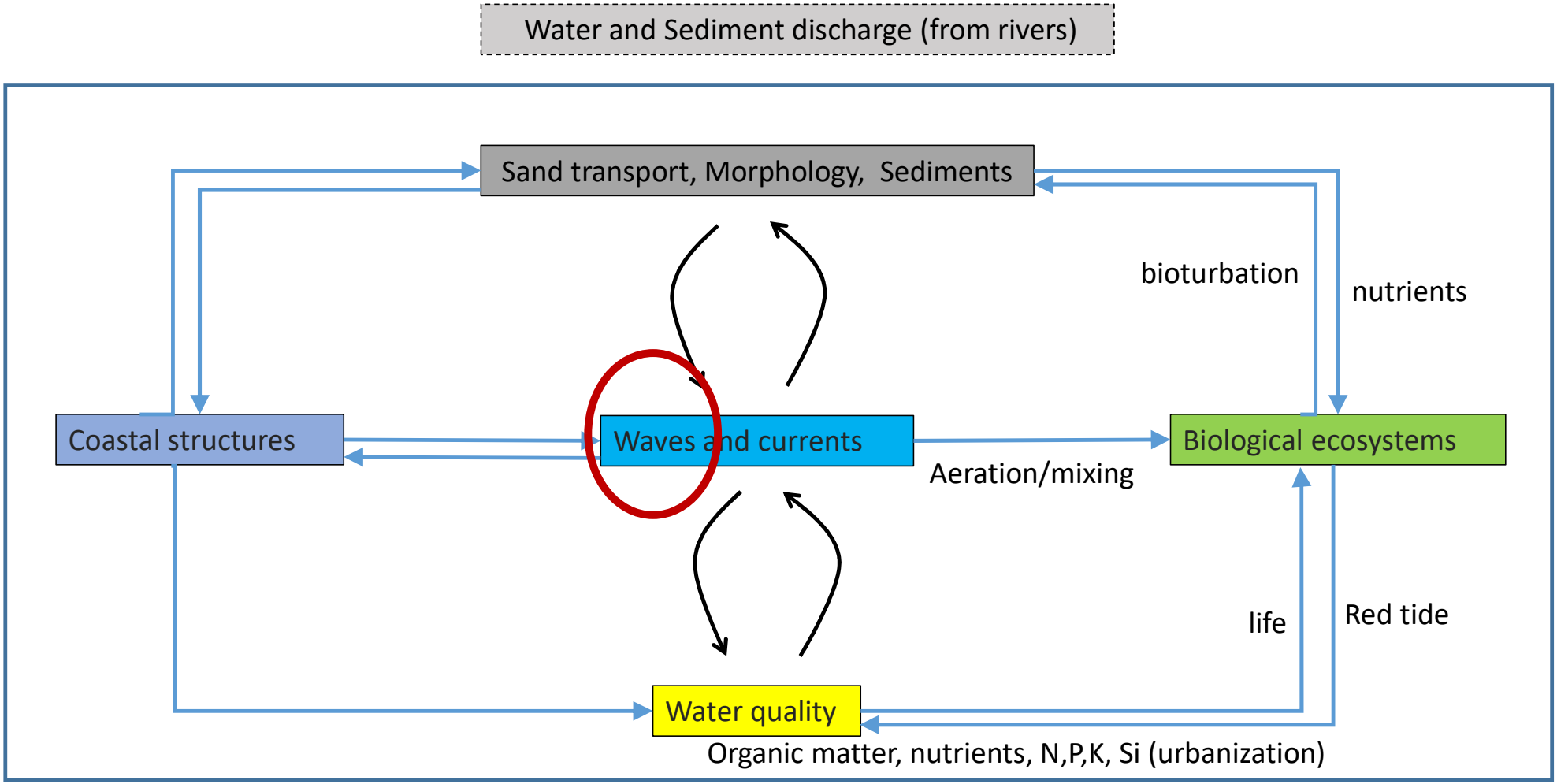
EWC 471: Basic Coastal Engineering

Lecture 7 and 8

Topic: Small Amplitude Wave theory and Application Examples

From previous lecture..



