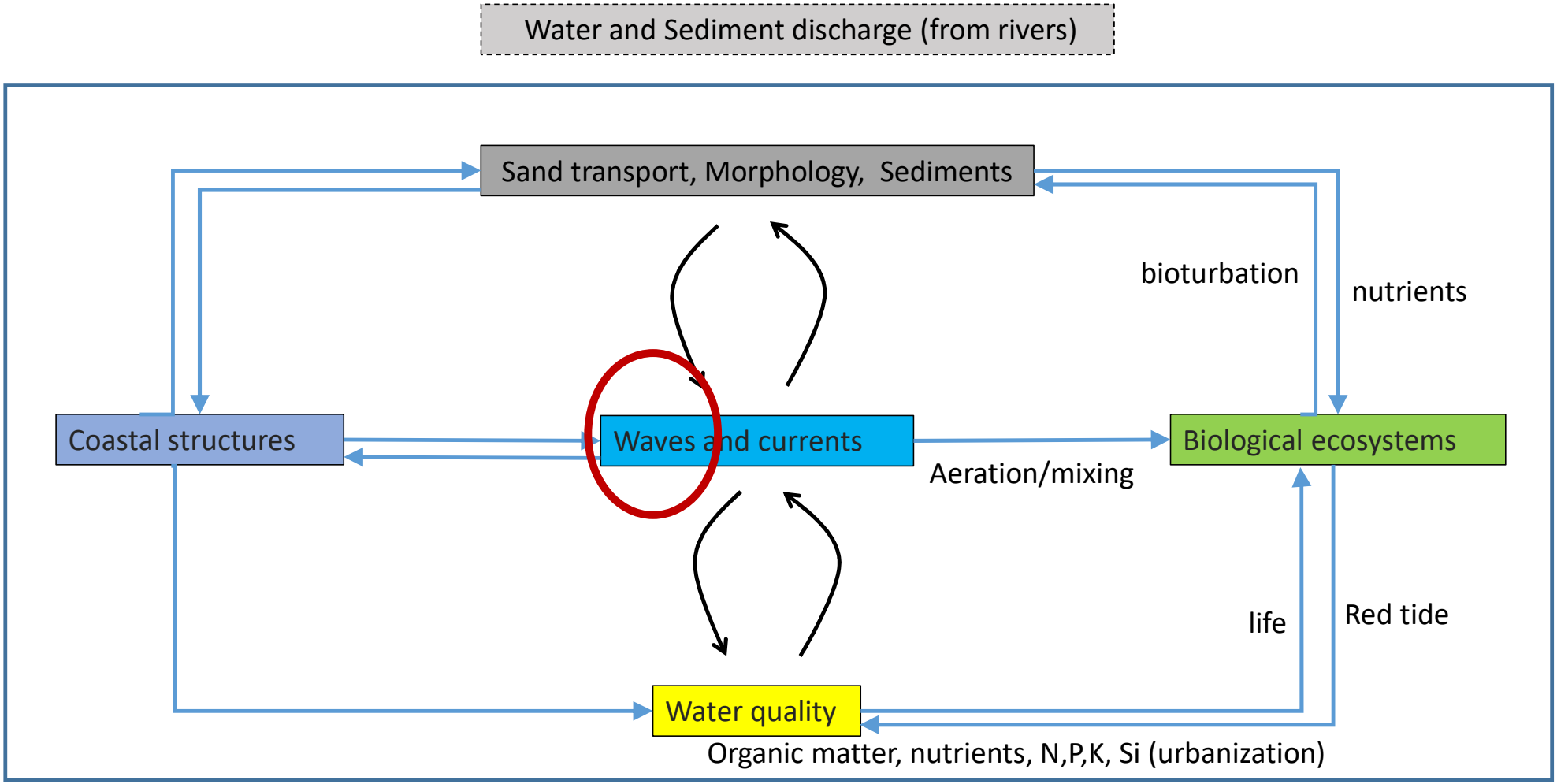
EWC 471: Basic Coastal Engineering

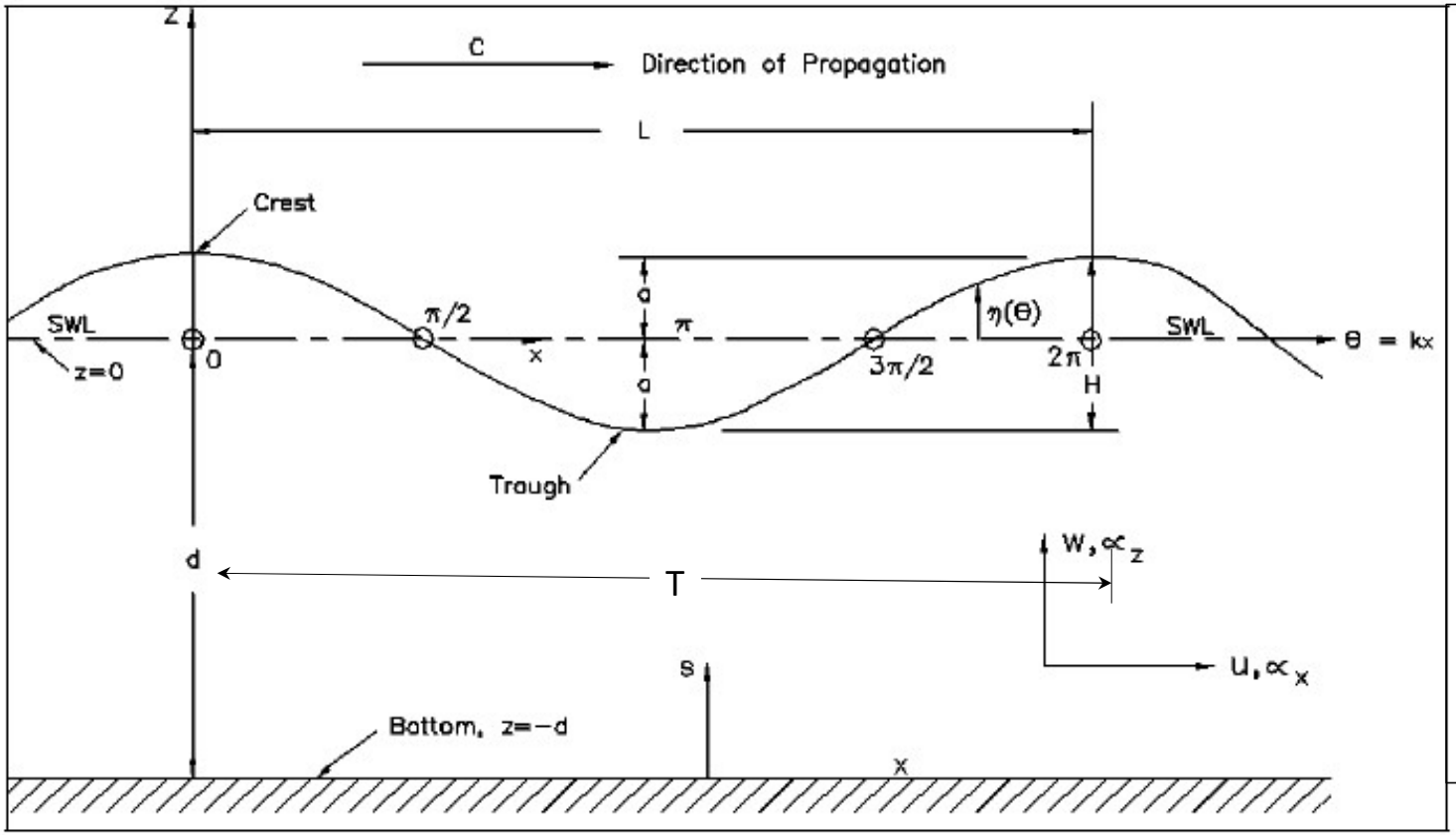
Lecture 5 and 6

Topic: Small Amplitude Wave theory and Application Examples

From previous lecture..



Basic terminology/ Definition of wave parameters



L= wave length
H= wave height
T= wave period
d=water depth
η=surface elevation

Crest = highest point
