiffness Matrix

$$K_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{12EI}{16^3} & \frac{6EI}{16^2} & -\frac{12EI}{16^3} & \frac{6EI}{16^2} \\ \frac{6EI}{16^2} & \frac{4EI}{16} & -\frac{6EI}{16^2} & \frac{2EI}{16} \\ -\frac{12EI}{16^3} & -\frac{6EI}{16^2} & \frac{12EI}{16^3} & -\frac{6EI}{16^2} \\ \frac{6EI}{16^2} & \frac{2EI}{16} & -\frac{6EI}{16^2} & \frac{4EI}{16} \end{bmatrix} \end{array}$$

Element-2 Stiffness Matrix

$$K_2 = \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} \frac{24EI}{20^3} & \frac{12EI}{20^2} & -\frac{24EI}{20^3} & \frac{12EI}{20^2} \\ \frac{12EI}{20^2} & \frac{8EI}{20} & -\frac{12EI}{20^2} & \frac{4EI}{20} \\ -\frac{24EI}{20^3} & -\frac{12EI}{20^2} & \frac{24EI}{20^3} & -\frac{12EI}{20^2} \\ \frac{12EI}{20^2} & \frac{4EI}{20} & -\frac{12EI}{20^2} & \frac{8EI}{20} \end{bmatrix} \end{array}$$

Global Stiffness Matrix

	1	2	3	4	5	6
1	$\frac{12EI}{16^3}$	$\frac{6EI}{16^2}$	$-\frac{12EI}{16^3}$	$\frac{6EI}{16^2}$		
2	$\frac{6EI}{16^2}$	$\frac{4EI}{16}$	$-\frac{6EI}{16^2}$	$\frac{2EI}{16}$		
3	$-\frac{12EI}{16^3}$	$-\frac{6EI}{16^2}$	$\frac{12EI}{16^3}+\frac{24EI}{20^3}$	$-\frac{6EI}{16^2}+\frac{12EI}{20^2}$	$-\frac{24EI}{20^3}$	$\frac{12EI}{20^2}$
4	$\frac{6EI}{16^2}$	$\frac{2EI}{16}$	$-\frac{6EI}{16^2}+\frac{12EI}{20^2}$	$\frac{4EI}{16}+\frac{8EI}{20}$	$-\frac{12EI}{20^2}$	$\frac{4EI}{20}$
5			$-\frac{24EI}{20^3}$	$-\frac{12EI}{20^2}$	$+\frac{24EI}{20^3}$	$-\frac{12EI}{20^2}$
6			$\frac{12EI}{20^2}$	$\frac{4EI}{20}$	$-\frac{12EI}{20^2}$	$+\frac{8EI}{20}$

TWO-SPAN BEAM EXAMPLE: Direct Stiffness Formulation

Compare global stiffness matrix formed by different methods.

Global stiffness matrix by principle of superposition

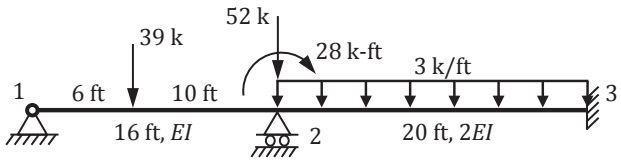
$$EI \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} \frac{12}{16^3} & \frac{6}{16^2} & -\frac{12}{16^3} & \frac{6}{16^2} & 0 & 0 \\ \frac{6}{16^2} & \frac{4}{16} & -\frac{6}{16^2} & \frac{2}{16} & 0 & 0 \\ -\frac{12}{16^3} & -\frac{6}{16^2} & \frac{12}{16^3}+\frac{24}{20^3} & -\frac{6}{16^2}+\frac{12}{20^2} & -\frac{24}{20^3} & +\frac{12}{20^2} \\ \frac{6}{16^2} & \frac{2}{16} & -\frac{6}{16^2}+\frac{12}{20^2} & \frac{4}{16}+\frac{8}{20} & -\frac{12}{20^2} & \frac{4}{20} \\ 0 & 0 & -\frac{24}{20^3} & -\frac{12}{20^2} & \frac{24}{20^3} & -\frac{12}{20^2} \\ 0 & 0 & \frac{12}{20^2} & \frac{4}{20} & -\frac{12}{20^2} & \frac{8}{20} \end{bmatrix} \end{array}$$

Global stiffness matrix by direct assembly

	1	2	3	4	5	6
1	$\frac{12EI}{16^3}$	$\frac{6EI}{16^2}$	$-\frac{12EI}{16^3}$	$\frac{6EI}{16^2}$		
2	$\frac{6EI}{16^2}$	$\frac{4EI}{16}$	$-\frac{6EI}{16^2}$	$\frac{2EI}{16}$		
3	$-\frac{12EI}{16^3}$	$-\frac{6EI}{16^2}$	$\frac{12EI}{16^3}+\frac{24EI}{20^3}$	$-\frac{6EI}{16^2}+\frac{12EI}{20^2}$	$-\frac{24EI}{20^3}$	$\frac{12EI}{20^2}$
4	$\frac{6EI}{16^2}$	$\frac{2EI}{16}$	$-\frac{6EI}{16^2}+\frac{12EI}{20^2}$	$\frac{4EI}{16}+\frac{8EI}{20}$	$-\frac{12EI}{20^2}$	$\frac{4EI}{20}$
5			$-\frac{24EI}{20^3}$	$-\frac{12EI}{20^2}$	$+\frac{24EI}{20^3}$	$-\frac{12EI}{20^2}$
6			$\frac{12EI}{20^2}$	$\frac{4EI}{20}$	$-\frac{12EI}{20^2}$	$+\frac{8EI}{20}$

TWO-SPAN BEAM EXAMPLE: Direct Stiffness Formulation

Shear and moments at member ends

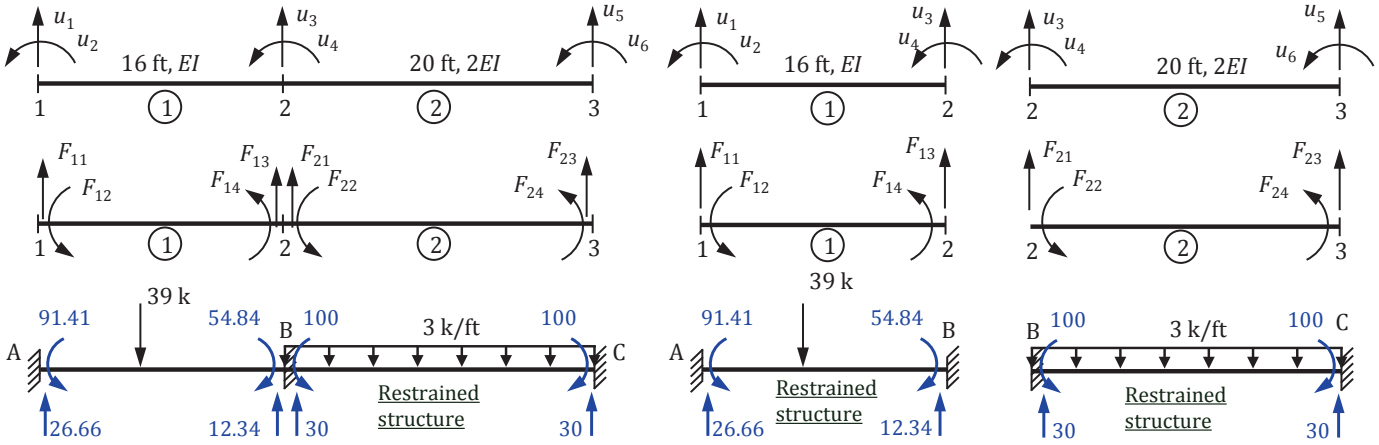


Apply the superposition equation

$\mathbf{F_e = F_{me} + K_e U_e}$

for individual members.