Linear wave theory: Wave energy

- Wave energy and power: The total energy of a wave system is the sum of its kinetic energy and its potential energy.
- The kinetic energy is that part of the total energy due to water particle velocities associated with wave motion. The kinetic energy per unit length of wave crest for a wave defined with the linear theory can be found from:

$$\bar{E}_k = \int_x^{x+L} \int_{-d}^{\eta} \rho \frac{u^2 + w^2}{2} \, dz \, dx$$

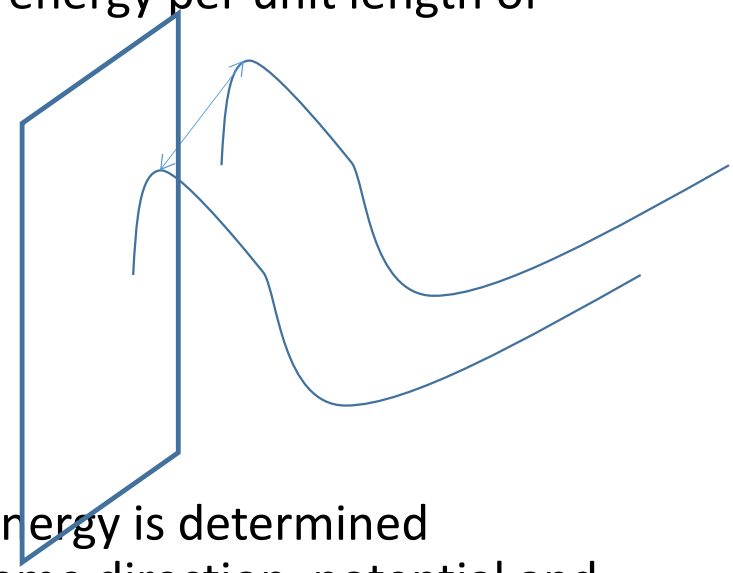
which, upon integration, gives:

$$\bar{E}_k = \frac{1}{16} \rho \, g \, H^2 \, L$$

Linear wave theory: Wave energy

- Potential energy is that part of the energy resulting from part of the fluid mass being above the trough: the wave crest. The potential energy per unit length of wave crest for a linear wave is given by:

$$\bar{E}_p = \int_x^{x+L} \rho\, g \left[\frac{(\eta + d)^2}{2} - \frac{d^2}{2} \right] dx$$



which, upon integration, gives:

$$\bar{E}_p = \frac{1}{16} \rho\, g\, H^2\, L$$

- According to the Linear wave theory, if the potential energy is determined relative to SWL, and all waves are propagated in the same direction, potential and kinetic energy components are equal, and the total wave energy in one wavelength per unit crest width is given by:

$$E = E_k + E_p = \frac{\rho g H^2 L}{16} + \frac{\rho g H^2 L}{16} = \frac{\rho g H^2 L}{8}$$

Linear wave theory: Wave energy

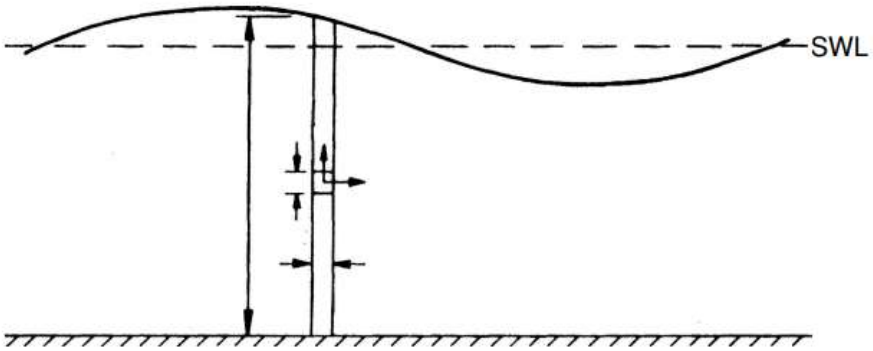


Figure 2.4. Definition sketch for wave energy derivation.

$$E_k = \int_o^L \int_{-d}^{o} \frac{1}{2}\rho dx dz(u^2 + w^2)$$

$$E_k = \frac{\rho g H^2 L}{16}$$

$$E_p = \int_o^L \rho g(d+\eta)\left(\frac{d+\eta}{2}\right)dx - \rho g L d\left(\frac{d}{2}\right)$$

$$E_p = \frac{\rho g H^2 L}{16}$$

Total energy:

$$E = E_k + E_p = \frac{\rho g H^2 L}{8}$$

Energy density or specific energy:

$$\bar{E} = \frac{E}{L(1)} = \frac{\rho g H^2}{8}$$

Linear wave theory: Wave power

Power: wave energy per unit time transmitted in the direction of wave propagation.
Power= force*velocity

$$p = \frac{1}{T}\int_o^T \int_{-d}^{o} (p + \rho g z) u dz dt$$

$$P = \frac{\rho g H^2 L}{16T}\left(1 + \frac{2kd}{\sinh 2kd}\right)$$

$$P = \frac{E}{2T}\left(1 + \frac{2kd}{\sinh 2kd}\right)$$

$$n = \frac{1}{2}\left(1 + \frac{2kd}{\sinh 2kd}\right)$$

$$P = \frac{nE}{T}$$

$$P = \left(\frac{nE}{T}\right)_1 = \left(\frac{nE}{T}\right)_2 = \text{constant}$$

n is the fraction of mechanical energy that is transmitted forward in each wave period.
n increases from 0.5 at deep water to 1.0 at shallow water