Trough = lowest point

SWL= still water level
For small amplitude waves
Wave amplitude a=H/2

Angular or radian frequency ω (or σ)=2π/T
Wave number k = 2π/L
Phase velocity or wave celerity C= L/T= ω /k

Wave steepness ε = H/L
Relative depth = d/L
Relative wave height = H/d

Small amplitude wave theory: wave classification

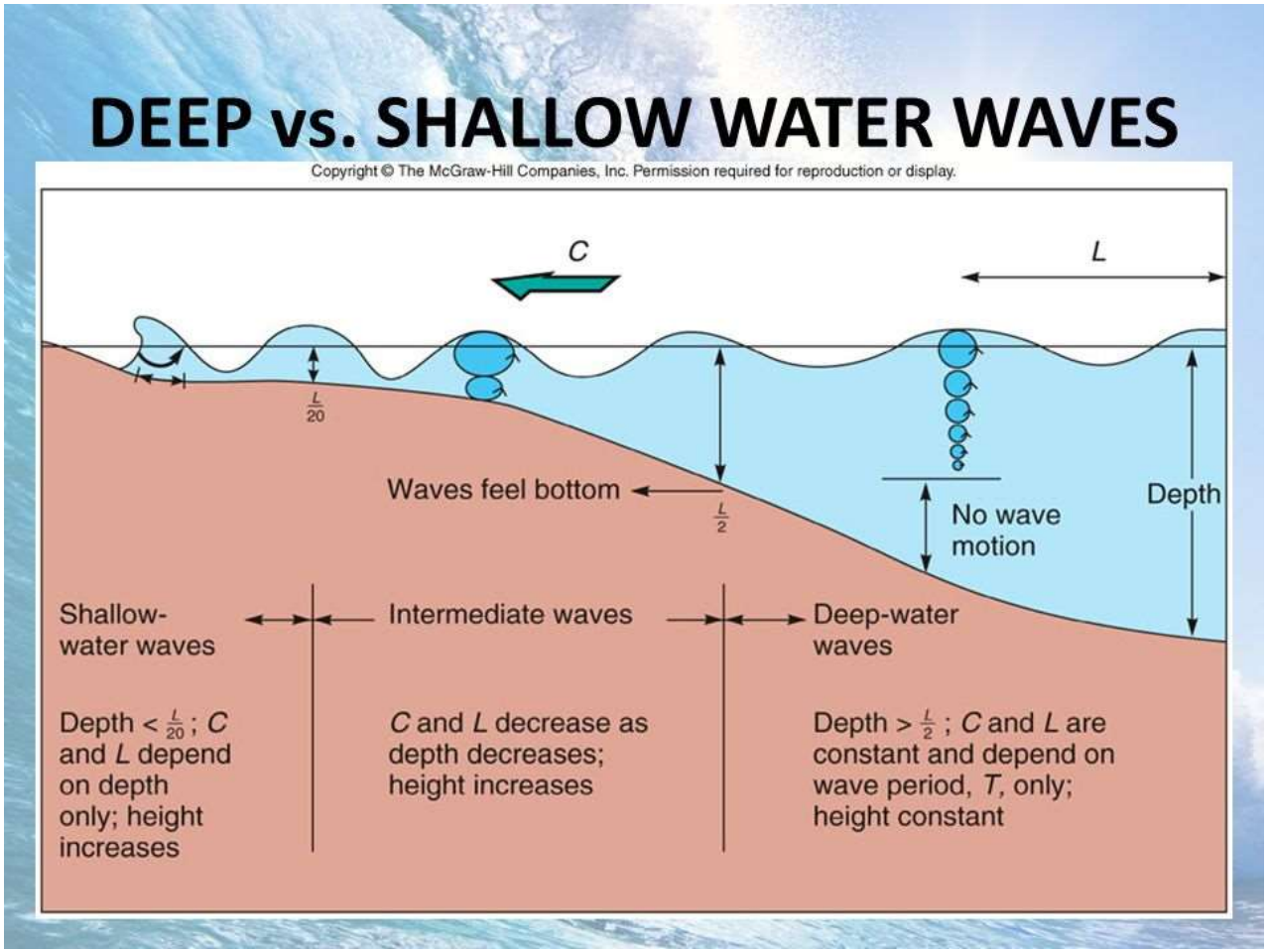
Shallow water waves:

$$C = \sqrt{gd}$$

$$L = \sqrt{gd}T$$

Intermediate waves:

$$\frac{C}{C_o} = \frac{L}{L_o} = \tanh\frac{2\pi d}{L}$$



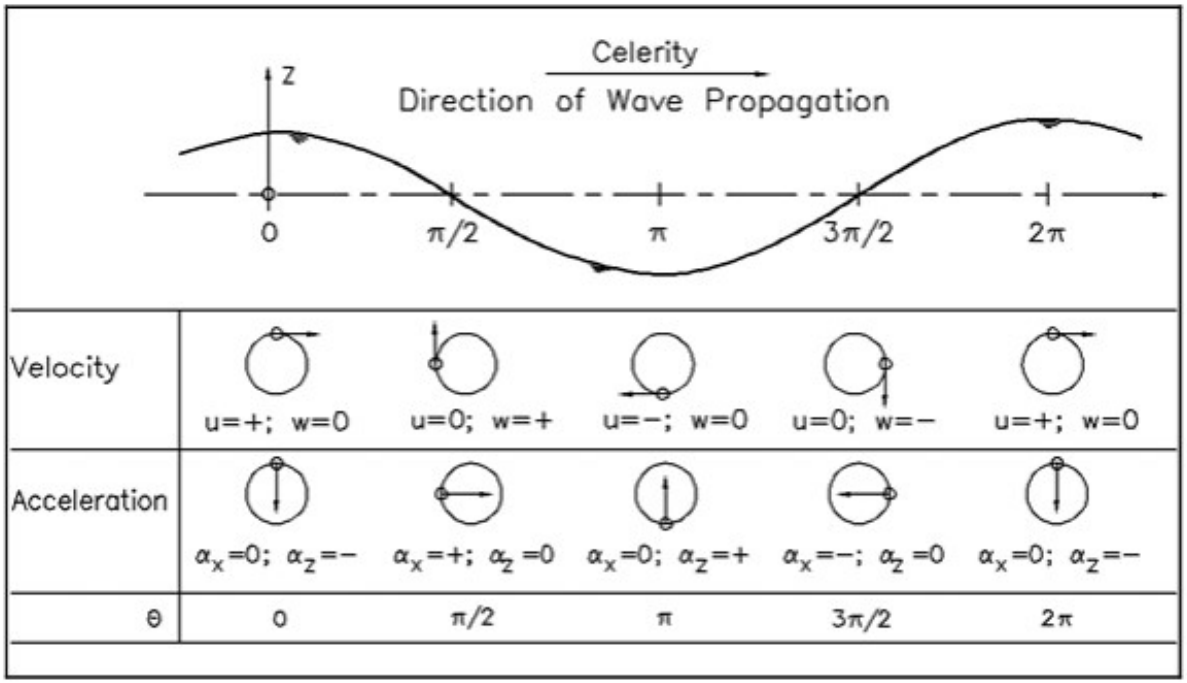
Deep water waves:

$$C_o = \sqrt{\frac{gL_o}{2\pi}}$$

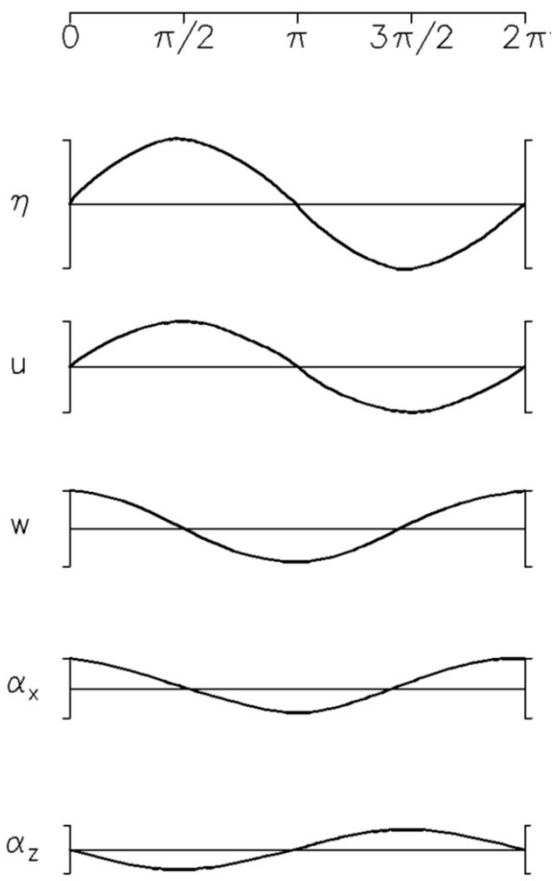
$$C_o = \frac{gT}{2\pi}$$

$$L_o = \frac{gT^2}{2\pi}$$

Linear wave theory/ small amplitude wave theory



Velocity	u=+; w=0	u=0; w=+	u=−; w=0	u=0; w=−	u=+; w=0
Acceleration	α_x=0; α_z=−	α_x=+; α_z=0	α_x=0; α_z=+	α_x=−; α_z=0	α_x=0; α_z=−
θ	0	π/2	π	3π/2	2π



Profiles of particle velocity and acceleration by Linear wave theory in relation to the surface elevation