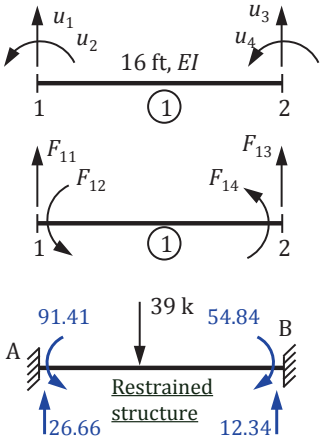
For member-1: For member-2:



TWO-SPAN BEAM EXAMPLE: Direct Stiffness Formulation

Shear and moments at member ends

For member-1: $\mathbf{F_e = F_{me} + K_e U_e}$



$$\begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{14} \end{bmatrix} = \begin{bmatrix} F_{m11} \\ F_{m12} \\ F_{m13} \\ F_{m14} \end{bmatrix} + \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$\begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{14} \end{bmatrix} = \begin{bmatrix} 26.66 \\ 91.41 \\ 12.34 \\ -54.84 \end{bmatrix} + \begin{bmatrix} \frac{12EI}{16^3} & \frac{6EI}{16^2} & -\frac{12EI}{16^3} & \frac{6EI}{16^2} \\ \frac{6EI}{16^2} & \frac{4EI}{16} & -\frac{6EI}{16^2} & \frac{2EI}{16} \\ -\frac{12EI}{16^3} & -\frac{6EI}{16^2} & \frac{12EI}{16^3} & -\frac{6EI}{16^2} \\ \frac{6EI}{16^2} & \frac{2EI}{16} & -\frac{6EI}{16^2} & \frac{4EI}{16} \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{342.3}{EI} \\ 0 \\ -\frac{46.73}{EI} \end{bmatrix} = \begin{bmatrix} 17.54 \\ 0 \\ 21.46 \\ -109.4 \end{bmatrix}$$

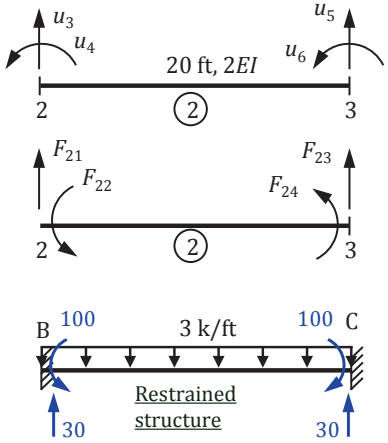
← shear-1
← moment-1
← shear-2
← moment-2

TWO-SPAN BEAM EXAMPLE: Direct Stiffness Formulation

Shear and moments at member ends

$$\mathbf{F_e = F_{me} + K_e U_e}$$

For member-2:



$$\begin{bmatrix} F_{21} \\ F_{22} \\ F_{23} \\ F_{24} \end{bmatrix} = \begin{bmatrix} F_{m21} \\ F_{m22} \\ F_{m23} \\ F_{m24} \end{bmatrix} + \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}$$

$$\begin{bmatrix} F_{21} \\ F_{22} \\ F_{23} \\ F_{24} \end{bmatrix} = \begin{bmatrix} 30 \\ 100 \\ 30 \\ -100 \end{bmatrix} + \begin{bmatrix} \frac{24EI}{20^3} & \frac{12EI}{20^2} & -\frac{24EI}{20^3} & \frac{12EI}{20^2} \\ \frac{12EI}{20^2} & \frac{8EI}{20} & -\frac{12EI}{20^2} & \frac{4EI}{20} \\ -\frac{24EI}{20^3} & -\frac{12EI}{20^2} & \frac{24EI}{20^3} & -\frac{12EI}{20^2} \\ \frac{12EI}{20^2} & \frac{4EI}{20} & -\frac{12EI}{20^2} & \frac{8EI}{20} \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{46.73}{EI} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 28.6 \\ 81.31 \\ 31.4 \\ -109.35 \end{bmatrix} \begin{matrix} \leftarrow \text{shear-1} \\ \leftarrow \text{moment-1} \\ \leftarrow \text{shear-2} \\ \leftarrow \text{moment-2} \end{matrix}$$

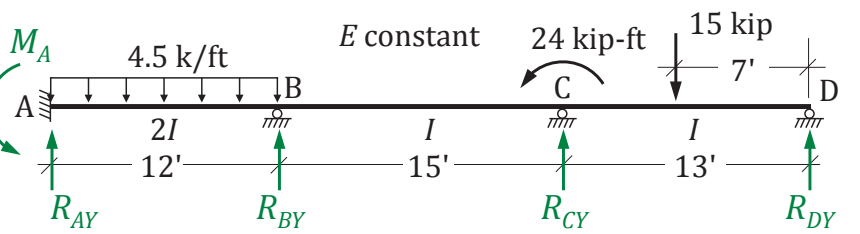
TWO-SPAN BEAM EXAMPLE: Direct Stiffness Formulation

It is observed that the global stiffness matrix can be formed by direct assembly of individual element stiffness matrices with appropriate destination vector.

Once the global stiffness matrix is formed, rest of the procedure is same as before –

a. we do matrix partitioning on the basis of known and unknown DOFs and form two matrix eq. (1) and (2)
b. solve for the unknown DOFs from eq. (1)
c. find the unknown support reactions using the eq. (2)
d. find shear force and bending moment at member ends
e. draw the shear force and bending moment diagrams.

THREE-SPAN BEAM EXAMPLE: Direct Stiffness Formulation



$$\mathbf{P_j} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \end{bmatrix} = \begin{bmatrix} R_{AY} \\ M_A \\ R_{BY} \\ 0 \\ R_{CY} \\ 24 \\ R_{DY} \\ 0 \end{bmatrix}$$

Total dof = 8

Unknown dof=3

Global matrix size = 8×8

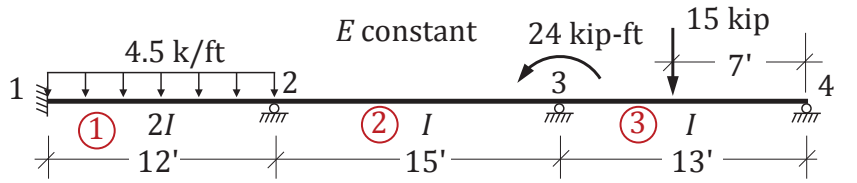






$$\mathbf{P_m} = \begin{bmatrix} P_{m1} \\ P_{m2} \\ P_{m3} \\ P_{m4} \\ P_{m5} \\ P_{m6} \\ P_{m7} \\ P_{m8} \end{bmatrix} = \begin{bmatrix} 27 \\ 54 \\ 27 \\ -54 \\ 8.364 \\ 26.1 \\ 6.636 \\ -22.37 \end{bmatrix} \quad \mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ u_4 \\ 0 \\ u_6 \\ 0 \\ u_8 \end{bmatrix}$$