Linear wave theory: Wave power

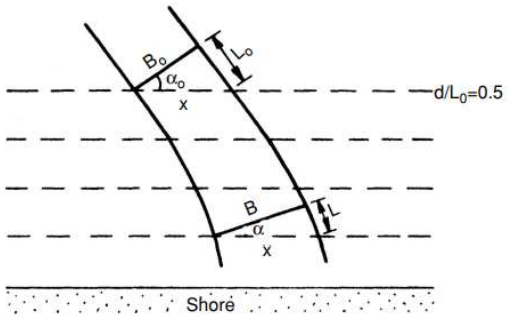


Figure 4.3. Wave refraction over straight parallel bottom contours.

$$\left(\frac{BnE}{T}\right)_1 = \left(\frac{BnE}{T}\right)_2 = \text{constant}$$

$$\frac{H_1}{H_2} = \sqrt{\frac{n_2 L_2}{n_1 L_1}}\sqrt{\frac{B_2}{B_1}}$$

$$\frac{H}{H_o} = \sqrt{\frac{L_o}{2nL}}\sqrt{\frac{B_o}{B}} \qquad \text{Eq. (2')]}$$

$$\frac{H}{H_o} = \frac{H}{H_o'}\sqrt{\frac{B_o}{B}}$$

Linear wave theory: Wave power

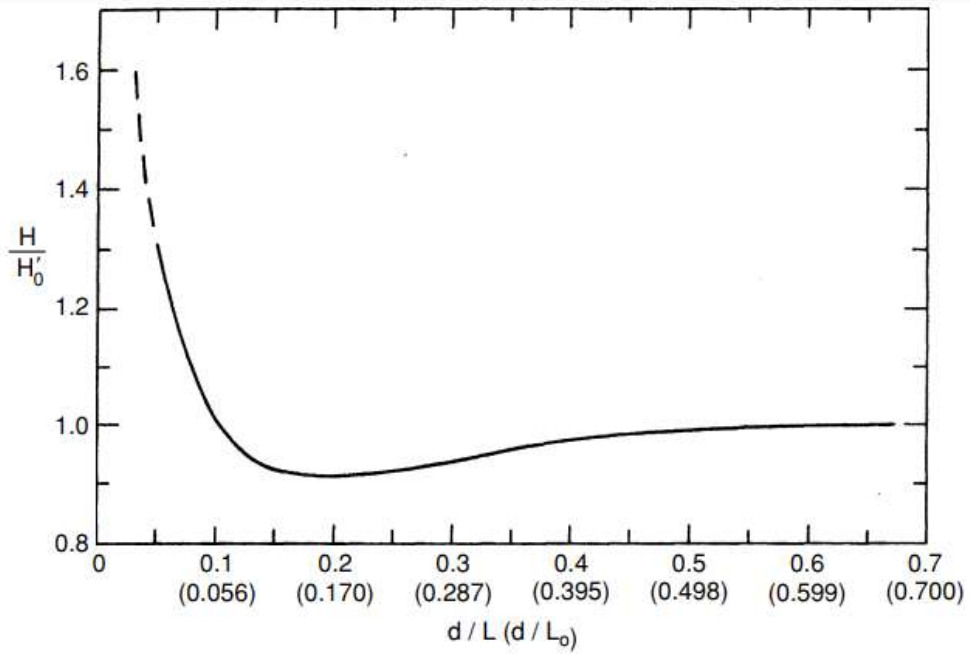


Figure 2.5. Dimensionless wave height versus relative depth for two-dimensional wave transformation.

Figure 2.5 is a plot of H/H'_0 versus d/L and d/L_o from deep to shallow water. Initially, as a wave enters intermediate water depths the wave height decreases because n increases at a faster rate than L decreases [see Eq. 2']. H/H'_0 reaches a minimum value of 0.913 at $d/L = 0.189$ ($d/L_0 = 0.157$). Shoreward of this point the wave height grows at an ever-increasing rate until the wave becomes unstable and **breaks**.