Linear wave theory/ small amplitude wave theory

- In wave force studies, the local fluid velocities and accelerations for various values of *z* and *t* during the passage of a wave must often be found. The horizontal component *u* and the vertical component *w* of the local fluid velocity are given by the following equations

$$u = \frac{H}{2}\frac{gT}{L}\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\cos\theta \qquad w = \frac{H}{2}\frac{gT}{L}\frac{\sinh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\sin\theta$$

- The local fluid particle accelerations are:

$$\alpha_x = \frac{g\pi H}{L}\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\sin\theta = \frac{\partial u}{\partial t}$$

$$\alpha_z = -\frac{g\pi H}{L}\frac{\sinh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\cos\theta = \frac{\partial w}{\partial t}$$

Linear wave theory/ small amplitude wave theory

- Water particle displacements: Another important aspect of linear wave theory deals with the displacement of individual water particles within the wave. Water particles generally move in elliptical paths in shallow or transitional depth water and in circular paths in deep water. If the mean particle position is considered to be at the center of the ellipse or circle, then vertical particle displacement with respect to the mean position cannot exceed one-half the wave height. Thus, since the wave height is assumed to be small, the displacement of any fluid particle from its mean position must be small.

$$\xi = -\frac{HgT^2}{4\pi L}\,\frac{\cosh\left(\frac{2\pi(z+d)}{L}\right)}{\cosh\left(\frac{2\pi d}{L}\right)}\,\sin\,\theta$$

$$\zeta = +\frac{HgT^2}{4\pi L}\,\frac{\sinh\left(\frac{2\pi(z\,+\,d)}{L}\right)}{\cosh\left(\frac{2\pi d}{L}\right)}\,\cos\,\theta$$

Linear wave theory/ small amplitude wave theory

- Simplified equations:

$$\xi = -\frac{H}{2}\frac{\cosh\left(\frac{2\pi(z+d)}{L}\right)}{\sinh\left(\frac{2\pi d}{L}\right)}\sin\theta$$

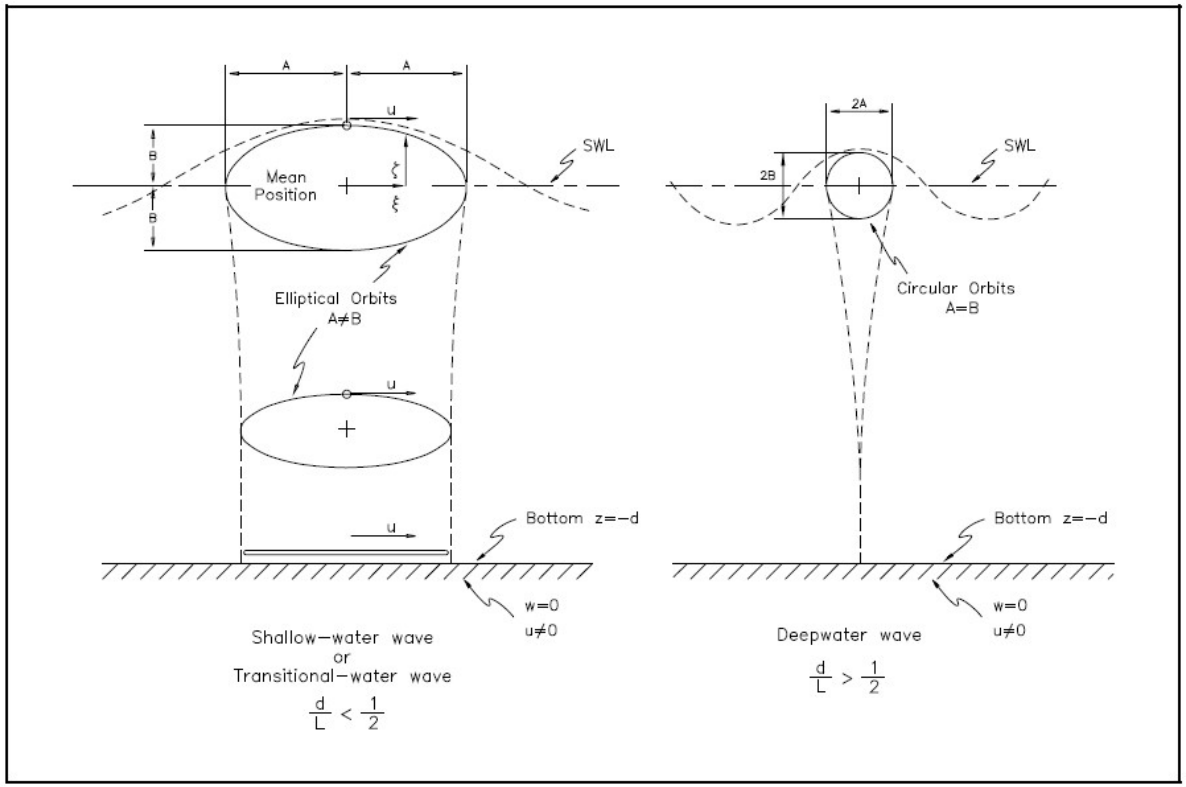
$$\zeta = +\frac{H}{2}\frac{\sinh\left(\frac{2\pi(z+d)}{L}\right)}{\sinh\left(\frac{2\pi d}{L}\right)}\cos\theta$$

$$\sin^2\theta = \left[\frac{\xi}{a}\frac{\sinh\left(\frac{2\pi d}{L}\right)}{\cosh\left(\frac{2\pi(z+d)}{L}\right)}\right]^2$$

$$\cos^2\theta = \left[\frac{\zeta}{a}\frac{\sinh\left(\frac{2\pi d}{L}\right)}{\sinh\left(\frac{2\pi(z+d)}{L}\right)}\right]^2$$

and adding gives

$$\frac{\xi^2}{A^2} + \frac{\zeta^2}{B^2} = 1 \quad \text{.....(A)}$$



Water particle displacements from mean position for shallow-water and deep water waves