THREE-SPAN BEAM EXAMPLE: Direct Stiffness Formulation





Joint co-ordinate matrix

Node	x	y
1	0	0
2	12	0
3	27	0
4	40	0

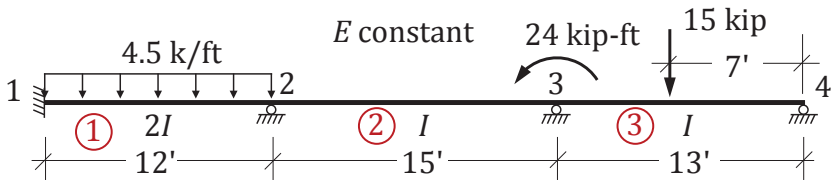
Member connectivity matrix

Memb	i	j
1	1	2
2	2	3
3	3	4

Member destination vector

Memb	$2i$-1	$2i$	$2j$-1	$2j$
1	1	2	3	4
2	3	4	5	6
3	5	6	7	8

THREE-SPAN BEAM EXAMPLE: Direct Stiffness Formulation





$$K_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{24EI}{12^3} & \frac{12EI}{12^2} & -\frac{24EI}{12^3} & \frac{12EI}{12^2} \\ \frac{12EI}{12^2} & \frac{8EI}{12} & -\frac{12EI}{12^2} & \frac{4EI}{12} \\ -\frac{24EI}{12^3} & -\frac{12EI}{12^2} & \frac{24EI}{12^3} & -\frac{12EI}{12^2} \\ \frac{12EI}{12^2} & \frac{4EI}{12} & -\frac{12EI}{12^2} & \frac{8EI}{12} \end{bmatrix} \end{array}$$

$$K_2 = \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} \frac{12EI}{15^3} & \frac{6EI}{15^2} & -\frac{12EI}{15^3} & \frac{6EI}{15^2} \\ \frac{6EI}{15^2} & \frac{4EI}{15} & -\frac{6EI}{15^2} & \frac{2EI}{15} \\ -\frac{12EI}{15^3} & -\frac{6EI}{15^2} & \frac{12EI}{15^3} & -\frac{6EI}{15^2} \\ \frac{6EI}{15^2} & \frac{2EI}{15} & -\frac{6EI}{15^2} & \frac{4EI}{15} \end{bmatrix} \end{array}$$

$$K_3 = \begin{array}{c} \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} \frac{12EI}{13^3} & \frac{6EI}{13^2} & -\frac{12EI}{13^3} & \frac{6EI}{13^2} \\ \frac{6EI}{13^2} & \frac{4EI}{13} & -\frac{6EI}{13^2} & \frac{2EI}{13} \\ -\frac{12EI}{13^3} & -\frac{6EI}{13^2} & \frac{12EI}{13^3} & -\frac{6EI}{13^2} \\ \frac{6EI}{13^2} & \frac{2EI}{13} & -\frac{6EI}{13^2} & \frac{4EI}{13} \end{bmatrix} \end{array}$$

THREE-SPAN BEAM EXAMPLE: Direct Stiffness Formulation

Element-1 stiffness matrix

$$K_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{24EI}{12^3} & \frac{12EI}{12^2} & -\frac{24EI}{12^3} & \frac{12EI}{12^2} \\ \frac{12EI}{12^2} & \frac{8EI}{12} & -\frac{12EI}{12^2} & \frac{4EI}{12} \\ -\frac{24EI}{12^3} & -\frac{12EI}{12^2} & \frac{24EI}{12^3} & -\frac{12EI}{12^2} \\ \frac{12EI}{12^2} & \frac{4EI}{12} & -\frac{12EI}{12^2} & \frac{8EI}{12} \end{bmatrix} \end{array}$$

Global stiffness matrix

	1	2	3	4	5	6	7	8
1	$\frac{24}{12^3}$	$\frac{12}{12^2}$	$-\frac{24}{12^3}$	$\frac{12}{12^2}$				
2	$\frac{12}{12^2}$	$\frac{8}{12}$	$-\frac{12}{12^2}$	$\frac{4}{12}$				
3	$-\frac{24}{12^3}$	$-\frac{12}{12^2}$	$\frac{24}{12^3}$	$-\frac{12}{12^2}$				
4	$\frac{12}{12^2}$	$\frac{4}{12}$	$-\frac{12}{12^2}$	$\frac{8}{12}$				
5								
6								
7								
8								

THREE-SPAN BEAM EXAMPLE: Direct Stiffness Formulation

Element-2 stiffness matrix

$$K_2 = \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} \frac{12EI}{15^3} & \frac{6EI}{15^2} & -\frac{12EI}{15^3} & \frac{6EI}{15^2} \\ \frac{6EI}{15^2} & \frac{4EI}{15} & -\frac{6EI}{15^2} & \frac{2EI}{15} \\ -\frac{12EI}{15^3} & -\frac{6EI}{15^2} & \frac{12EI}{15^3} & -\frac{6EI}{15^2} \\ \frac{6EI}{15^2} & \frac{2EI}{15} & -\frac{6EI}{15^2} & \frac{4EI}{15} \end{bmatrix} \end{array}$$

Global stiffness matrix

	1	2	3	4	5	6	7	8
1	$\frac{24}{12^3}$	$\frac{12}{12^2}$	$-\frac{24}{12^3}$	$\frac{12}{12^2}$				
2	$\frac{12}{12^2}$	$\frac{8}{12}$	$-\frac{12}{12^2}$	$\frac{4}{12}$				
3	$-\frac{24}{12^3}$	$-\frac{12}{12^2}$	$\frac{24}{12^3}+\frac{12}{15^3}$	$-\frac{12}{12^2}+\frac{6}{15^2}$	$-\frac{12}{15^3}$	$+\frac{6}{15^2}$		
4	$\frac{12}{12^2}$	$\frac{4}{12}$	$-\frac{12}{12^2}+\frac{6}{15^2}$	$\frac{8}{12}+\frac{4}{15}$	$-\frac{6}{15^2}$	$+\frac{2}{15}$		
5			$-\frac{12}{15^3}$	$-\frac{6}{15^2}$	$\frac{12}{15^3}$	$-\frac{6}{15^2}$		
6			$+\frac{6}{15^2}$	$+\frac{2}{15}$	$-\frac{6}{15^2}$	$+\frac{4}{15}$		
7								
8								

THREE-SPAN BEAM EXAMPLE: Direct Stiffness Formulation

Element-3 stiffness matrix

$$K_3 = \begin{array}{c} \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} \frac{12EI}{13^3} & \frac{6EI}{13^2} & -\frac{12EI}{13^3} & \frac{6EI}{13^2} \\ \frac{6EI}{13^2} & \frac{4EI}{13} & -\frac{6EI}{13^2} & \frac{2EI}{13} \\ -\frac{12EI}{13^3} & -\frac{6EI}{13^2} & \frac{12EI}{13^3} & -\frac{6EI}{13^2} \\ \frac{6EI}{13^2} & \frac{2EI}{13} & -\frac{6EI}{13^2} & \frac{4EI}{13} \end{bmatrix} \end{array}$$

Global stiffness matrix

	1	2	3	4	5	6	7	8
1	$\frac{24}{12^3}$	$\frac{12}{12^2}$	$-\frac{24}{12^3}$	$\frac{12}{12^2}$				
2	$\frac{12}{12^2}$	$\frac{8}{12}$	$-\frac{12}{12^2}$	$\frac{4}{12}$				
3	$-\frac{24}{12^3}$	$-\frac{12}{12^2}$	$\frac{24}{12^3}+\frac{12}{15^3}$	$-\frac{12}{12^2}+\frac{6}{15^2}$	$-\frac{12}{15^3}$	$+\frac{6}{15^2}$		
4	$\frac{12}{12^2}$	$\frac{4}{12}$	$-\frac{12}{12^2}+\frac{6}{15^2}$	$\frac{8}{12}+\frac{4}{15}$	$-\frac{6}{15^2}$	$+\frac{2}{15}$		
5			$-\frac{12}{15^3}$	$-\frac{6}{15^2}$	$\frac{12}{15^3}+\frac{12}{13^3}$	$-\frac{6}{15^2}+\frac{6}{13^2}$	$-\frac{12}{13^3}$	$\frac{6}{13^2}$
6			$+\frac{6}{15^2}$	$+\frac{2}{15}$	$-\frac{6}{15^2}+\frac{6}{13^2}$	$+\frac{4}{15}+\frac{4}{13}$	$-\frac{6}{13^2}$	$+\frac{2}{13}$
7					$-\frac{12}{13^3}$	$-\frac{6}{13^2}$	$+\frac{12}{13^3}$	$-\frac{6}{13^2}$
8					$\frac{6}{13^2}$	$+\frac{2}{13}$	$-\frac{6}{13^2}$	$+\frac{4}{13}$