Linear wave theory: Wave energy

- The subscripts *k* and *p* refer to kinetic and potential energies. Total average wave energy per unit surface area, termed the *specific energy* or *energy density*, is given by:

$$\bar{E} = \frac{E}{L} = \frac{\rho g H^2}{8}$$

- *Wave energy flux* is the rate at which energy is transmitted in the direction of wave propagation across a vertical plane perpendicular to the direction of wave advance and extending down the entire depth. Assuming linear theory holds, the average energy flux per unit wave crest width transmitted across a vertical plane perpendicular to the direction of wave advance is:

$$\bar{P} = \frac{1}{T}\int_{t}^{t+T} \int_{-d}^{\eta} p\, u\, dz\, dt$$

which, upon integration, gives: $\bar{P} = \bar{E}nC = \bar{E}C_g$

- where $\bar{P}$ is frequently called *wave power*, and the variable *n* has been defined earlier

Linear wave theory: Wave energy

- For deep and shallow water: $\bar{P}_0 = \frac{1}{2} \bar{E}_0 C_o$ (deep water)

$$\bar{P} = \bar{E} C_g = \bar{E} C \text{ (shallow water)}$$

- An energy balance for a region through which waves are passing will reveal that, for steady state, the amount of energy entering the region will equal the amount leaving the region provided no energy is added or removed. Therefore, when the waves are moving so that their crests are parallel to the bottom contours: $\bar{E}_0 n_0 C_0 = \bar{E} n C$

since $n_0 = \frac{1}{2}$, so $\frac{1}{2} \bar{E}_0 C_0 = \bar{E} n C$

Linear wave theory: Wave energy

- When the wave crests are not parallel to the bottom contours, some parts of the wave will be traveling at different speeds and the wave will be refracted; in this case the previous Equation does not apply.
- The rate of energy transmission is important for coastal design, and it requires knowledge of C_g to determine how fast waves move toward shore. The mean rate of energy transmission associated with waves propagating into an area of calm water provides a different physical description of the concept of group velocity.
- The previous Equation establishes a relationship between the ratio of the wave height at some arbitrary depth and the deep water wave height. This ratio, known as the *shoaling coefficient*, is dependent on the wave steepness. The variation of shoaling coefficient with wave steepness as a function of relative water depth d/L_0 is an important parameter for solving practical coastal engineering problems.

Linear wave theory: Application example #6

Consider the wave from previous example when it has propagated into a water depth of 10 m without refracting and assuming energy gains and losses can be ignored. Determine the wave height and the water particle velocity and pressure at a point 1 m below the still water level under the wave crest. (Assume fresh water.)

Solution:

From Example 2.3–1 we have $L_0 = 156\,\text{m}$ and Eq. (2.14) gives

$$L = \frac{9.81(10)^2}{2\pi} \tanh \frac{2\pi(10)}{L}$$

which can be solved by trial to yield $L = 93.3\,\text{m}$. Then, $k = 2\pi/93.3 = 0.0673\,\text{m}^{-1}$ and from Eq. (2.38)

$$n = \frac{1}{2}\left(1 + \frac{2(0.0673)(10)}{\sinh(2(0.0673)(10)}\right) = 0.874$$

Linear wave theory: Application example #6

With $K_r = 1$, Eq. (2.42) yields

$$H = 2\sqrt{\frac{156}{2(0.874)(93.3)}} = 1.97\,\text{m}$$

At the crest of the wave $\cos(kx - \sigma t) = 1$, and $z = -1$, so Eq. (2.21) gives

$$u = \frac{\pi(1.97)}{10}\left[\frac{\cosh(0.0673)(9)}{\sinh(0.0673)(10)}\right] = 1.01\ \text{m/s}$$

which is the total particle velocity since $w = 0$ under the wave crest. Equation (2.32) gives

$$P = -1000(9.81)(-1) + \frac{1000(9.81)(1.97)}{2}\left[\frac{\cosh(0.0673)9}{\cosh(0.0673)10}\right]$$