Linear wave theory/ small amplitude wave theory

- in which *A* and *B* are

$$A = \frac{H}{2} \frac{\cosh\left(\frac{2\pi(z + d)}{L}\right)}{\sinh\left(\frac{2\pi d}{L}\right)} \qquad B = \frac{H}{2} \frac{\sinh\left(\frac{2\pi(z + d)}{L}\right)}{\sinh\left(\frac{2\pi d}{L}\right)}$$

- Equation (A) is the equation of an ellipse with a major- (horizontal) semi-axis equal to *A* and a minor (vertical) semi-axis equal to *B*. The lengths of *A* and *B* are measures of the horizontal and vertical displacements of the water particles (Figure in previous slide). Thus, the water particles are predicted to move in closed orbits by linear wave theory; i.e., a fluid particle returns to its initial position after each wave cycle. Comparing laboratory measurements of particle orbits with this theory shows that particle orbits are not completely closed. This difference between linear theory and observations is due to the mass transport phenomenon. It shows that linear theory is inadequate to explain wave motion completely.

Linear wave theory/ small amplitude wave theory

- for deep water conditions, *A* and *B* are equal and particle paths are circular (see Figure). These equations become

$$A = B = \frac{H}{2} e^{\left(\frac{2\pi z}{L}\right)} \quad \text{for } \frac{\mathrm{d}}{\mathrm{L}} > \frac{1}{2} \text{ (i.e., deepwater limit)}$$

- For shallow-water conditions (d/L < 1/20), the equations become

$$A = \frac{H}{2}\frac{L}{2\pi d} \quad \text{and} \quad B = \frac{H}{2}\left(1 + \frac{z}{d}\right)$$

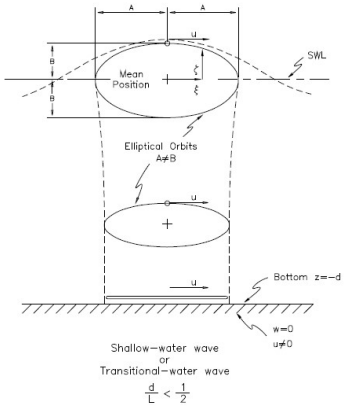
Linear wave theory/ small amplitude wave theory

- Thus, in deep water, the water particle orbits are circular, in transitional and shallow water, the orbits are elliptical.
- The more shallow the water, the flatter the ellipse. The amplitude of the water particle displacement decreases exponentially with depth and in deep water regions becomes small relative to the wave height at a depth equal to one-half the wavelength below the free surface; i.e., when $z = L_0/2$.
- For shallow regions, horizontal particle displacement near the bottom can be large. In fact, this is apparent in offshore regions seaward of the breaker zone where wave action and turbulence lift bottom sediments into suspension. The vertical displacement of water particles varies from a minimum of zero at the bottom to a maximum equal to one-half the wave height at the surface.

Linear wave theory: Application example

GIVEN: A wave with a period T = 8 sec, in a water depth d = 15 m , and a wave height H = 5.5 m.

FIND: The local horizontal and vertical velocities u and w, and accelerations α_x and α_z , and displacements at an elevation z = -5 m below the SWL when $\theta = 2\pi x/L - 2\pi t/T = \pi/3$ (or 60^0).



Solution:

Deep water wavelength, $L_0 = 1.56\ T^2 = 1.56*(8)^2 = 99.84$ m

When d = 15m, $d/L_0 = 15/99.84 = 0.15$ i.e. $1/20 < d/L_0 < ½$ i.e. transitional water condition.

Through trial and error, L = 81.7 m

$$u = \frac{H}{2}\frac{gT}{L}\frac{\cosh\left[\frac{2\pi(z+d)}{L}\right]}{\cosh\frac{2\pi d}{L}}\cos\theta$$

$$w = \frac{H}{2}\frac{gT}{L}\frac{\sinh\left[\frac{2\pi(z+d)}{L}\right]}{\cosh\frac{2\pi d}{L}}\sin\theta$$

$$a_x = \frac{g\pi H}{L}\frac{\cosh\left[\frac{2\pi(z+d)}{L}\right]}{\cosh\frac{2\pi d}{L}}\sin\theta$$

$$a_z = -\frac{g\pi H}{L}\frac{\sinh\left[\frac{2\pi(z+d)}{L}\right]}{\cosh\frac{2\pi d}{L}}\cos\theta$$

Linear wave theory: Application example

Solution (contd.):

$$u = \frac{H}{2}\frac{gT}{L}\frac{\cosh\left[\frac{2\pi(z+d)}{L}\right]}{\cosh\frac{2\pi d}{L}}cos\theta$$

$$u = \frac{5.5}{2}\frac{9.81*8}{81.7}\frac{\cosh\left[\frac{2*3.14(-5+15)}{81.7}\right]}{1.742}\cos(60)$$

u=0.9935 m/s (Ans.)

$$w = \frac{H}{2}\frac{gT}{L}\frac{\sinh\left[\frac{2\pi(z+d)}{L}\right]}{\cosh\frac{2\pi d}{L}}sin\theta$$

$$w = \frac{5.5}{2}\frac{9.81*8}{81.7}\frac{\sinh\left[\frac{2*3.14(-5+15)}{81.7}\right]}{1.742}\sin(60)$$

w=1.11 m/s (Ans.)

$$a_x = \frac{g\pi H}{L}\frac{\cosh\left[\frac{2\pi(z+d)}{L}\right]}{\cosh\frac{2\pi d}{L}}sin\theta$$

$$a_x = \frac{9.81*3.14*5.5}{81.7}\frac{\cosh\left[\frac{2*3.14(-5+15)}{81.7}\right]}{1.742}\sin(60)$$

a_x=1.35 m/s^2 (Ans.)

$$a_z = -\frac{g\pi H}{L}\frac{\sinh\left[\frac{2\pi(z+d)}{L}\right]}{\cosh\frac{2\pi d}{L}}cos\theta$$

$$a_z = -\frac{9.81*3.14*5.5}{81.7}\frac{\sinh\left[\frac{2*3.14(-5+15)}{81.7}\right]}{1.742}cos(60)$$

a_z= - 0.5 m/s^2 (Ans.)