THREE-SPAN BEAM EXAMPLE: Direct Stiffness Formulation

$\mathbf{P_j = P_m + KU}$

R_{AY}		27		1	$\frac{24}{12^3}$	$\frac{12}{12^2}$	$-\frac{24}{12^3}$	$\frac{12}{12^2}$					0
M_A		54		2	$\frac{12}{12^2}$	$\frac{8}{12}$	$-\frac{12}{12^2}$	$\frac{4}{12}$					0
R_{BY}		27		3	$-\frac{24}{12^3}$	$-\frac{12}{12^2}$	$\frac{24}{12^3}+\frac{12}{15^3}$	$-\frac{12}{12^2}+\frac{6}{15^2}$	$-\frac{12}{15^3}$	$+\frac{6}{15^2}$			0
0		−54		4	$\frac{12}{12^2}$	$\frac{4}{12}$	$-\frac{12}{12^2}+\frac{6}{15^2}$	$\frac{8}{12}+\frac{4}{15}$	$-\frac{6}{15^2}$	$+\frac{2}{15}$			u_4
R_{CY}	=	8.364	+	5			$-\frac{12}{15^3}$	$-\frac{6}{15^2}$	$\frac{12}{15^3}+\frac{12}{13^3}$	$-\frac{6}{15^2}+\frac{6}{13^2}$	$-\frac{12}{13^3}$	$\frac{6}{13^2}$	0
24		26.1		6			$+\frac{6}{15^2}$	$+\frac{2}{15}$	$-\frac{6}{15^2}+\frac{6}{13^2}$	$+\frac{4}{15}+\frac{4}{13}$	$-\frac{6}{13^2}$	$+\frac{2}{13}$	u_6
R_{DY}		6.636		7					$-\frac{12}{13^3}$	$-\frac{6}{13^2}$	$+\frac{12}{13^3}$	$-\frac{6}{13^2}$	0
0		−22.37		8					$\frac{6}{13^2}$	$+\frac{2}{13}$	$-\frac{6}{13^2}$	$+\frac{4}{13}$	u_8
					1	2	3	4	5	6	7	8	

THREE-SPAN BEAM EXAMPLE: Direct Stiffness Formulation

Shifting rows 4, 6 and 8 to top

0		−54		1	$\frac{12}{12^2}$	$\frac{4}{12}$	$-\frac{12}{12^2}+\frac{6}{15^2}$	$\frac{8}{12}+\frac{4}{15}$	$-\frac{6}{15^2}$	$+\frac{2}{15}$			0
24		26.1		2			$+\frac{6}{15^2}$	$+\frac{2}{15}$	$-\frac{6}{15^2}+\frac{6}{13^2}$	$+\frac{4}{15}+\frac{4}{13}$	$-\frac{6}{13^2}$	$+\frac{2}{13}$	0
0		−22.37		3					$\frac{6}{13^2}$	$+\frac{2}{13}$	$-\frac{6}{13^2}$	$+\frac{4}{13}$	0
R_{AY}		27		4	$\frac{24}{12^3}$	$\frac{12}{12^2}$	$-\frac{24}{12^3}$	$\frac{12}{12^2}$					u_4
M_A	=	54	+	5	$\frac{12}{12^2}$	$\frac{8}{12}$	$-\frac{12}{12^2}$	$\frac{4}{12}$					0
R_{BY}		27		6	$-\frac{24}{12^3}$	$-\frac{12}{12^2}$	$\frac{24}{12^3}+\frac{12}{15^3}$	$-\frac{12}{12^2}+\frac{6}{15^2}$	$-\frac{12}{15^3}$	$+\frac{6}{15^2}$			u_6
R_{CY}		8.364		7			$-\frac{12}{15^3}$	$-\frac{6}{15^2}$	$\frac{12}{15^3}+\frac{12}{13^3}$	$-\frac{6}{15^2}+\frac{6}{13^2}$	$-\frac{12}{13^3}$	$\frac{6}{13^2}$	0
R_{DY}		6.636		8					$-\frac{12}{13^3}$	$-\frac{6}{13^2}$	$+\frac{12}{13^3}$	$-\frac{6}{13^2}$	u_8
					1	2	3	4	5	6	7	8	

THREE-SPAN BEAM EXAMPLE: Direct Stiffness Formulation

Shifting columns 4, 6 and 8 to left and partitioning

				4	6	8	1	2	3	5	7	
0		−54		4	$\frac{8}{12}+\frac{4}{15}$	$+\frac{2}{15}$		$\frac{12}{12^2}$	$\frac{4}{12}$	$-\frac{12}{12^2}+\frac{6}{15^2}$	$-\frac{6}{15^2}$		u_4
24		26.1		6	$+\frac{2}{15}$	$+\frac{4}{15}+\frac{4}{13}$	$+\frac{2}{13}$			$+\frac{6}{15^2}$	$-\frac{6}{15^2}+\frac{6}{13^2}$	$-\frac{6}{13^2}$	u_6
0		−22.37		8		$+\frac{2}{13}$	$+\frac{4}{13}$				$\frac{6}{13^2}$	$-\frac{6}{13^2}$	u_8
R_{AY}	=	27	+	1	$\frac{12}{12^2}$			$\frac{24}{12^3}$	$\frac{12}{12^2}$	$-\frac{24}{12^3}$			0
M_A		54		2	$\frac{4}{12}$			$\frac{12}{12^2}$	$\frac{8}{12}$	$-\frac{12}{12^2}$			0
R_{BY}		27		3	$-\frac{12}{12^2}+\frac{6}{15^2}$	$+\frac{6}{15^2}$		$-\frac{24}{12^3}$	$-\frac{12}{12^2}$	$\frac{24}{12^3}+\frac{12}{15^3}$	$-\frac{12}{15^3}$		0
R_{CY}		8.364		5	$-\frac{6}{15^2}$	$-\frac{6}{15^2}+\frac{6}{13^2}$	$\frac{6}{13^2}$			$-\frac{12}{15^3}$	$\frac{12}{15^3}+\frac{12}{13^3}$	$-\frac{12}{13^3}$	0
R_{DY}		6.636		7		$-\frac{6}{13^2}$	$-\frac{6}{13^2}$				$-\frac{12}{13^3}$	$+\frac{12}{13^3}$	0

THREE-SPAN BEAM EXAMPLE: Direct Stiffness Formulation

$$\begin{bmatrix} \mathbf{P_{j,u}} \\ -- \\ \mathbf{P_{j,r}} \end{bmatrix} = \begin{bmatrix} \mathbf{P_{m,u}} \\ -- \\ \mathbf{P_{m,r}} \end{bmatrix} + \left[\begin{array}{c|c} \mathbf{K_{uu}} & \mathbf{K_{ur}} \\ --- & --- \\ \mathbf{K_{ru}} & \mathbf{K_{rr}} \end{array}\right] \begin{bmatrix} \mathbf{u_u} \\ -- \\ \mathbf{u_r} \end{bmatrix}$$