$$= 19,113\ \text{N/m}^2$$

Linear wave theory: group velocity

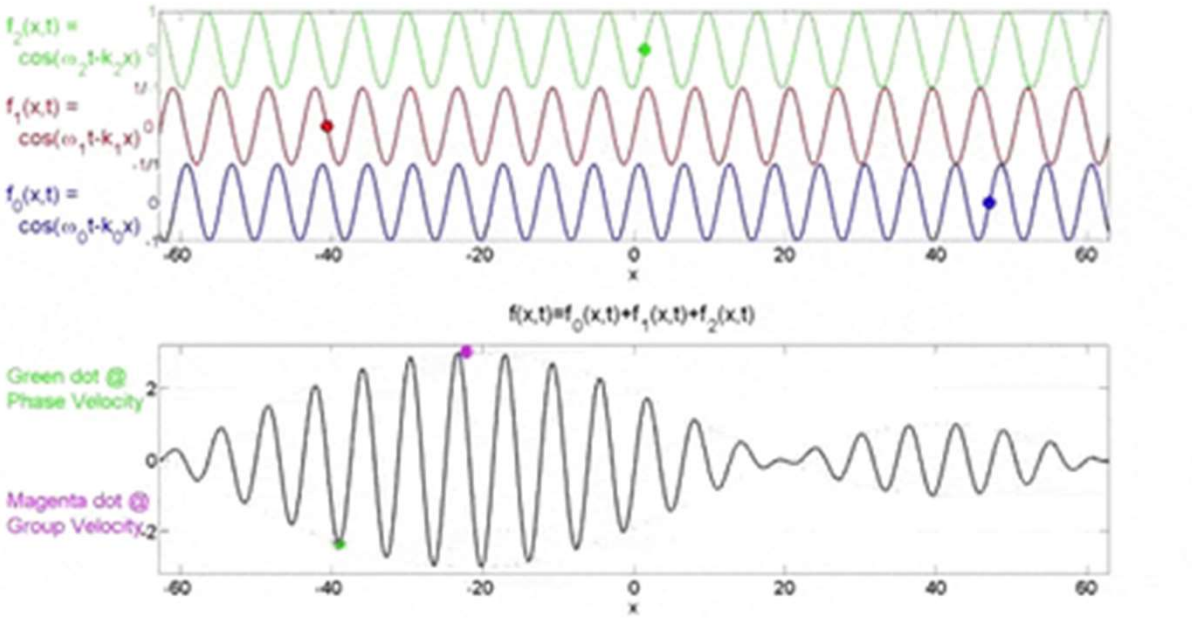
- Group velocity: It is desirable to know how fast wave energy is moving. One way to determine this is to look at the speed of wave groups that represents propagation waves or a wave train travels is generally not identical to the speed with which individual waves within the group travel. The group speed is termed the *group velocity Cg*; the individual wave speed is the *phase velocity* or *wave celerity* given by following two Equations. For waves propagating in deep or transitional water with gravity as the primary restoring force, the group velocity will be less than the phase velocity.

$$C = \sqrt{\frac{gL}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)}$$

- For those waves, propagated primarily under the influence of surface tension (i.e., capillary waves), the group velocity may exceed the velocity of an individual wave.

Linear wave theory: group velocity

$$\eta = \eta_1 + \eta_2 = \frac{H}{2}\cos\left(\frac{2\pi x}{L_1} - \frac{2\pi t}{T_1}\right) + \frac{H}{2}\cos\left(\frac{2\pi x}{L_2} - \frac{2\pi t}{T_2}\right)$$



Linear wave theory: group velocity

- The concept of group velocity can be described by considering the interaction of two sinusoidal wave trains moving in the same direction with slightly different wavelengths and periods. The equation of the water surface is given by

$$\eta = \eta_1 + \eta_2 = \frac{H}{2}\cos\left(\frac{2\pi x}{L_1} - \frac{2\pi t}{T_1}\right) + \frac{H}{2}\cos\left(\frac{2\pi x}{L_2} - \frac{2\pi t}{T_2}\right)$$

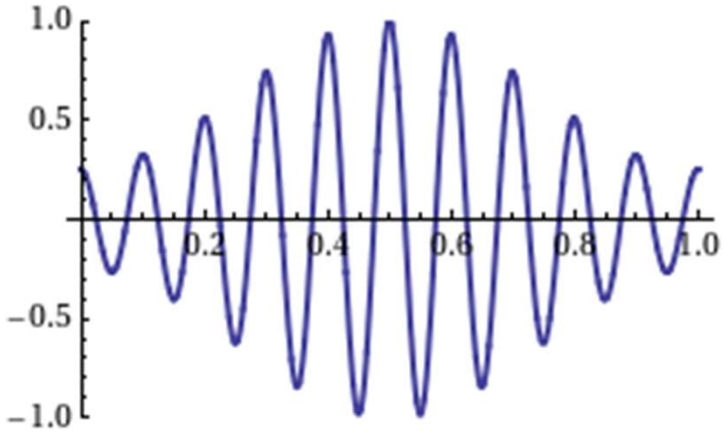
Animation: https://www.fiberoptics4sale.com/blogs/wave-optics/100149702-phase-velocity-and-group-velocity