Linear wave theory: Application example

Solution (contd.):

$$\xi = \frac{H}{2}\frac{\cosh\left[\frac{2\pi(z+d)}{L}\right]}{\sinh\frac{2\pi d}{L}}sin\theta$$

$$\xi = \frac{5.5}{2}\frac{\cosh\left[\frac{2*3.14(-5+15)}{81.7}\right]}{1.43}sin(60)$$

ξ = 2.19 m (Ans.)

$$\zeta = \frac{H}{2}\frac{\sinh\left[\frac{2\pi(z+d)}{L}\right]}{\sinh\frac{2\pi d}{L}}cos\theta$$

$$\zeta = \frac{5.5}{2}\frac{\sinh\left[\frac{2*3.14(-5+15)}{81.7}\right]}{1.43}cos(60)$$

ζ = 0.82 m (Ans.)

Calculate the same six parameters for z=-10m and z=-15m;
Check with excel file

$$\xi = -\frac{H}{2}\frac{\cosh\left(\frac{2\pi(z+d)}{L}\right)}{\sinh\left(\frac{2\pi d}{L}\right)}\sin\theta$$

$$\zeta = +\frac{H}{2}\frac{\sinh\left(\frac{2\pi(z+d)}{L}\right)}{\sinh\left(\frac{2\pi d}{L}\right)}\cos\theta$$