We can write,

$$\mathbf{P_{j,u} = P_{m,u} + K_{uu}u_u + K_{ur}u_r} \quad (1)$$

$$\mathbf{P_{j,r} = P_{m,r} + K_{ru}u_u + K_{rr}u_r} \quad (2)$$

For the present case, $\mathbf{u_r = [0]}$, therefore, eq.(1) becomes

$\mathbf{P_{j,u} = P_{m,u} + K_{uu}u_u}$

Putting values of the matrices,

$$\begin{bmatrix} 0 \\ 24 \\ 0 \end{bmatrix} = \begin{bmatrix} -54 \\ 26.1 \\ -22.37 \end{bmatrix} + \begin{bmatrix} \frac{8}{12}+\frac{4}{15} & \frac{2}{15} & \\ \frac{2}{15} & \frac{4}{15}+\frac{4}{13} & \frac{2}{13} \\ & \frac{2}{13} & \frac{4}{13} \end{bmatrix} EI \begin{bmatrix} u_4 \\ u_6 \\ u_8 \end{bmatrix} \Rightarrow \begin{bmatrix} u_4 \\ u_6 \\ u_8 \end{bmatrix} = \begin{bmatrix} 64.13 \\ -43.9 \\ 94.65 \end{bmatrix} \frac{1}{EI}$$

THREE-SPAN BEAM EXAMPLE: Direct Stiffness Formulation

Support reactions:

Since $\mathbf{u_r = [0]}$, therefore, eq.(2) becomes
$\mathbf{P_{j,r} = P_{m,r} + K_{ru}u_u}$

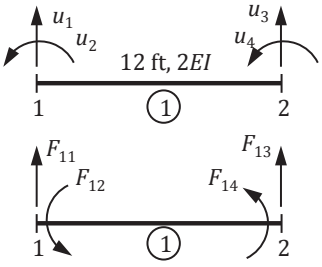
Putting values of the matrices,

$$\begin{bmatrix} R_{AY} \\ M_A \\ R_{BY} \\ R_{CY} \\ R_{DY} \end{bmatrix} = \begin{bmatrix} 27 \\ 54 \\ 27 \\ 8.364 \\ 6.636 \end{bmatrix} + \begin{bmatrix} \frac{12}{12^2} & 0 & 0 \\ \frac{4}{12} & 0 & 0 \\ -\frac{12}{12^2}+\frac{6}{15^2} & \frac{6}{15^2} & 0 \\ -\frac{6}{15^2} & -\frac{6}{15^2}+\frac{6}{13^2} & \frac{6}{13^2} \\ 0 & -\frac{6}{13^2} & -\frac{6}{13^2} \end{bmatrix} EI \begin{bmatrix} 64.13 \\ -43.9 \\ 94.65 \end{bmatrix} \frac{1}{EI} \quad \Rightarrow \quad \begin{bmatrix} R_{AY} \\ M_A \\ R_{BY} \\ R_{CY} \\ R_{DY} \end{bmatrix} = \begin{bmatrix} 32.34 \\ 75.38 \\ 22.2 \\ 9.63 \\ 4.83 \end{bmatrix}$$

THREE-SPAN BEAM EXAMPLE: Direct Stiffness Formulation

Shear and moments at member ends

For member-1:



Superposition eqn.: $\mathbf{F_e = F_{me} + K_eU_e}$

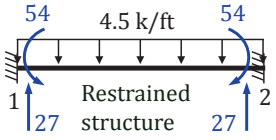
$$\begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{14} \end{bmatrix} = \begin{bmatrix} F_{m11} \\ F_{m12} \\ F_{m13} \\ F_{m14} \end{bmatrix} + \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$\mathbf{F_e}$ = Member end force/moment vector

$\mathbf{F_{me}}$ = Member end force/moment due to span load at fixed condition.

$\mathbf{K_e}$ = Member stiffness matrix

$\mathbf{U_e}$ = Member end displacement vector

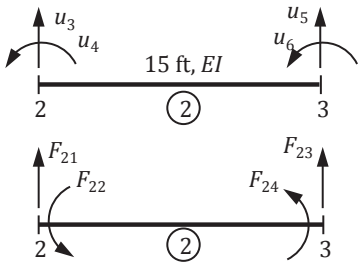


$$\begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{14} \end{bmatrix} = \begin{bmatrix} 27 \\ 54 \\ 27 \\ -54 \end{bmatrix} + \begin{bmatrix} \frac{24EI}{12^3} & \frac{12EI}{12^2} & -\frac{24EI}{12^3} & \frac{12EI}{12^2} \\ \frac{12EI}{12^2} & \frac{8EI}{12} & -\frac{12EI}{12^2} & \frac{4EI}{12} \\ -\frac{24EI}{12^3} & -\frac{12EI}{12^2} & \frac{24EI}{12^3} & -\frac{12EI}{12^2} \\ \frac{12EI}{12^2} & \frac{4EI}{12} & -\frac{12EI}{12^2} & \frac{8EI}{12} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{64.13}{EI} \end{bmatrix} = \begin{bmatrix} 32.34 \\ 75.38 \\ 21.66 \\ -11.25 \end{bmatrix} \begin{matrix} \leftarrow \text{shear-1} \\ \leftarrow \text{moment-1} \\ \leftarrow \text{shear-2} \\ \leftarrow \text{moment-2} \end{matrix}$$

THREE-SPAN BEAM EXAMPLE: Direct Stiffness Formulation

Shear and moments at member ends

For member-2:



$$\mathbf{F_e = F_{me} + K_e U_e}$$

$$\begin{bmatrix} F_{21} \\ F_{22} \\ F_{23} \\ F_{24} \end{bmatrix} = \begin{bmatrix} F_{m21} \\ F_{m22} \\ F_{m23} \\ F_{m24} \end{bmatrix} + \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}$$

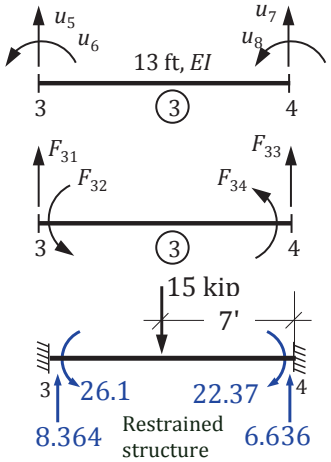


$$\begin{bmatrix} F_{21} \\ F_{22} \\ F_{23} \\ F_{24} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{12EI}{15^3} & \frac{6EI}{15^2} & -\frac{12EI}{15^3} & \frac{6EI}{15^2} \\ \frac{6EI}{15^2} & \frac{4EI}{15} & -\frac{6EI}{15^2} & \frac{2EI}{15} \\ -\frac{12EI}{15^3} & -\frac{6EI}{15^2} & \frac{12EI}{15^3} & -\frac{6EI}{15^2} \\ \frac{6EI}{15^2} & \frac{2EI}{15} & -\frac{6EI}{15^2} & \frac{4EI}{15} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{64.13}{EI} \\ 0 \\ -\frac{43.9}{EI} \end{bmatrix} = \begin{bmatrix} 0.54 \\ 11.25 \\ -0.54 \\ -3.16 \end{bmatrix} \begin{matrix} \leftarrow \text{shear-1} \\ \leftarrow \text{moment-1} \\ \leftarrow \text{shear-2} \\ \leftarrow \text{moment-2} \end{matrix}$$

THREE-SPAN BEAM EXAMPLE: Direct Stiffness Formulation

Shear and moments at member ends

For member-3:

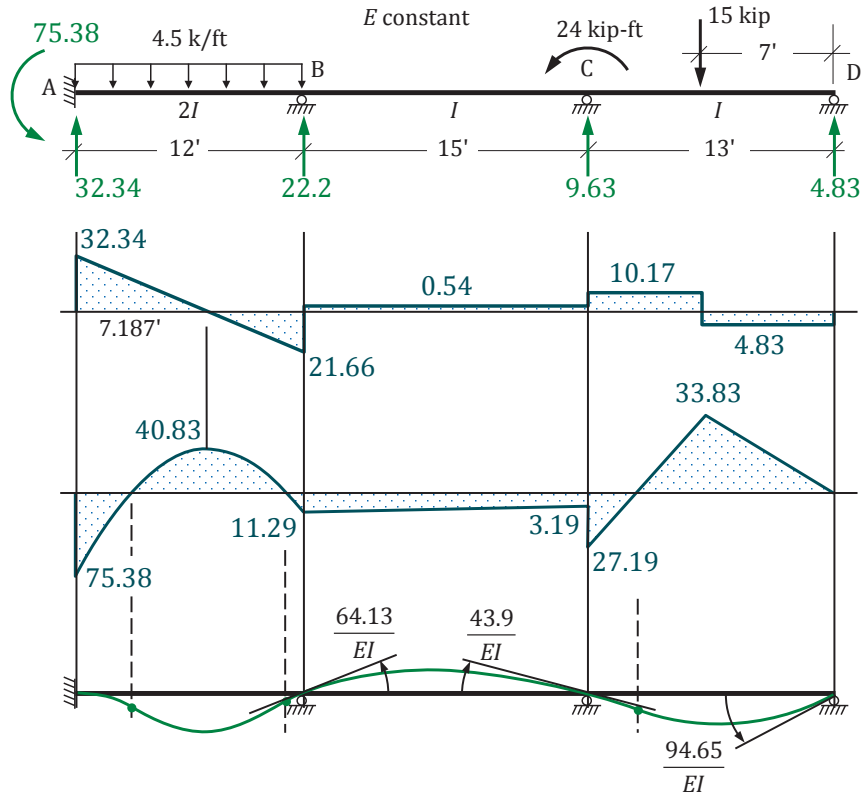


$$\mathbf{F_e = F_{me} + K_e U_e}$$

$$\begin{bmatrix} F_{31} \\ F_{32} \\ F_{33} \\ F_{34} \end{bmatrix} = \begin{bmatrix} F_{m31} \\ F_{m32} \\ F_{m33} \\ F_{m34} \end{bmatrix} + \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix}$$

$$\begin{bmatrix} F_{31} \\ F_{32} \\ F_{33} \\ F_{34} \end{bmatrix} = \begin{bmatrix} 8.364 \\ 26.1 \\ 6.636 \\ -22.37 \end{bmatrix} + \begin{bmatrix} \frac{12EI}{13^3} & \frac{6EI}{13^2} & -\frac{12EI}{13^3} & \frac{6EI}{13^2} \\ \frac{6EI}{13^2} & \frac{4EI}{13} & -\frac{6EI}{13^2} & \frac{2EI}{13} \\ -\frac{12EI}{13^3} & -\frac{6EI}{13^2} & \frac{12EI}{13^3} & -\frac{6EI}{13^2} \\ \frac{6EI}{13^2} & \frac{2EI}{13} & -\frac{6EI}{13^2} & \frac{4EI}{13} \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{43.9}{EI} \\ 0 \\ \frac{94.65}{EI} \end{bmatrix} = \begin{bmatrix} 10.17 \\ 27.15 \\ 4.834 \\ 0 \end{bmatrix} \begin{matrix} \leftarrow \text{shear-1} \\ \leftarrow \text{moment-1} \\ \leftarrow \text{shear-2} \\ \leftarrow \text{moment-2} \end{matrix}$$

THREE-SPAN BEAM EXAMPLE: Direct Stiffness Formulation
Shear force, bending moment diagrams and deflected shape.



$$\begin{bmatrix} u_4 \\ u_6 \\ u_8 \end{bmatrix} = \begin{bmatrix} 64.13 \\ -43.9 \\ 94.65 \end{bmatrix} \frac{1}{EI}$$

Direct Stiffness Formulation: Support settlement problems

Solving support settlement problems in direct stiffness approach does not require any additional theoretical treatment.

After partitioning the global superposition equation we get

$$\begin{bmatrix} \mathbf{P}_{j,u} \\ -- \\ \mathbf{P}_{j,r} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{m,u} \\ -- \\ \mathbf{P}_{m,r} \end{bmatrix} + \left[\begin{array}{c|c} \mathbf{K}_{uu} & \mathbf{K}_{ur} \\ \hline \mathbf{K}_{ru} & \mathbf{K}_{rr} \end{array}\right] \begin{bmatrix} \mathbf{u}_u \\ -- \\ \mathbf{u}_r \end{bmatrix}$$

We can write,

$$\mathbf{P}_{j,u} = \mathbf{P}_{m,u} + \mathbf{K}_{uu}\mathbf{u}_u + \mathbf{K}_{ur}\mathbf{u}_r \quad (1)$$

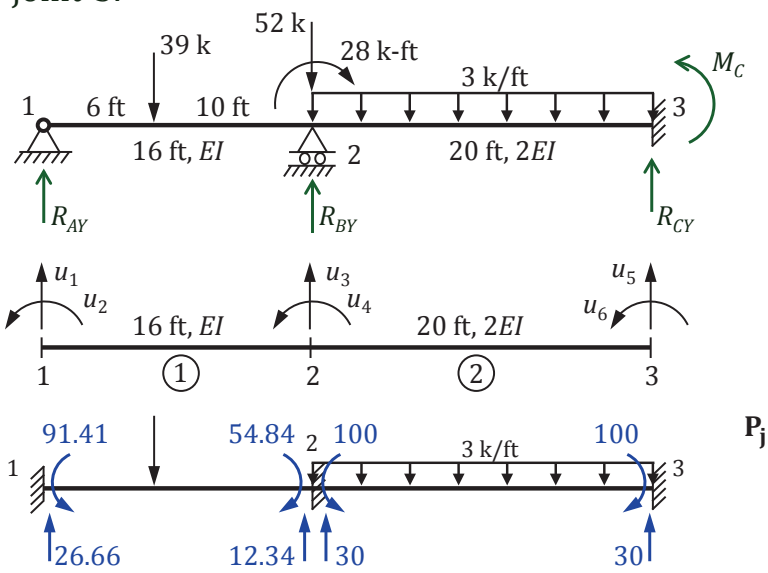
$$\mathbf{P}_{j,r} = \mathbf{P}_{m,r} + \mathbf{K}_{ru}\mathbf{u}_u + \mathbf{K}_{rr}\mathbf{u}_r \quad (2)$$

In the above equations, $\mathbf{u}_r$ represents the known degrees of freedom. In the previous examples, we had $\mathbf{u}_r = \mathbf{[0]}$, which means supports were unyielding (no displacements).

For support settlement problems, a known amount of support movement (displacement or rotation) are applied at joints. This means the corresponding elements of $\mathbf{u}_r$ are known and non-zero. Therefore we can solve eq.(1) and determine $\mathbf{u}_u$.

TWO-SPAN BEAM WITH SUPPORT SETTLEMENT: Direct Stiffness Formulation

We consider the two-span beam problem discussed earlier. In addition to loads, we apply 0.15ft upward movement at joint-2 and 0.04 radian anticlockwise rotation at joint-3.