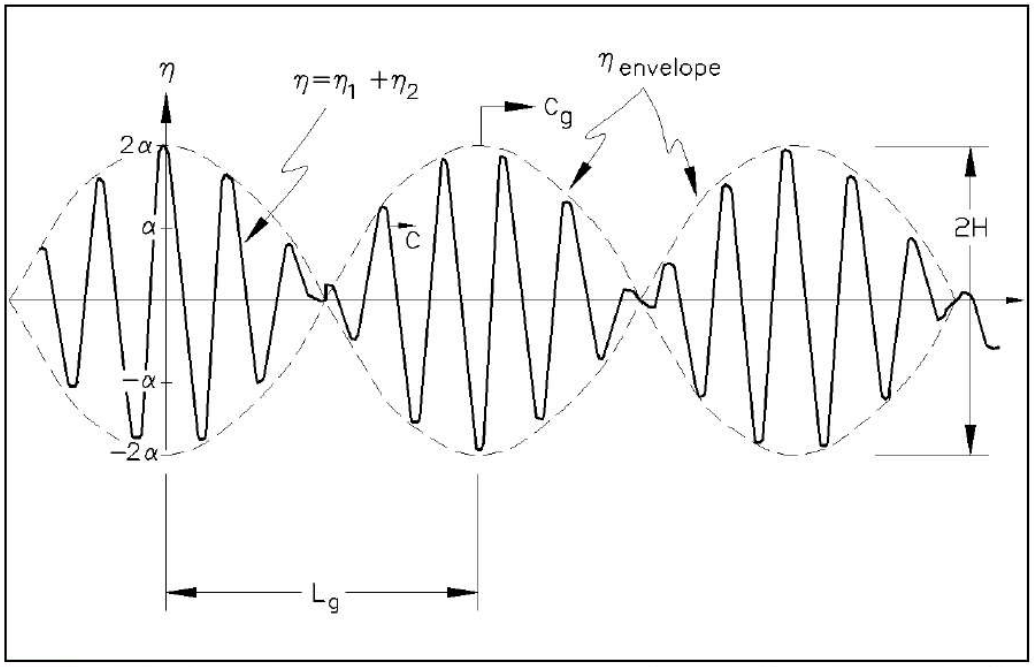
- where and η_1 are η_2 the two components. They may be summed since superposition of solutions is permissible when the linear wave theory is used. For simplicity, the heights of both wave components have been assumed equal. Since the wavelengths of the two component waves, L1 and L2, have been assumed slightly different for some values of x at a given time, the two components will be in phase and the wave height observed will be 2H; for some other values of x, the two waves will be completely out of phase and

Linear wave theory: group velocity

- the resultant wave height will be zero. The surface profile made up of the sum of the two sinusoidal waves is given by following Equation and is shown in Figure. The waves shown in the Figure appear to be traveling in groups described by the equation of the **envelope** curves

$$\eta_{envelope} = \pm H \cos\left[\pi\left(\frac{L_2 - L_1}{L_1 L_2}\right)x - \pi\left(\frac{T_2 - T_1}{T_1 T_2}\right)t\right]$$





Characteristics of a wave group formed by the addition of sinusoids with different periods

Linear wave theory: group velocity

- It is the speed of these groups (i.e., the velocity of propagation of the envelope curves) defined in $\eta_{envelop}$ Equation that represents the group velocity. The limiting speed of the wave groups as they become large (i.e., as the wavelength L_1 approaches L_2 and consequently the wave period T_1 approaches T_2) is the group velocity and can be shown to be equal to:

$$C_g = \frac{1}{2}\frac{L}{T}\left[1 + \frac{\frac{4\pi d}{L}}{\sinh\left(\frac{4\pi d}{L}\right)}\right] = nC$$

where

$$n = \frac{1}{2}\left[1 + \frac{\frac{4\pi d}{L}}{\sinh\left(\frac{4\pi d}{L}\right)}\right]$$

Linear wave theory: group velocity

- In deep water, the term (4πd/L)/sinh(4 π d/L) is approximately zero and *n* = 1/2, giving

$$C_{g_0} = \frac{1}{2}\frac{L_0}{T} = \frac{1}{2}C_0 \text{ (deep water)}$$

or in deep water the group velocity is one-half the phase velocity.