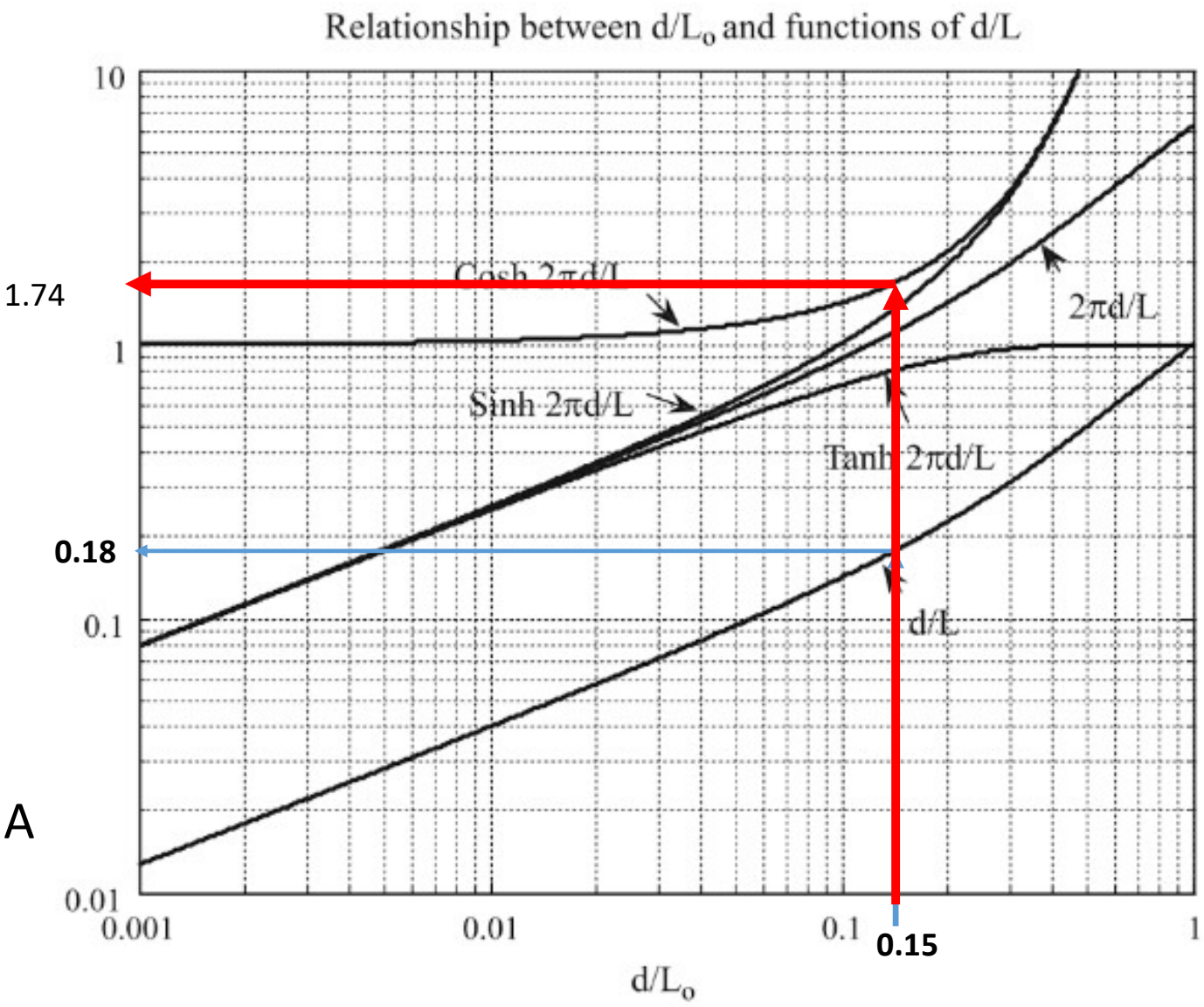


Figure A

You may also use
Available charts

Linear wave theory: wave pressure

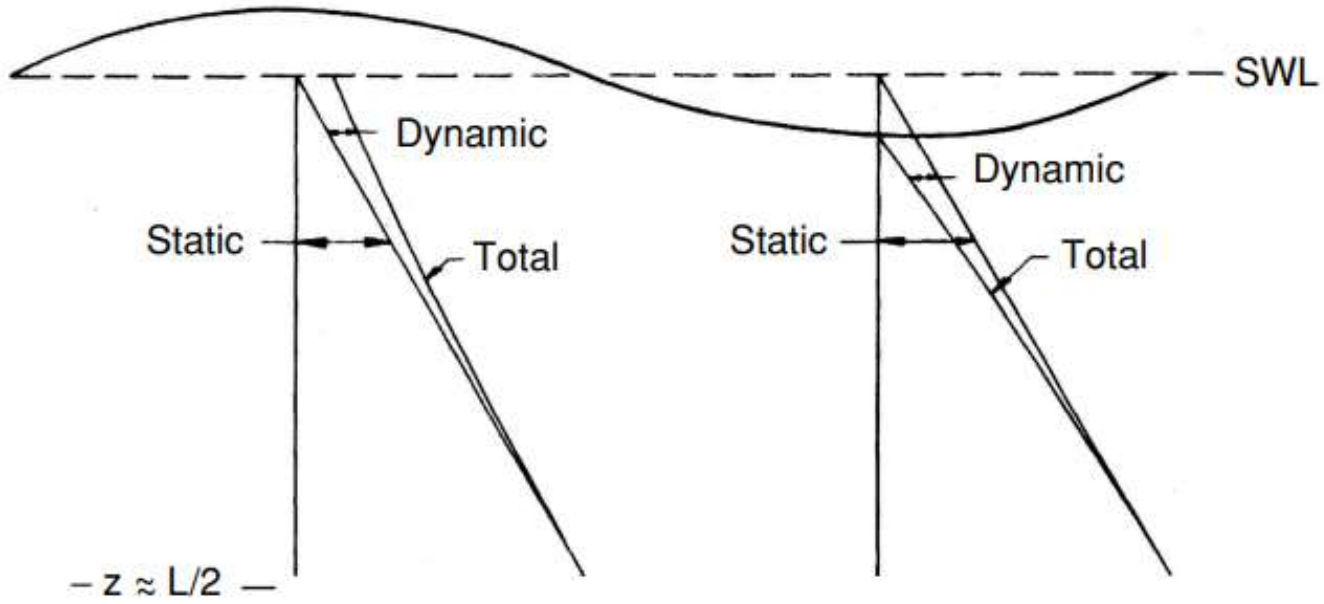
- Subsurface pressure: (a) Subsurface pressure under a wave is the sum of two contributing components, dynamic and static pressures, and is given by:

$$p' = \frac{\rho g H \cosh\left[\frac{2\pi(z+d)}{L}\right]}{2\cosh\left(\frac{2\pi d}{L}\right)} \cos\theta - \rho g z + p_a$$

- where p' is the total or absolute pressure, p_a is the atmospheric pressure, and ρ is the mass density of water (for salt water, ρ = 1,025 kg/m^3, for fresh water, ρ = 1,000 kg/m^3). The first term of Equation represents a dynamic component due to acceleration, while the second term is the static component of pressure. For convenience, the pressure is usually taken as the gauge pressure defined as:

$$p = p' - p_a = \frac{\rho g H \cosh\left[\frac{2\pi(z+d)}{L}\right]}{2\cosh\left(\frac{2\pi d}{L}\right)} \cos\theta - \rho g z$$

Linear wave theory: wave pressure



Deep water wave vertical pressure distributions

Linear wave theory: wave pressure

- The above equation can be written as:

$$p = \rho g \eta \frac{\cosh\left[\frac{2\pi(z + d)}{L}\right]}{\cosh\left(\frac{2\pi d}{L}\right)} - \rho g z$$

Since

$$\eta = \frac{H}{2}\cos\left(\frac{2\pi x}{L} - \frac{2\pi t}{T}\right) = \frac{H}{2}\cos\theta$$

- Now the ratio $K_z = \frac{\cosh\left[\frac{2\pi(z + d)}{L}\right]}{\cosh\left(\frac{2\pi d}{L}\right)}$ is termed as the *pressure response factor*.