$\Delta_2 = 0.15\text{ft} \uparrow$, $\theta_3 = 0.04$ rad $\circlearrowleft$

Assume EI = 4000 k-ft^2

$$\mathbf{P_j} = \begin{bmatrix} R_{AY} \\ 0 \\ R_{BY} - 52 \\ -28 \\ R_{CY} \\ M_C \end{bmatrix} \quad \mathbf{P_m} = \begin{bmatrix} 26.66 \\ 91.41 \\ 42.34 \\ 45.16 \\ 30 \\ -100 \end{bmatrix} \quad \mathbf{U} = \begin{bmatrix} 0 \\ u_2 \\ \boxed{0.15} \\ u_4 \\ 0 \\ \boxed{0.04} \end{bmatrix}$$

TWO-SPAN BEAM WITH SUPPORT SETTLEMENT: Direct Stiffness Formulation

$$\mathbf{P_j = P_m + KU}$$

After assembling the stiffness matrix.....

$$\begin{bmatrix} R_{AY} \\ 0 \\ R_{BY} - 52 \\ -28 \\ R_{CY} \\ M_C \end{bmatrix} = \begin{bmatrix} 26.66 \\ 91.41 \\ 42.34 \\ 45.16 \\ 30 \\ -100 \end{bmatrix} + EI \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \overbrace{\begin{array}{c} \begin{array}{cccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} & \color{red}{6} \end{array} \\ \begin{bmatrix} \frac{12}{16^3} & \frac{6}{16^2} & -\frac{12}{16^3} & \frac{6}{16^2} & 0 & 0 \\ \frac{6}{16^2} & \frac{4}{16} & -\frac{6}{16^2} & \frac{2}{16} & 0 & 0 \\ -\frac{12}{16^3} & -\frac{6}{16^2} & \frac{12}{16^3}+\frac{24}{20^3} & -\frac{6}{16^2}+\frac{12}{20^2} & -\frac{24}{20^3} & +\frac{12}{20^2} \\ \frac{6}{16^2} & \frac{2}{16} & -\frac{6}{16^2}+\frac{12}{20^2} & \frac{4}{16}+\frac{8}{20} & -\frac{12}{20^2} & \frac{4}{20} \\ 0 & 0 & -\frac{24}{20^3} & -\frac{12}{20^2} & \frac{24}{20^3} & -\frac{12}{20^2} \\ 0 & 0 & \frac{12}{20^2} & \frac{4}{20} & -\frac{12}{20^2} & \frac{8}{20} \end{bmatrix} \end{array}}^{\text{Global Stiffness Matrix, K}} \begin{bmatrix} 0 \\ u_2 \\ 0.15 \\ u_4 \\ 0 \\ 0.04 \end{bmatrix}$$

applied support settlement

applied support settlement

TWO-SPAN BEAM WITH SUPPORT SETTLEMENT: Direct Stiffness Formulation

After row and column interchange and partitioning...

$$\begin{matrix} \\ \text{Known nodal forces} \\ \\ \\ \text{Unknown nodal forces} \end{matrix} \begin{bmatrix} 0 \\ -28 \\ \hline R_{AY} \\ R_{BY}-52 \\ R_{CY} \\ M_C \end{bmatrix} = \begin{bmatrix} 91.41 \\ 45.16 \\ \hline 26.66 \\ 42.34 \\ 30 \\ -100 \end{bmatrix} + \begin{array}{c} \begin{matrix} 2 & 4 & 1 & 3 & 5 & 6 \end{matrix} \\ \left[\begin{array}{cc|cccc} \frac{4}{16} & \frac{2}{16} & \frac{6}{16^2} & -\frac{6}{16^2} & 0 & 0 \\ \frac{2}{16} & \frac{4}{16}+\frac{8}{20} & \frac{6}{16^2} & -\frac{6}{16^2}+\frac{12}{20^2} & -\frac{12}{20^2} & \frac{4}{20} \\ \hline \frac{6}{16^2} & \frac{6}{16^2} & \frac{12}{16^3} & -\frac{12}{16^3} & 0 & 0 \\ -\frac{6}{16^2} & -\frac{6}{16^2}+\frac{12}{20^2} & \frac{12}{16^3} & \frac{12}{16^3}+\frac{24}{20^3} & -\frac{24}{20^3} & +\frac{12}{20^2} \\ 0 & -\frac{12}{20^2} & 0 & -\frac{24}{20^3} & \frac{24}{20^3} & -\frac{12}{20^2} \\ 0 & \frac{4}{20} & 0 & \frac{12}{20^2} & -\frac{12}{20^2} & \frac{8}{20} \end{array}\right] \end{array} EI \begin{bmatrix} u_2 \\ u_4 \\ \hline 0 \\ 0.15 \\ 0 \\ 0.04 \end{bmatrix} \begin{matrix} 2 \\ 4 \\ \\ 1 \\ 3 \\ 5 \\ 6 \end{matrix} \begin{matrix} \\ \text{Unknown nodal d.o.f} \\ \\ \\ \text{known nodal d.o.f} \end{matrix}$$

We can write,

$$\mathbf{P_{j,u} = P_{m,u} + K_{uu}u_u + K_{ur}u_r} \quad (1)$$

$$\mathbf{P_{j,r} = P_{m,r} + K_{ru}u_u + K_{rr}u_r} \quad (2)$$

TWO-SPAN BEAM WITH SUPPORT SETTLEMENT: Direct Stiffness Formulation

$\mathbf{P_{j,u} = P_{m,u} + K_{uu}u_u + K_{ur}u_r}$ (1) Assume EI = 4000 k-ft^2

$$\underset{\mathbf{P_{j,u}}}{\begin{bmatrix} 0 \\ -28 \end{bmatrix}} = \underset{\mathbf{P_{m,u}}}{\begin{bmatrix} 91.41 \\ 45.16 \end{bmatrix}} + \underbrace{\begin{bmatrix} \frac{4}{16} & \frac{2}{16} \\ \frac{2}{16} & \frac{13}{20} \end{bmatrix}}_{\mathbf{K_{uu}}} EI \underbrace{\begin{bmatrix} u_2 \\ u_4 \end{bmatrix}}_{\mathbf{u_u}} + \underbrace{\begin{bmatrix} \frac{6}{16^2} & -\frac{6}{16^2} & 0 & 0 \\ \frac{6}{16^2} & -\frac{6}{16^2}+\frac{12}{20^2} & -\frac{12}{20^2} & \frac{4}{20} \end{bmatrix}}_{\mathbf{K_{ur}}} EI \underbrace{\begin{bmatrix} 0 \\ 0.15 \\ 0 \\ 0.04 \end{bmatrix}}_{\mathbf{u_r}}$$

$$\Rightarrow \begin{bmatrix} -77.35 \\ -109.1 \end{bmatrix} = \begin{bmatrix} \frac{4}{16} & \frac{2}{16} \\ \frac{2}{16} & \frac{13}{20} \end{bmatrix} 4000 \begin{bmatrix} u_2 \\ u_4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_2 \\ u_4 \end{bmatrix} = \begin{bmatrix} -0.0624 \\ -0.03 \end{bmatrix}$$

Therefore, the displacement vector can be written as

$$\mathbf{U} = \begin{bmatrix} 0 \\ -0.0624 \\ \boxed{0.15} \\ -0.03 \\ 0 \\ \boxed{0.04} \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

TWO-SPAN BEAM WITH SUPPORT SETTLEMENT: Direct Stiffness Formulation

Support reactions:

$$\mathbf{P_{j,r} = P_{m,r} + K_{ru}u_u + K_{rr}u_r} \qquad (2)$$

Putting values of the matrices and setting $EI = 4000$,

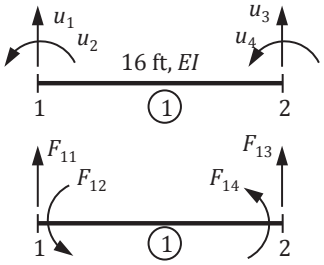
$$\begin{bmatrix} R_{AY} \\ R_{BY}-52 \\ R_{CY} \\ M_C \end{bmatrix} = \begin{bmatrix} 26.66 \\ 42.34 \\ 30 \\ -100 \end{bmatrix} + \begin{bmatrix} \frac{6}{16^2} & \frac{6}{16^2} \\ -\frac{6}{16^2} & -\frac{6}{16^2}+\frac{12}{20^2} \\ 0 & -\frac{12}{20^2} \\ 0 & \frac{4}{20} \end{bmatrix} EI \begin{bmatrix} -0.0624 \\ -0.03 \end{bmatrix} + \begin{bmatrix} \frac{12}{16^3} & -\frac{12}{16^3} & 0 & 0 \\ -\frac{12}{16^3} & \frac{12}{16^3}+\frac{24}{20^3} & -\frac{24}{20^3} & +\frac{12}{20^2} \\ 0 & -\frac{24}{20^3} & \frac{24}{20^3} & -\frac{12}{20^2} \\ 0 & \frac{12}{20^2} & -\frac{12}{20^2} & \frac{8}{20} \end{bmatrix} EI \begin{bmatrix} 0 \\ 0.15 \\ 0 \\ 0.04 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} R_{AY} \\ R_{BY}-52 \\ R_{CY} \\ M_C \end{bmatrix} = \begin{bmatrix} 26.66 \\ 42.34 \\ 30 \\ -100 \end{bmatrix} + \begin{bmatrix} -8.663 \\ 5.06 \\ 3.6 \\ -24 \end{bmatrix} + \begin{bmatrix} -1.76 \\ 8.36 \\ -6.6 \\ 82 \end{bmatrix} \Rightarrow \begin{bmatrix} R_{AY} \\ R_{BY} \\ R_{CY} \\ M_C \end{bmatrix} = \begin{bmatrix} 16.24 \\ 107.76 \\ 27 \\ -42 \end{bmatrix}$$