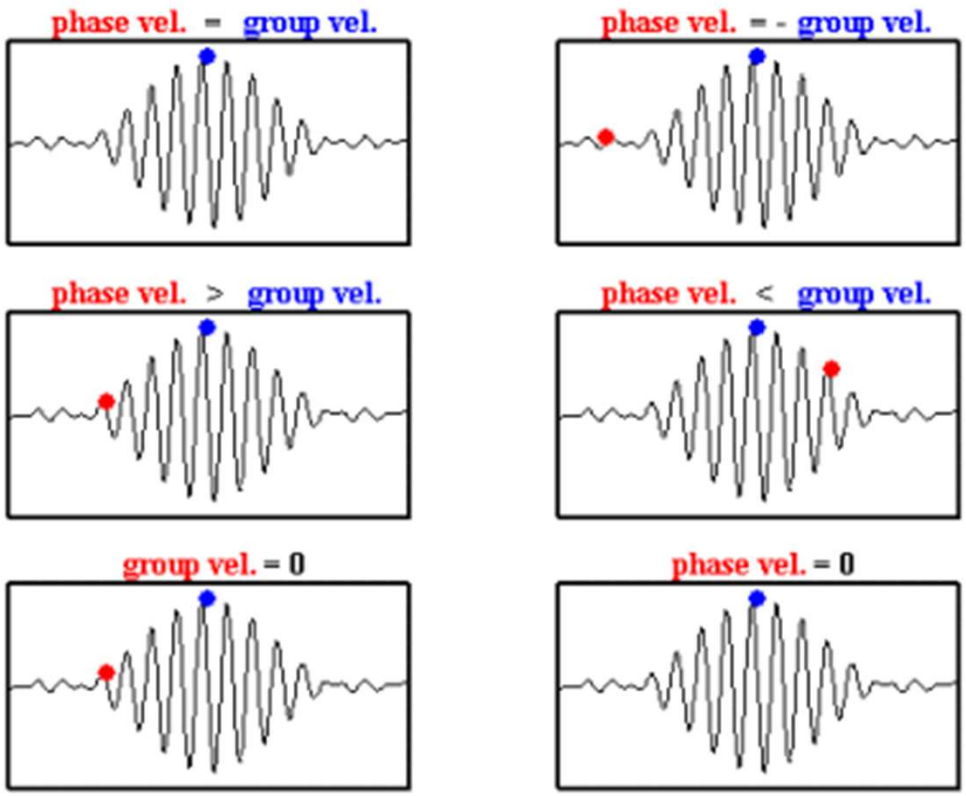
- In shallow water, sinh(4πd/L) ≈ 4πd/L and $C_{g_s} = \frac{L}{T} = C \approx \sqrt{gd}$ (shallow water)

- Hence, in shallow water the group and phase velocities are equal. Thus, in shallow water, because wave celerity is determined by the depth, all component waves in a wave train will travel at the same speed precluding the alternate reinforcing and canceling of components. In deep and transitional water, wave celerity depends on wavelength; hence, slightly longer waves travel slightly faster and produce the small phase differences resulting in wave groups. These waves are said to be *dispersive* or propagating in a *dispersive medium*; i.e., in a medium where their celerity is dependent on wavelength.

Linear wave theory: group velocity

- Outside of shallow water, the phase velocity of gravity waves is greater than the group velocity. An observer that follows a group of waves at group velocity will see waves that originate at the rear of the group move forward through the group traveling at the phase velocity and disappear at the front of the wave group.
- Group velocity is important because it is with this velocity that wave energy is propagated. Although mathematically the group velocity can be shown rigorously from the interference of two or more waves (Lamb 1945), the physical significance is not as obvious as it is in the method based on the consideration of wave energy. Therefore an additional explanation of group velocity is provided on wave energy and energy transmission.

Linear wave theory: group velocity



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Linear wave theory: Application example

A wave with period T = 10 s moves from a water depth of 200 m to a depth of 3 m. Calculate the wave group velocity, Cg, at both depths (i.e. at 200 m and 3 m, respectively).