Linear wave theory: wave pressure

- Hence, the previous equation can be written as

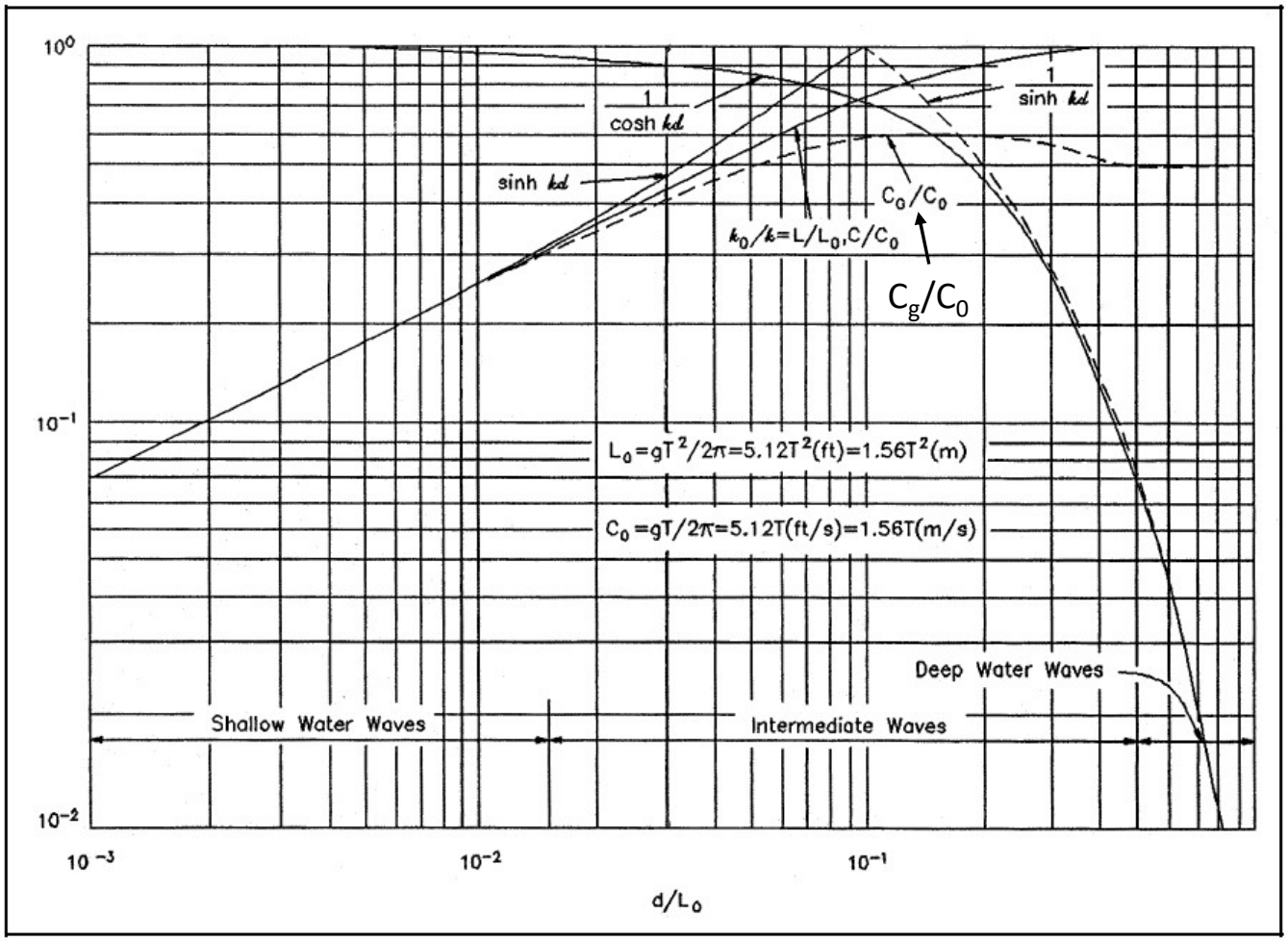
$$p = \rho g(\eta K_z - z)$$

- The pressure response factor *K* for the pressure at the bottom when *z* = -d,

$$K_z = K = \frac{1}{\cosh\left(\frac{2\pi d}{L}\right)}$$

- It is presented as function of d/L_0 *in* the following Figure. This figure is a convenient graphic means to determine intermediate and shallow-water values of the bottom pressure response factor *K*, the ratio C/C_0 ($=L/L_0 = k_0/k$), and a number of other variables commonly occurring in water wave calculations.

Variation of wave parameters with d/L_0 (Dean and Dalrymple 1991)



Linear wave theory: wave pressure

- It is often necessary to determine the height of surface waves based on subsurface measurements of pressure. For this purpose, it is convenient to rewrite Equation

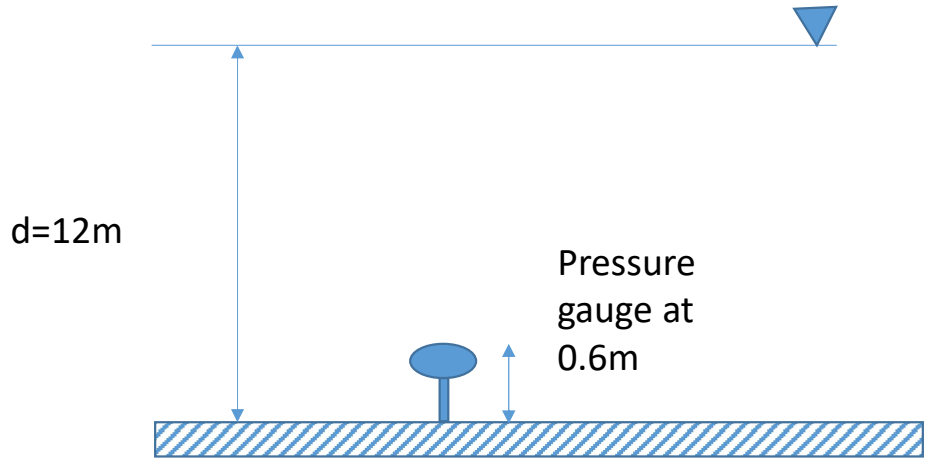
$$\eta = \frac{N(p + \rho g z)}{\rho g K_z}$$

where z is the depth below the SWL of the pressure gauge, and N a correction factor equal to unity if the linear theory applies

Linear wave theory: Application example

GIVEN: An average maximum pressure p = 124 kilo newtons per square meter is measured by a subsurface pressure gauge located in salt water 0.6 meter above the bed in depth d = 12 m. The average frequency f = 0.06666 cycles per second (Hertz).

FIND: The height of the wave H assuming that linear theory applies, and the average frequency corresponds to the average wave amplitude.



Linear wave theory: Application example 5

Solution:

Time period, T=1/f=1/0.06666=15 sec

$L_0 = 1.56T^2 = 1.56\ (15)^2 = 351$ m

$d/L_0 = 12/351 = 0.0342 < 1/20$, i.e. shallow water condition.

From equ $L = \sqrt{gd}T$

From L = 162.75 m

Also $\cosh \frac{2\pi d}{L} = 1.11$

Linear wave theory: Application example 5

Solution (contd.): z=-(12-0.6) = -11.4m

Pressure response factor, $k_z = \frac{cosh\left(\frac{2\pi(z+d)}{L}\right)}{cosh\left(\frac{2\pi d}{L}\right)} = \frac{cosh\left(\frac{2*3.14*(-11.4+12)}{162.75}\right)}{1.11} = 0.8948$

Height of water surface, $\eta = \frac{N(P+\rho g z)}{\rho g k_z} = \frac{1*\left(124+1.025*9.81*(-11.4)\right)}{1.025*9.81*0.8948} = 1.04$

For linear waves, N=1 and for salt water, density=1.025 kg/L

As, $\frac{H}{2} = \eta, H = 2 * 1.04 = 2.08m\ (Ans.)$