TWO-SPAN BEAM WITH SUPPORT SETTLEMENT: Direct Stiffness Formulation

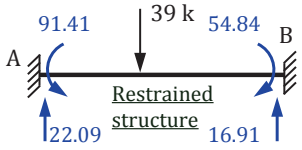
Shear and moments at member ends

For member-1: $\mathbf{F_e = F_{me} + K_eU_e}$



$$\begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{14} \end{bmatrix} = \begin{bmatrix} F_{m11} \\ F_{m12} \\ F_{m13} \\ F_{m14} \end{bmatrix} + \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} 0 \\ -0.0624 \\ 0.15 \\ -0.03 \\ 0 \\ 0.04 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$



Putting values of the matrices and setting $EI = 4000$,

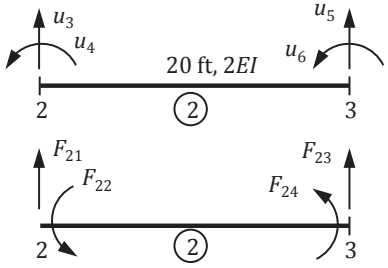
$$\begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{14} \end{bmatrix} = \begin{bmatrix} 26.66 \\ 91.41 \\ 12.34 \\ -54.84 \end{bmatrix} + \begin{bmatrix} \frac{12EI}{16^3} & \frac{6EI}{16^2} & -\frac{12EI}{16^3} & \frac{6EI}{16^2} \\ \frac{6EI}{16^2} & \frac{4EI}{16} & -\frac{6EI}{16^2} & \frac{2EI}{16} \\ -\frac{12EI}{16^3} & -\frac{6EI}{16^2} & \frac{12EI}{16^3} & -\frac{6EI}{16^2} \\ \frac{6EI}{16^2} & \frac{2EI}{16} & -\frac{6EI}{16^2} & \frac{4EI}{16} \end{bmatrix} \begin{bmatrix} 0 \\ -0.0624 \\ 0.15 \\ -0.03 \end{bmatrix} = \begin{bmatrix} 16.24 \\ 0 \\ 22.76 \\ -130.1 \end{bmatrix} \begin{matrix} \leftarrow \text{shear-1} \\ \leftarrow \text{moment-1} \\ \leftarrow \text{shear-2} \\ \leftarrow \text{moment-2} \end{matrix}$$

TWO-SPAN BEAM WITH SUPPORT SETTLEMENT: Direct Stiffness Formulation

Shear and moments at member ends

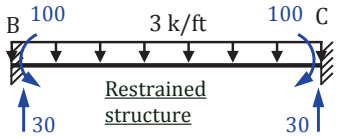
$$\mathbf{F_e = F_{me} + K_e U_e}$$

For member-2:



$$\begin{bmatrix} F_{21} \\ F_{22} \\ F_{23} \\ F_{24} \end{bmatrix} = \begin{bmatrix} F_{m21} \\ F_{m22} \\ F_{m23} \\ F_{m24} \end{bmatrix} + \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} 0 \\ -0.0624 \\ 0.15 \\ -0.03 \\ 0 \\ 0.04 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$



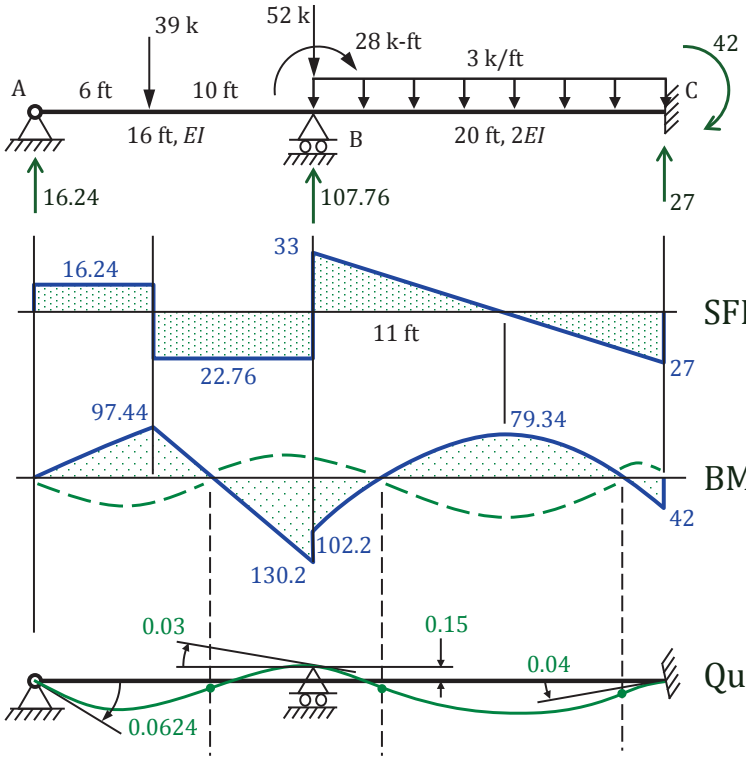
Putting values of the matrices and setting EI = 4000,

$$\begin{bmatrix} F_{21} \\ F_{22} \\ F_{23} \\ F_{24} \end{bmatrix} = \begin{bmatrix} 30 \\ 100 \\ 30 \\ -100 \end{bmatrix} + \begin{bmatrix} \frac{24EI}{20^3} & \frac{12EI}{20^2} & -\frac{24EI}{20^3} & \frac{12EI}{20^2} \\ \frac{12EI}{20^2} & \frac{8EI}{20} & -\frac{12EI}{20^2} & \frac{4EI}{20} \\ -\frac{24EI}{20^3} & -\frac{12EI}{20^2} & \frac{24EI}{20^3} & -\frac{12EI}{20^2} \\ \frac{12EI}{20^2} & \frac{4EI}{20} & -\frac{12EI}{20^2} & \frac{8EI}{20} \end{bmatrix} \begin{bmatrix} 0.15 \\ -0.03 \\ 0 \\ 0.04 \end{bmatrix} = \begin{bmatrix} 33 \\ 102 \\ 27 \\ -42 \end{bmatrix}$$

← shear-1
← moment-1
← shear-2
← moment-2

TWO-SPAN BEAM WITH SUPPORT SETTLEMENT

Shear force and bending moment diagrams



$$\Rightarrow \begin{bmatrix} R_{AY} \\ R_{BY} \\ R_{CY} \\ M_C \end{bmatrix} = \begin{bmatrix} 16.24 \\ 107.76 \\ 27 \\ -42 \end{bmatrix} \qquad \mathbf{U} = \begin{bmatrix} 0 \\ -0.0624 \\ 0.15 \\ -0.03 \\ 0 \\ 0.04 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$