Solution:

Deep water wavelength, $L_0 = 1.56\ T^2 = 1.56*(10)^2 = 156$ m

d = 200m, d/L_0 = 200/156 i.e. >1/2, deep water condition

$$C = C_0 = \frac{gT}{2\pi} = \frac{9.81 * 10}{2 * 3.14} = 15.6\ m/s$$

$$C_g = \frac{C}{2} = \frac{15.6}{2} = 7.8 m/s \quad \text{(Ans.)}$$

When d= 3m, d/L_0 = 3/156 = 0.0192 i.e. < 1/20, shallow water condition.

$$C_g = C = \sqrt{gd} = \sqrt{9.81 * 3} = 5.42 m/s \quad \text{(Ans.)}$$

Linear wave theory: summary

Relative Depth	Shallow Water $\frac{d}{L} < \frac{1}{20}$ $kd < \frac{\pi}{10}$	Transitional Water $\frac{1}{20} < \frac{d}{L} < \frac{1}{2}$ $\frac{\pi}{10} < kd < \frac{\pi}{2}$	Deep Water $\frac{d}{L} > \frac{1}{2}$ $kd > \frac{\pi}{2}$
1. Wave profile	Same As >	$\eta = \frac{H}{2}\cos\left[\frac{2\pi x}{L} - \frac{2\pi t}{T}\right] = \frac{H}{2}\cos\theta$	< Same As
2. Wave celerity	$C = \frac{L}{T} = \sqrt{gd}$	$C = \frac{L}{T} = \frac{gT}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)$	$C = C_0 = \frac{L}{T} = \frac{gT}{2\pi}$
3. Wavelength	$L = T\sqrt{gd} = CT$	$L = \frac{gT^2}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)$	$L = L_0 = \frac{gT^2}{2\pi} = C_0 T$
4. Group velocity	$C_g = C = \sqrt{gd}$	$C_g = nC = \frac{1}{2}\left[1 + \frac{4\pi d/L}{\sinh(4\pi d/L)}\right]C$	$C_g = \frac{1}{2}C = \frac{gT}{4\pi}$
5. Water particle velocity (a) Horizontal	$u = \frac{H}{2}\sqrt{\frac{g}{d}}\cos\theta$	$u = \frac{H}{2}\frac{gT}{L}\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\cos\theta$	$u = \frac{\pi H}{T}e^{\left(\frac{2\pi z}{L}\right)}\cos\theta$
(b) Vertical	$w = \frac{H\pi}{T}\left(1+\frac{z}{d}\right)\sin\theta$	$w = \frac{H}{2}\frac{gT}{L}\frac{\sinh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\sin\theta$	$w = \frac{\pi H}{T}e^{\left(\frac{2\pi z}{L}\right)}\sin\theta$

Linear wave theory: summary

Relative Depth	Shallow Water $\frac{d}{L} < \frac{1}{20}$ $kd < \frac{\pi}{10}$	Transitional Water $\frac{1}{20} < \frac{d}{L} < \frac{1}{2}$ $\frac{\pi}{10} < kd < \frac{\pi}{2}$	Deep Water $\frac{d}{L} > \frac{1}{2}$ $kd > \frac{\pi}{2}$
6. Water particle accelerations			
(a) Horizontal	$a_x = \frac{H\pi}{T}\sqrt{\frac{g}{d}}\sin\theta$	$a_x = \frac{g\pi H}{L}\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\sin\theta$	$a_x = 2H\left(\frac{\pi}{T}\right)^2 e^{\left(\frac{2\pi z}{L}\right)}\sin\theta$
(b) Vertical	$a_z = -2H\left(\frac{\pi}{T}\right)^2\left(1+\frac{z}{d}\right)\cos\theta$	$a_z = -\frac{g\pi H}{L}\frac{\sinh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\cos\theta$	$a_z = -2H\left(\frac{\pi}{T}\right)^2 e^{\left(\frac{2\pi z}{L}\right)}\cos\theta$
7. Water particle displacements			
(a) Horizontal	$\xi = -\frac{HT}{4\pi}\sqrt{\frac{g}{d}}\sin\theta$	$\xi = -\frac{H}{2}\frac{\cosh[2\pi(z+d)/L]}{\sinh(2\pi d/L)}\sin\theta$	$\xi = -\frac{H}{2}e^{\left(\frac{2\pi z}{L}\right)}\sin\theta$
(b) Vertical	$\zeta = \frac{H}{2}\left(1+\frac{z}{d}\right)\cos\theta$	$\zeta = \frac{H}{2}\frac{\sinh[2\pi(z+d)/L]}{\sinh(2\pi d/L)}\cos\theta$	$\zeta = \frac{H}{2}e^{\left(\frac{2\pi z}{L}\right)}\cos\theta$
8. Subsurface pressure	$p = \rho g(\eta - z)$	$p = \rho g\eta\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)} - \rho g z$	$p = \rho g\eta e^{\left(\frac{2\pi z}{L}\right)} - \rho g z$

Thank You